

Math 140C Fall 2006  
Assignments 1-26

**Assignment 1** (Due September 29)

1. Read sections 1.2, 1.3, 1.4 in Buck (The lectures will continue with section 1.5). Do not waste your time reading about the concepts *angle*, *orthogonal*, *hyperplane*, *normal vector*, *line*, *convexity*, which are discussed in section 1.3 of Buck. We have no immediate use for them. Thus, you may skip pages 15-18 and 21-27 for now.
2.
  - Buck [§1.2 page 10 #5, 10, 23]
  - Buck [§1.3 page 19 #1, 2, 5, 6]

**Assignment 2** (Due October 6)

- Buck [§1.4 page 27 #3, 15, 16]
- Buck [§1.5 page 36 #1, 5, 9, 13]

**Assignment 3** (Due October 6)

Fix  $p \in \mathbf{R}^n$ . Show that  $\{q \in \mathbf{R}^n : |q - p| > 2\}$  is an open set.

**Assignment 4** (Due October 13)

Recall Young's inequality (Theorem 3.1):

Let  $\varphi$  be differentiable and strictly increasing on  $[0, \infty)$ ,  $\varphi(0) = 0$ ,  $\lim_{u \rightarrow \infty} \varphi(u) = \infty$ ,  $\psi := \varphi^{-1}$ ,  $\Phi(x) := \int_0^x \varphi(u) du$ ,  $\Psi(x) := \int_0^x \psi(u) du$ . Then for all  $a, b \in [0, \infty)$ ,

$$ab \leq \Phi(a) + \Psi(b). \quad (1)$$

Moreover, equality holds in (1) if and only if  $b = \varphi(a)$ .

Give a rigorous proof of Young's inequality. More precisely,

**Step 1** First establish, for  $c \in [0, \infty)$ , the formula

$$\int_0^c \varphi(u) du + \int_0^{\varphi(c)} \psi(v) dv = c\varphi(c). \quad (2)$$

**Step 2** Use (2) to prove (1).

**Step 3** Prove the “moreover” statement.

**Assignment 5** (Due October 13)

- Show that every open set in  $\mathbf{R}^n$  is the union of a countable collection of open balls. (Hint: The answer is somewhere in the minutes for my 140C class of Fall 2005)

- Show that in  $\mathbf{R}^1$ , the open balls can be assumed to be disjoint

**Assignment 6** (Due October 6) Prove the following assertions:

- (a)  $\text{int } S = \cup \{G : G \text{ is open, } G \subset S\}$
- (b)  $\overline{S} = \cap \{F : F \text{ is closed, } S \subset F\}$

**Assignment 7** (Due October 6) [Buck §1.5 page 36 #2,6,10,11]

**Assignment 8** (Due October 13) Prove directly the following three assertions. The fourth assertion will be proved in class.

- (a) If  $S$  satisfies BW, then  $S$  is a closed set.
- (b) If  $S$  satisfies BW, then  $S$  is a bounded set.
- (c) If  $S$  satisfies HB, then  $S$  is a bounded set.
- (d) (This will be done in class, not part of the homework—it is included here for comparison purposes only) If  $S$  satisfies HB, then  $S$  is a closed set.

Assertions (c) and (d) are stated in Buck as [§1.8 page 69 #1,2]

**Assignment 9** (Due October 27) [Buck, §2.2 page 80 #1 or 2, #3 or 4, #7 or 8, #12 or 13, #14 or 17] You are to hand in 5 problems, one from each of these 5 pairs. You will of course be responsible for all of the problems.

**Assignment 10** (Due October 20) [Buck, §1.6 page 54 #1, 2, 3, 4, 32, 35]

**Assignment 11** (Due October 27) [Buck, §2.3 page 88 #1–7]

**Assignment 12** (Due October 27) [Buck, §3.3 page 134 #4,5]

**Assignment 13** (Due November 3 Hint: This was done in class)

- (A) Give a proof of Problem 3(c) on page 37 of Buck using Corollary 14.2.
- (B) Use Corollary 14.2 to prove that if  $A$  and  $B$  are closed sets in  $\mathbf{R}$ , then  $A \times B$  is a closed set in  $\mathbf{R}^2$ .

**Assignment 14** (Due November 3) Let  $S$  be any subset of  $\mathbf{R}^n$ . Using only the definitions of cluster point and boundary, prove the following statements.

- $\text{cl}(\text{cl } S) \subset \text{cl } S$  (Hint: See the solution to Problem 8(b) on the first midterm for Math 140C, Fall 2005, which is at the top of page 26 of the minutes)
- $\mathbf{R}^n - \text{cl } S$  is open
- $\text{bdy}(\text{cl } S) \subset \text{cl } S$

- $\{\lim_k p_k : p_k \in \text{cl } S\} \subset \text{cl } S$

**Assignment 15** (Due November 3) Prove Theorem 16.4.

**Assignment 16** (Due November 3) Show that a linear transformation (see [Buck, Section 7.3]) is uniformly continuous. (Hint: Use [Buck, Theorem 8, page 338])

**Assignment 17** (Due November 10) From (19) and (20), show that  $D_j f(p_0)$  exists. Hint: Use the property  $\mathbf{D}_{-e_j} f = -\mathbf{D}_{e_j} f$  of directional derivatives (see Buck, page 126)

**Assignment 18** (Due November 10) [Buck page 351 #1,2,7,8]

**Assignment 19** (Due November 17) Show that the function of Problem 4 on page 135 of Buck, namely,  $f(x, y) = xy/(x^2 + y^2)$  for  $(x, y) \neq (0, 0)$  and  $f(0, 0) = 0$  is not differentiable at  $(0, 0)$ .

**Assignment 20** (Due November 17)

(a) Use Theorem 20.3 to prove the following theorem from [Buck, section 3.4].

**Theorem** [Theorem 14, page 136 of Buck] Let  $F(t) = f(x, y)$ , where  $x = g(t)$ ,  $y = h(t)$ , the functions  $g, h$  are assumed to be of class  $C^1$  on an open interval containing  $t_0 \in \mathbf{R}$ , and the function  $f$  is assumed to be of class  $C^1$  in an open ball with center  $p_0 = (x_0, y_0) = (g(t_0), h(t_0))$ . Then  $F$  is of class  $C^1$  on an open interval containing  $t_0 \in \mathbf{R}$ , and for  $t$  in that interval,

$$F'(t) = \frac{\partial f}{\partial x}(g(t), h(t))g'(t) + \frac{\partial f}{\partial y}(g(t), h(t))h'(t).$$

(b) Let  $F(x, y) = f(g(x, y), h(x, y))$ , where  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ ,  $g : \mathbf{R}^2 \rightarrow \mathbf{R}$ , and  $h : \mathbf{R}^2 \rightarrow \mathbf{R}$  are differentiable. Use Theorem 20.3 to prove that  $F$  is differentiable and

$$D_1 F(x, y) = D_1 f(g(x, y), h(x, y))D_1 g(x, y) + D_2 f(g(x, y), h(x, y))D_1 h(x, y)$$

and

$$D_2 F(x, y) = D_1 f(g(x, y), h(x, y))D_2 g(x, y) + D_2 f(g(x, y), h(x, y))D_2 h(x, y).$$

**Assignment 21** (Due November 17) [Buck page 145 #1,2]

**Assignment 22** (Due November 22) [Buck page 154 #18] (Look in the index of Buck to find the definitions of *convex* and *Lipschitz condition*)

**Assignment 23** (Due November 22) [Buck page 361 #11] (The answer to the question is NO. Look at the hint in Buck to construct a proof)

**Assignment 24** (Due December 1) Show that, for  $F = x^2 + y^2 - 1$ ,  $T = T_F$  is not one-to-one on  $D = \mathbf{R}^2$  and  $T(\mathbf{R}^2)$  is not an open subset of  $\mathbf{R}^2$ .

**Assignment 25** (Due December 1) [Buck page 366 #2,5,9,11]

**Assignment 26** (Due December 1) Prove that if  $K$  is a compact set in  $\mathbf{R}^n$  and  $q \notin K$ , then

$$\inf\{|p - q| : p \in K\} > 0.$$