Complex Analysis Math 147—Winter 2008
Assignment 13
due March 20, 2008

1. Let the nonconstant function f be continuous on the closed unit disk and analytic on the open unit disk. Suppose $|f(z)|$ is a constant on $|z| = 1$. Prove that f has a zero in $|z| < 1$.

   Hint: Denote the constant by c, so that $|f(z)| = c$ for all $|z| = 1$ and consider the two cases $c = 0$ and $c > 0$. In the latter case apply the Maximum modulus theorem to both f and $1/f$ (Assuming f has no zeros).

2. Let f be a continuous function from the closed unit disk to the closed unit disk which is analytic on the open unit disk and such that $|f(z)| = 1$ whenever $|z| = 1$. Show that f is a rational function, that is, a quotient of polynomials.

   Hint: Recall that if a is a zero of order k of f, then $g(z) := f(z)/(z-a)^k$, defined for $z \neq a$ has a removable singularity at $z = a$ and that the analytic extension of g to a doesn't vanish at a. Proceed by proving the following statements.

   (a) If a is a zero of order k of f, show that the function $h(z) := f(z)/\left(\frac{z-a}{1-\bar{a}z}\right)^k$ has a removable singularity at $z = a$ and that the analytic extension of h to a doesn't vanish at a, and furthermore satisfies $|h(z)| = 1$ whenever $|z| = 1$.

   (b) Let $a_1, \ldots, a_n$ be all the distinct zeros of f in the open unit disk, and let k_j be the order of a_j. Show that

$$f(z) = c \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a}_j z} \right)^{k_j}$$

   for some constant $c \in \mathbf{C}$ and for all $|z| \leq 1$. (Use Problem 1 above.)