Complex Analysis Math 147—Winter 2008
Assignment 14
due March 20, 2008

Suppose f is analytic in $B(a, R) - \{a\}$ except for a sequence of poles $\{z_1, z_2, \ldots\}$ which converges to a. (Note that although f is not assumed to be analytic at a, nevertheless, a is not an isolated singularity of f.) Show that $f(B(a, R) - \{a, z_1, \ldots\})$ is dense in the complex plane.

> Hint: If it is not true, consider $g(z) = 1/(f(z) - w)$ for $z \in G := B(a, R) - \{a, z_1, z_2, \ldots\}$, where $w \in \mathbf{C}$ and $\delta > 0$ are such that $|f(z) - w| > \delta$ for all $z \in G$. Then obtain a contradiction by proving the following statements.

(a) There is an analytic extension h of g to $B(a, R) - \{a\}$ which vanishes at each z_n.

> Hint: Apply Riemann's Removable Singularity Theorem successively to obtain a sequence of functions $g_1, g_2, \ldots$, such that g_1 is analytic, vanishes at z_1, and extends g to z_1; g_2 is analytic, vanishes at z_2, and extends g_1 to z_2, and hence extends g to z_1 and z_2; and so forth, so that g_n is analytic, vanishes at $z_1, \ldots, z_n$, and extends g to $z_1, \ldots, z_n$, for $n = 1, 2, \ldots$. Now you can define h (consistently) on $B(a, R) - \{a\}$.

(b) The point a is an isolated singularity of h which is a removable singularity of h, and the analytic extension of h to a vanishes at a.