Complex Analysis Math 147—Winter 2008
Assignments 7,8,9,10
due March 7, 2008

Assignment 7 Complete the proof of Proposition 22.3 as follows. Suppose that $G = A \cup B$ where A, B are open, non-empty, and disjoint. Let $z_0, z_1, \ldots, z_n$ be a finite sequence of points in D with $z_0 = a$, $z_n = b$ and such that the line segments $[z_{k-1}, z_k] := \{sz_k + (1-s)z_{k-1} : s \in [0,1]\}$ all lie in D. Choose one of these segments which has one endpoint in A and the other in B, and denote it by $[p, q]$. Then $[0,1] = S \cup T$ where $S = \{s \in [0,1] : sq + (1-s)p \in A\}$ and $T = \{t \in [0,1] : tq + (1-t)p \in B\}$. S and T are each non-empty, since $0 \in S$ and $1 \in T$.

(a) Prove that S is equal to a set of the form $\{0\} \cup S'$ where S' is open. Prove that T is equal to a set of the form $\{1\} \cup T'$ where T' is open.

(b) Complete the proof of Proposition 22.3 by deriving a contradiction (Hint: Consider $\alpha = \sup S$ and the three cases $\alpha = 0, \alpha = 1, 0 < \alpha < 1$).

Assignment 8 If f is analytic on the open unit disk and $f(0) = 0$ and $|f(z)| \le 1$ for all $|z| < 1$, then the function g defined on the unit disk by $g(z) = f(z)/z$ for $z \ne 0$ and $g(0) = f'(0)$, is analytic at 0.

Assignment 9 For a fixed complex number a with $|a| < 1$, define a function φ_a by

$$\varphi_a(z) = \frac{z + a}{1 + \overline{a}z}.$$

Although $\varphi_a(z)$ is defined for all $z \ne -1/\overline{a}$, we shall consider it as a function on the closed unit disk $|z| \le 1$. Prove the following statements.

(a) If $|z| < 1$ then $|\varphi_a(z)| < 1$.

(b) If $|z| = 1$ then $|\varphi_a(z)| = 1$.

(c) φ_a is a one to one function, that is, if $|z_1| < 1, |z_2| < 1$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$.

(d) φ_a is an onto function, that is, if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $f(z_0) = w_0$.

(e) What is the inverse of φ_a?

Assignment 10 Let f be an arbitrary analytic function on the unit disk $|z| < 1$ which is one to one and onto, that is, if $|z_1| < 1, |z_2| < 1$ and if $f(z_1) = f(z_2)$, then $z_1 = z_2$; and if $|w_0| < 1$, then there is a z_0 with $|z_0| < 1$ and $f(z_0) = w_0$. Prove the following statements.

(a) If $f(0) = 0$, then $f(z) = e^{i\theta}z$ for some real θ.

(b) If $f(0) = a \ne 0$, let $g(z)$ be defined by $g(z) = \varphi_{-a}(f(z))$. Then $g(z) = e^{i\theta}z$ for some real θ.

(c) The function f has the form

$$f(z) = e^{i\theta}\varphi_a(z),$$

for some θ real and $|a| < 1$.