Assignment 7  Complete the proof of Proposition 22.3 as follows. Suppose that \( G = A \cup B \) where \( A, B \) are open, non-empty, and disjoint. Let \( z_0, z_1, \ldots, z_n \) be a finite sequence of points in \( D \) with \( z_0 = a, z_n = b \) and such that the line segments \( [z_{k-1}, z_k] := \{ sz_{k-1} + (1-s)z_k : s \in [0,1] \} \) all lie in \( D \). Choose one of these segments which has one endpoint in \( A \) and the other in \( B \), and denote it by \([p,q]\). Then \([0,1] = S \cup T \) where \( S = \{ s \in [0,1] : sq + (1-s)p \in A \} \) and \( T = \{ t \in [0,1] : tq + (1-t)p \in B \} \). \( S \) and \( T \) are each non-empty, since \( 0 \in S \) and \( 1 \in T \).

(a) Prove that \( S \) is equal to a set of the form \( \{0\} \cup S' \) where \( S' \) is open. Prove that \( T \) is equal to a set of the form \( \{1\} \cup T' \) where \( T' \) is open.

(b) Complete the proof of Proposition 22.3 by deriving a contradiction (Hint: Consider \( \alpha = \text{sup} S \) and the three cases \( \alpha = 0, \alpha = 1, 0 < \alpha < 1 \)).

Assignment 8 If \( f \) is analytic on the open unit disk and \( f(0) = 0 \) and \( |f(z)| \leq 1 \) for all \( |z| < 1 \), then the function \( g \) defined on the unit disk by \( g(z) = f(z)/z \) for \( z \neq 0 \) and \( g(0) = f'(0) \), is analytic at \( 0 \).

Assignment 9 For a fixed complex number \( a \) with \( |a| < 1 \), define a function \( \varphi_a \) by

\[
\varphi_a(z) = \frac{z + a}{1 + \overline{a}z}.
\]

Although \( \varphi_a(z) \) is defined for all \( z \neq -1/\overline{a} \), we shall consider it as a function on the closed unit disk \( |z| \leq 1 \). Prove the following statements.

(a) If \( |z| < 1 \) then \( |\varphi_a(z)| < 1 \).

(b) If \( |z| = 1 \) then \( |\varphi_a(z)| = 1 \).

(c) \( \varphi_a \) is a one to one function, that is, if \( |z_1| < 1, |z_2| < 1 \) and if \( f(z_1) = f(z_2) \), then \( z_1 = z_2 \).

(d) \( \varphi_a \) is an onto function, that is, if \( |w_0| < 1 \), then there is a \( z_0 \) with \( |z_0| < 1 \) and \( f(z_0) = w_0 \).

(e) What is the inverse of \( \varphi_a \)?

Assignment 10 Let \( f \) be an arbitrary analytic function on the unit disk \( |z| < 1 \) which is one to one and onto, that is, if \( |z_1| < 1, |z_2| < 1 \) and if \( f(z_1) = f(z_2) \), then \( z_1 = z_2 \); and if \( |w_0| < 1 \), then there is a \( z_0 \) with \( |z_0| < 1 \) and \( f(z_0) = w_0 \). Prove the following statements.

(a) If \( f(0) = 0 \), then \( f(z) = e^{i\theta} z \) for some real \( \theta \).

(b) If \( f(0) = a \neq 0 \), let \( g(z) \) be defined by \( g(z) = \varphi_{-a}(f(z)) \). Then \( g(z) = e^{i\theta} z \) for some real \( \theta \).

(c) The function \( f \) has the form

\[
f(z) = e^{i\theta} \varphi_a(z),
\]

for some \( \theta \) real and \( |a| < 1 \).