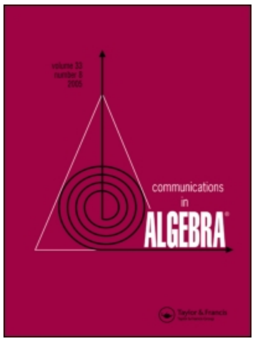

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Representability of actions in the category of (Pre)crossed modules in Leibniz algebras

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ABSTRACT

We investigate the representability of actions in the category of (pre)crossed modules in Leibniz algebras. For this, we construct an actor of a (pre)cat^1-Leibniz algebra and then by using the natural equivalence of the categories of (pre)cat^1-Leibniz algebras and that of (pre)crossed modules, we construct the split extension classifier of the corresponding (pre)crossed module.

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1. Introduction

In an algebraic category, many homological properties such as obstruction theory of the objects depend on the representability of actions in the category. Representability of actions in semi-abelian categories was investigated in [3]. A different study of this problem in the categories of interest was given in [7] with a combinatorial approach. The same was given for modified categories of interest in [8]. The definition of split extension classifier (object which represents actions), is formulated in [2] for semi-abelian categories in terms of categorical notions of internal object action and semidirect product. Categories of interest are semi-abelian categories. As an application of [4], in this special case these notions coincide with the ones given in [12]. Analogous situation exists in the case of modified category of interest and categories equivalent to them. Split extension classifiers (which called actors in [8, 10]) play an important role by analogy with automorphism group of a group in the category of groups or derivation algebra of a Lie algebra in the category of Lie algebras.

Many well known categories of algebraic structures such as (pre)cat^1-Leibniz algebras (Lie algebras, associative algebras, associative commutative algebras), commutative von Neumann rings etc. are not categories of interest. These kinds of notions are unified under the name of modified categories of interest, which satisfy all axioms of a category of groups with operations in [13] except one, which is replaced by a new axiom; these categories satisfy as well two additional axioms introduced in [12].

The category of (pre)crossed modules in Leibniz algebras (which is equivalent to a modified category of interest, namely, the category of (pre)cat^1-Leibniz algebras) was introduced in [6]. In [9], it is shown that the third dimensional cohomology of Leibniz n-algebras classifies crossed modules of Leibniz n-algebras. The notion of crossed modules in Leibniz algebras can be thought as a generalization of Leibniz algebras. For any Leibniz algebra M, we have the crossed module $M \xrightarrow{id} M$ and so the category of Leibniz algebras is a full subcategory of crossed modules in Leibniz algebras. The same thing is true for precrossed modules, namely, $M \longrightarrow 0$ is a precrossed module. Naturally, it will be important to

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investigate the representability of actions, in other words, investigate the existence and construction of split extension classifiers in the category of (pre)crossed modules.

At this vein, for a given (pre)crossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$, we found a condition under which we construct an actor of the corresponding (pre)cat^1-Leibniz algebra $(M_1 \rtimes M_0, \omega_1, \omega_0)$ by using the general construction of universal strict general actor of $(M_1 \rtimes M_0, \omega_1, \omega_0)$ given in [5]. Then applying the equivalence of the categories $(\mathbf{Pre})\mathbf{Cat^1}\text{-}\mathbf{Lbnz} \simeq (\mathbf{Pre})\mathbf{XLbnz}$ of (pre)cat^1-Leibniz algebras and (pre)crossed modules, we carry the construction of an actor of $(M_1 \rtimes M_0, \omega_1, \omega_0)$ to the category of (pre)crossed modules, which is a split extension classifier of the (pre)crossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ under the appropriate condition on it. Therefore we found a new example of a category and individual objects there with representable actions. This problem is stated in [3] (Problem 2).

The outline of the paper is as follows: in Section 2, we give some needed notions from literature and introduce the notions such as biderivations and (generalized) crossed biderivations of a (pre)crossed module. In Section 3, we construct an actor of a precat^1-Leibniz algebra and consequently in Section 4, we construct the split extension classifier of a precrossed module by using the natural equivalence between the categories of precat^1-Leibniz algebras and precrossed modules. In Section 5, we give the same constructions for cat^1-Leibniz algebras and crossed modules with additional modifications. In Section 6, we give a comparison between the split extension classifiers of precrossed modules and crossed modules in the categories of Lie algebras and that of Leibniz algebras.

2. Preliminaries

In this section we will recall some basic definitions and properties about (pre)crossed modules in Leibniz algebras which needed in the rest of the paper. Additionally, we define new notions such as biderivations and (generalized) crossed biderivations of a (pre)crossed module and some related results. Also we will recall the notion of modified category of interest, some related definitions and results from [5]. In addition, we will give the construction of universal strict general actor of a precat^1-Leibniz algebra (M, w_1^M, w_0^M) by using the general construction given for modified categories of interest in [5].

2.1. (Pre)crossed modules in Leibniz algebras

Leibniz algebras, which are a non-antisymmetric generalization of Lie algebras, were introduced in 1965 by Bloh [1], who called them D-algebras and in 1993 Loday [11] made them popular and studied their (co)homology.

Let k be a commutative ring with unit. In the rest of the paper all Leibniz algebras and Lie algebras will be over k.

Definition 2.1. Let L be a k-module and $[-,-] : L \times L \to L$ be a bilinear map. If the Leibniz identity

$$\left[x, [y, z]\right] = \left[\left[x, y\right], z\right] - \left[[x, z], y\right]$$

is satisfied, for all $x, y, z \in L$, then L is called a Leibniz algebra with the bracket $[-,-]$.

Definition 2.2. Let L, L' be Leibniz algebras. A k-linear map $h : L \longrightarrow L'$ is called a homomorphism of Leibniz algebras if $h\left[l, l'\right] = \left[h(l), h(l')\right]$, for all $l, l' \in L$.

Consequently, we have the category of Leibniz algebras which will be denoted here by **Lbnz**.

Now, we recall the construction of the Leibniz algebra $Bider(L)$ of biderivations of a Leibniz algebra L introduced by Loday [11]. Here, we use the notation given in [7].

Definition 2.3. An element φ of $Bider(L)$ is a pair $\varphi := (\varphi_l, \varphi_r)$ consists of k-linear maps $\varphi_l : L \longrightarrow L$, $\varphi_r : L \longrightarrow L$ with

$$\begin{aligned} \varphi_r[x,x'] &= [x,\varphi_r(x')] + [\varphi_r(x),x'], \\ \varphi_l[x,x'] &= [\varphi_l(x),x'] - [\varphi_l(x'),x], \\ [x,\varphi_r(x')] &= -[x,\varphi_l(x')], \end{aligned}$$

for all $x, x' \in L$.

$Bider(L)$ is a Leibniz algebra with the bracket

$$[\varphi,\varphi'] = ([\varphi,\varphi']_l, [\varphi,\varphi']_r)$$

where $[\varphi,\varphi']_l = \varphi_l\varphi'_l + \varphi'_r\varphi_l$, $[\varphi,\varphi']_r = \varphi'_r\varphi_r - \varphi_r\varphi'_r$, for all $\varphi,\varphi' \in Bider(L)$.

Let $M, N \in$ **Lbnz**. Recall from [7] that an action (i.e. derived action) of N on M is a pair of bilinear maps $N \times M \longrightarrow M$, $(n,m) \longmapsto [n,m]$ and $M \times N \longrightarrow M$, $(m,n) \longmapsto [m,n]$ such that

i) $[m,[m',n]] = [[m,m'],n] - [[m,n],m']$,
ii) $[m,[n,m']] = [[m,n],m'] - [[m,m'],n]$,
iii) $[n,[m,m']] = [[n,m],m'] - [[n,m'],m]$,
iv) $[m,[n,n']] = [[m,n],n'] - [[m,n'],n]$,
v) $[n,[m,n']] = [[n,m],n'] - [[n,n'],m]$,
vi) $[n,[n',m]] = [[n,n'],m] - [[n,m],n']$,

for all $m, m' \in M$, $n, n' \in N$.

It is well known that, for a Lie algebra $\mathfrak{g}$, the Lie algebra $Der(\mathfrak{g})$ of all derivations has an action on $\mathfrak{g}$, which was called as adjoint representation. A similar action exists for Leibniz algebras under certain conditions.

Remark 2.4. Let $M \in$ **Lbnz**. Consider the maps

$$\begin{aligned} Bider(M) \times M &\longrightarrow M \\ (\varphi, m) &\longmapsto \varphi_l(m) \end{aligned}$$

and

$$\begin{aligned} M \times Bider(M) &\longrightarrow M \\ (m, \varphi) &\longmapsto \varphi_r(m). \end{aligned}$$

In general, these maps do not define an action of $Bider(M)$ on M. Nevertheless, if M satisfies $Ann(M) = 0$ or $[M,M] = M$, then these maps define an action of $Bider(M)$ on M (See [7], for details).

Definition 2.5 ([6]). A precrossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ in Leibniz algebras consists of a Leibniz homomorphism $d : M_1 \longrightarrow M_0$, called boundary map, together with an action of M_0 on M_1 satisfying the identities

$$d([m_0,m_1]) = [m_0,d(m_1)], d([m_1,m_0]) = [d(m_1),m_0],$$

for all $m_1 \in M_1$, $m_0 \in M_0$. In addition, if

$$[d(m'_1),m_1] = [m'_1,m_1], \qquad [m_1,d(m'_1)] = [m_1,m'_1],$$

for all $m_1, m'_1 \in M_1$ then $\mathcal{M} : M_1 \xrightarrow{d} M_0$ is called a crossed module.

Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ and $\mathcal{M}' : M'_1 \xrightarrow{d'} M'_0$ be (pre)crossed modules. A homomorphism from M to M' is a pair (μ_1, μ_0) of Leibniz algebra homomorphisms $\mu_1 : M_1 \longrightarrow M'_1$, $\mu_0 : M_0 \longrightarrow M'_0$ such that $d'\mu_1 = \mu_0 d$, $\mu_1([m_0, m_1]) = [\mu_0(m_0), \mu_1(m_1)]$ and $\mu_1([m_1, m_0]) = [\mu_1(m_1), \mu_0(m_0)]$, for all $m_1 \in M_1$, $m_0 \in M_0$. Consequently, we have the category of precrossed modules and the category of crossed modules in the category of Leibniz algebras, which will be denoted here by **PXLbnz**, **XLbnz**, respectively.

Example 2.6.

(i) Let N be a (two-sided) ideal of a Leibniz algebra M, then $N \overset{inc.}{\hookrightarrow} M$ is a crossed module, where the action is given by the adjoint representation. Consequently, $M \xrightarrow{id} M$ and $0 \overset{inc.}{\hookrightarrow} M$ are crossed modules.

(ii) Let L' be a non-abelian Leibniz algebra with an action of L on L'. Then $L' \xrightarrow{0} L$ is a precrossed module which is not a crossed module. Consequently, $L' \xrightarrow{0} L'$ and $L' \xrightarrow{0} 0$ are precrossed modules.

(iii) Let M be a Leibniz algebra. Then $\mathcal{M} : M \times M \xrightarrow{\pi_1} M$ is a precrossed module with a componentwise action of M on $M \times M$ where π_1 is the projection. $M \times M \xrightarrow{\pi_1} M$ is not a crossed module.

(iv) Let M be a Leibniz algebra satisfying $Ann(M) = 0$ or $[M, M] = M$. Then

$$\begin{aligned} d : M &\longrightarrow Bider(M) \\ m &\longmapsto \varphi_m = ([m, -], [-, m]) \end{aligned}$$

is a crossed module with the action defined in Remark 2.4.

A (pre)crossed module $\mathcal{M}' : M_1' \xrightarrow{d'} M_0'$ is a (pre)crossed submodule (or a subobject in (**P**) **XLbnz**) of $\mathcal{M} : M_1 \xrightarrow{d} M_0$ if M_1', M_0' are Leibniz subalgebras of M_1 and M_0, respectively; $d' = d|_{M_1'}$ and the action of M_0' on M_1' is induced by the action of M_0 on M_1. This situation will be denoted by $\mathcal{M}' \leq \mathcal{M}$.

In addition, if M_1' and M_0' are ideals of M_1 and M_0, respectively, $[m_0, m_1'], [m_1', m_0] \in M_1'$, for all $m_0 \in M_0, m_1' \in M_1'$ and $[m_0', m_1], [m_1, m_0'] \in M_1'$, for all $m_0' \in M_0', m_1 \in M_1$, then $\mathcal{M}'$ is called an ideal of $\mathcal{M}$.

Let $\mathcal{M}' : M_1' \xrightarrow{d} M_0'$ be an ideal of a (pre)crossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$. Then the quotient (pre)crossed module $\mathcal{M}/\mathcal{M}'$ is the (pre)crossed module $M_1/M_1' \longrightarrow M_0/M_0'$ with the induced boundary map and action.

Definition 2.7. Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a (pre)crossed module. A biderivation of $\mathcal{M}$ is a pair (f, g) consists of biderivations f, g of M_1 and M_0 respectively, such that $df_l = g_l d$, $df_r = g_r d$ and
a) $f_l([m_1, m_0]) = -f_l([m_0, m_1]) = [f_l(m_1), m_0] - [g_l(m_0), m_1]$,
b) $f_r([m_0, m_1]) = [g_r(m_0), m_1] + [m_0, f_r(m_1)]$,
c) $f_r([m_1, m_0]) = [m_1, g_r(m_0)] + [f_r(m_1), m_0]$,
d) $[m_0, f_r(m_1)] = -[m_0, f_l(m_1)]$,
e) $[m_1, g_r(m_0)] = -[m_1, g_l(m_0)]$,
for all $m_0 \in M_0,\ m_1 \in M_1$.

Notation 2.8. *In the definition, instead of writing $d\alpha_l = \beta_l d$, $d\alpha_r = \beta_r d$, we may write these two equalities in one as $d\alpha_{l,r} = \beta_{l,r} d$. In the rest of the paper we will use this notation, for shortness.*

The set of all biderivations of a (pre)crossed module $\mathcal{M}$ will be denoted by $Bider(\mathcal{M})$. It can be easily checked that $Bider(\mathcal{M})$ is a Leibniz algebra with usual scalar multiplication, addition and the bracket defined by

$$[(f, g), (f', g')] = ([f, f'], [g, g']),$$

for all $(f, g), (f', g') \in Bider(\mathcal{M})$.

Definition 2.9. Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a precrossed module, $\alpha, \alpha^1 \in Bider(M_1)$ and the pair $\partial := (\partial_l, \partial_r)$ consists of k-linear maps $\partial_l, \partial_r : M_0 \longrightarrow M_1$ such that

1) a) $\alpha_l([m_1, m_0]) = -\alpha_l([m_0, m_1]) = [\alpha_l(m_1), m_0] - [\partial_l(m_0), m_1]$,
   b) $\alpha_r([m_0, m_1]) = [\partial_r(m_0), m_1] + [m_0, \alpha_r(m_1)]$,
   c) $\alpha_r([m_1, m_0]) = [m_1, \partial_r(m_0)] + [\alpha_r(m_1), m_0]$,
   d) $\partial_l[m_0, m_0'] = [\partial_l(m_0), m_0'] - [\partial_l(m_0'), m_0]$,
   e) $\partial_r[m_0, m_0'] = [\partial_r(m_0), m_0'] + [m_0, \partial_r(m_0')]$,
   f) $[m_0, \alpha_r(m_1)] = -[m_0, \alpha_l(m_1)]$,
   g) $[m_1, \partial_r(m_0)] = -[m_1, \partial_l(m_0)]$,
2) (α^1, β^1) is a biderivation of $\mathcal{M}$,
3) $\beta^1_{l,r} d(m_1) = d\alpha_{l,r}(m_1)$,

for all $m_0 \in M_0$, $m_1 \in M_1$ where $d\partial_{l,r} = \beta^1_{l,r}$. Then the triples $(\alpha, \partial, \alpha^1)$ will be called a generalized crossed biderivations of $\mathcal{M}$. The set of all generalized biderivations of $\mathcal{M}$ will be denoted by $Gbider(M_0, M_1)$.

Proposition 2.10. *Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a precrossed module. $Gbider(M_0, M_1)$ is non-empty.*

Proof. Choose $a_1 \in M_1$. Define

$$\begin{array}{llll} \alpha_l : M_1 \to M_1, & m_1 \mapsto [a_1, m_1], \alpha_r : M_1 \to M_0, & m_1 \mapsto [m_1, a_1], \\ \partial_l : M_0 \to M_1, & m_0 \mapsto [a_1, m_0], \partial_r : M_0 \to M_1, & m_0 \mapsto [m_0, a_1], \\ \alpha_l^1 : M_1 \to M_1, & m_1 \mapsto [d(a_1), m_1], \alpha_r^1 : M_1 \to M_1, & m_1 \mapsto [m_1, d(a_1)], \\ \beta_l^1 : M_0 \to M_0, & m_0 \mapsto [d(a_1), m_0], \beta_r^1 : M_0 \to M_0, & m_0 \mapsto [m_0, d(a_1)], \end{array}$$

for all $m_0 \in M_0, m_1 \in M_1$. Denote $\alpha := (\alpha_l, \alpha_r)$, $\partial := (\partial_l, \partial_r)$, $\alpha^1 := (\alpha_l^1, \alpha_r^1)$. We have that $(\alpha, \partial, \alpha^1) \in Gbider(M_0, M_1)$. Indeed $\alpha, \alpha^1 \in Bider(M_1)$ and $\alpha, \alpha^1, \partial$ satisfy the identities given in Definition 2.9. □

$Gbider(M_0, M_1)$ is a Leibniz algebra with usual scalar multiplication, addition and the bracket defined by

$$[(\alpha, \partial, \alpha^1), (\gamma, \lambda, \gamma^1)] = \big([\alpha, \gamma], [\partial, \lambda], [\alpha^1, \gamma^1]\big)$$

where $[\alpha, \gamma]$, $[\alpha^1, \gamma^1]$ are usual brackets of biderivations and

$$[\partial, \lambda]_l = \alpha_l \lambda_l + \gamma_r \partial_l, \ [\partial, \lambda]_r = \gamma_r \partial_r - \alpha_r \lambda_r,$$

for all $(\alpha, \partial, \alpha^1), (\gamma, \lambda, \gamma^1) \in Bider(M_0, M_1)$.

Remark 2.11. Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a precrossed module which is not a crossed module. So there exist some $m_1, m_1' \in M_1$ such that $[m_1, d(m_1')] \neq [m_1, m_1']$ or $[d(m_1'), m_1] \neq [m_1', m_1]$. Consider the triple $(\alpha, \partial, \alpha^1)$ defined by m_1 as in Proposition 2.10. Then we have $\alpha \neq \alpha^1$.

Definition 2.12. Let $\mathcal{K} : K_1 \xrightarrow{d} K_0$ be a crossed module. $Bider(K_0, K_1)$ is defined as the set of all pairs $\partial := (\partial_l, \partial_r)$ such that $\partial_l, \partial_r : K_0 \longrightarrow K_1$ are k-linear maps and

$$\begin{aligned} \partial_l[k_0, k_0'] &= [\partial_l(k_0), k_0'] - [\partial_l(k_0'), k_0], \\ \partial_r[k_0, k_0'] &= [\partial_r(k_0), k_0'] + [k_0, \partial_r(k_0')], \\ [k_0, \partial_r(k_0')] &= -[k_0, \partial_l(k_0')], \end{aligned}$$

for all $k_0, k_0' \in K_0$. This time, we will call the elements of $Bider(K_0, K_1)$ as crossed biderivations.

By a similar construction given in Proposition 2.10, $Bider(K_0, K_1)$ is non-empty. Also $Bider(K_0, K_1)$ is a Leibniz algebra with usual scalar multiplication, addition and the bracket defined by $[\partial, \lambda]_l = \partial_l d\lambda_l + \lambda_r d\partial_l$, $[\partial, \lambda]_r = \lambda_r d\partial_r - \partial_r d\lambda_r$, for all $\partial, \lambda \in Bider(K_0, K_1)$.

Remark 2.13. Definitions 2.7, 2.9 and 2.12 are deduced from the construction of biderivations of a semidirect product in Section 4. One can see that biderivations and (generalized) crossed biderivations of a (pre)crossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ can be easily obtained from certain type of biderivations of the semidirect product $M_1 \rtimes M_0$. In the case of Lie algebras these definitions coincide with the ones given in [8, 10].

Now we recall the categories of (pre)cat^1-Leibniz algebras and their functorial relation between the categories **PXLbnz** and **XLbnz**. Details can be found in [9].

A precat^1-Leibniz algebra is a Leibniz algebra with two additional unary operations $\omega_0, \omega_1 : M \longrightarrow M$ such that

$$\omega_0\omega_1 = \omega_1, \omega_1\omega_0 = \omega_0$$

where ω_0 and ω_1 are Leibniz algebra homomorphisms. We will denote such a precat^1-Leibniz algebra by (M, ω_0, ω_1). A homomorphism of precat^1-Leibniz algebras is a Leibniz algebra homomorphism compatible with unary operations. The resulting category will be denoted by **Precat^1-Lbnz**.

Let **Cat^1-Lbnz** be the full subcategory of **Precat^1-Lbnz**, consists of those objects satisfying;

$$[\ker w_0, \ker w_1] = 0, \ [\ker w_1, \ker w_0] = 0.$$

The objects of **Cat^1-Lbnz** are called as cat^1-Leibniz algebras.

Define a functor C : **PXLbnz** $\longrightarrow$ **Precat^1-Lbnz** as follows; for any precrossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$, $C(\mathcal{M}) := (M_1 \rtimes M_0, w_0, w_1)$ where $M_1 \rtimes M_0$ is a semi-direct product and

$$w_0(m_1, m_0) = (0, m_0), \ w_1(m_1, m_0) = (0, d(m_1) + m_0),$$

for all $m_1 \in M_1$, $m_0 \in M_0$. Note that to construct the semi-direct product we use the action of M_0 on M_1 from the precrossed module $\mathcal{M}$.

Conversely, define a functor P : **Precat^1-Lbnz** $\longrightarrow$ **PXLbnz** as follows; for any object (M, w_1, w_0) in **Precat^1-Lbnz**, $P((M, w_1, w_0))$ is the precrossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ where $M_1 = \ker w_0$, $M_0 = \mathrm{Im} w_0$ and $d = w_1|_{\ker w_0}$.

These two functors give rise to a natural equivalence between the categories **PXLbnz** and **Precat^1-Lbnz**. On the other hand, the obvious restrictions of the functors give a natural equivalence between the categories **XLbnz** and **Cat^1-Lbnz**.

2.2. *Modified category of interest*

Modified categories of interest were introduced in [5]. The condition (d) in the definition of modified categories of interest was "For each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identities $\omega(x+y) = \omega(x)+\omega(y)$ and $\omega(x * y) = \omega(x) * \omega(y)$". But this condition does not contain the scalar multiplication. The authors of [5] changed this condition. After this change all constructions and proofs in [5] are true. According to this corrected definition every category of interest is a modified category of interest as well.

Now we will recall the definition of modified category of interest with its revised form.

Let $\mathbb{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbb{E}$, such that $\mathbb{E}$ includes the group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) the group operations (written additively : $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \setminus \{+\}$, $\Omega_1' = \Omega_1 \setminus \{-\}$. Assume that if $* \in \Omega_2$, then Ω_2' contains $*^\circ$ defined by $x *^\circ y = y * x$ and assume $\Omega_0 = \{0\}$;

(c) for each $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $x * (y + z) = x * y + x * z$;

(d) for each $\omega \in \Omega_1'$ and $* \in \Omega_2'$, $\mathbb{E}$ includes the identity $\omega(x+y) = \omega(x) + \omega(y)$ and either the identity $\omega(x * y) = \omega(x) * \omega(y)$ or the identity $\omega(x * y) = \omega(x) * y$.

Let C be an object of $\mathbb{C}$ and $x_1, x_2, x_3 \in C$:

Axiom 1. $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$, for each $* \in \Omega_2'$.

Axiom 2. For each ordered pair $(*, \overline{*}) \in \Omega_2' \times \Omega_2'$ there is a word W such that

$$\begin{aligned}(x_1 * x_2)\overline{*}x_3 = W(&x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1,\\ &(x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),\end{aligned}$$

where each juxtaposition represents an operation in Ω_2'.

Definition 2.14. A category of groups with operations $\mathbb{C}$ satisfying conditions $(a) - (d)$, Axioms 1 and 2, is called a *modified category of interest.*

Let $\mathbb{E}_G$ be the subset of identities of $\mathbb{E}$ which includes the group identities and the identities (c) and (d). We denote by $\mathbb{C}_G$ the corresponding category of groups with operations. Thus we have $\mathbb{E}_G \hookrightarrow \mathbb{E}$, $\mathbb{C} = (\Omega, \mathbb{E})$, $\mathbb{C}_G = (\Omega, \mathbb{E}_G)$ and there is a full inclusion functor $\mathbb{C} \hookrightarrow \mathbb{C}_G$. $\mathbb{C}_G$ is called a general category of groups with operations of a modified category of interest $\mathbb{C}$.

Example 2.15. The categories $\mathbf{Cat^1}$-$\mathbf{Lbnz}$, and $\mathbf{PreCat^1}$-$\mathbf{Lbnz}$ are modified categories of interest, which are not categories of interest. Further examples can be found in [5].

Definition 2.16. Let $A, B \in \mathbb{C}$. An extension of B by A is a sequence

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} B \longrightarrow 0, \tag{3.1}$$

in which p is surjective and i is the kernel of p. We say that an extension is split if there is a morphism $s : B \longrightarrow E$ such that $ps = 1_B$.

Definition 2.17. For $A, B \in \mathbb{C}$, it is said that there is a set of actions of B on A, whenever there is a map $f_* : A \times B \longrightarrow A$, for each $* \in \Omega_2$.

A split extension of B by A, induces an action of B on A corresponding to the operations in $\mathbb{C}$. For a given split extension (3.1), we have

$$b \cdot a = s(b) + a - s(b), \tag{3.2}$$

$$b * a = s(b) * a, \tag{3.3}$$

for all $b \in B$, $a \in A$ and $* \in \Omega_2{}'$. Actions defined by (3.2) and (3.3) will be called derived actions of B on A. The notation $b \,\dot{*}\, a$, is used to denote both the dot and the star actions.

Definition 2.18. Given an action of B on A, a semidirect product $A \rtimes B$ is a universal algebra, whose underlying set is $A \times B$ and the operations are defined by

$$\begin{aligned}
&\omega(a, b) = (\omega(a), \omega(b)),\\
&(a', b') + (a, b) = (a' + b' \cdot a, b' + b),\\
&(a', b') * (a, b) = (a' * a + a' * b + b' * a, b' * b),
\end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$.

Theorem 2.19 ([5]). *An action of B on A is a derived action if and only if $A \rtimes B$ is an object of $\mathbb{C}$.*

Now we will define the actions in the categories $\mathbf{Precat^1}$-$\mathbf{Lbnz}$ and $\mathbf{Cat^1}$-$\mathbf{Lbnz}$ according to the definition of action in a modified category of interest.

Example 2.20. (i) Let $(M, \omega_1^M, \omega_0^M), (N, \omega_1^N, \omega_0^N) \in \mathbf{Precat^1}$-$\mathbf{Lbnz}$ and let $(N, \omega_1^N, \omega_0^N)$ has a derived action on $(M, \omega_1^M, \omega_0^M)$. By Definition 2.17 we have the identities (2.1) and also the identities;

$$\begin{aligned}
\left[\omega_i^N(n), \omega_i^M(m)\right] &= \omega_i^M[n, m],\\
\left[\omega_i^M(m), \omega_i^N(n)\right] &= \omega_i^M[m, n],\\
[\omega_j^N(n), \omega_i^M(m)] &= \omega_i^M[\omega_j^M(n), m],\\
[\omega_i^M(m), \omega_j^N(n)] &= \omega_i^M[m, \omega_j^N(n)],\\
[\omega_j^N(n), \omega_i^M(m)] &= \omega_j^M\left[n, \omega_i^M(m)\right],\\
[\omega_i^M(m), \omega_j^N(n)] &= \omega_j^M\left[\omega_i^M(m), n\right],
\end{aligned} \tag{3.4}$$

$i, j = 0, 1, i \neq j$, for any $m \in (M, \omega_1^M, \omega_0^M)$, $n \in (N, \omega_1^N, \omega_0^N)$.

(ii) Let $(M, \omega_1^M, \omega_0^M), (N, \omega_1^N, \omega_0^N) \in \mathbf{Cat^1}$-$\mathbf{Lbnz}$ and let $(N, \omega_1^N, \omega_0^N)$ has a derived action on $(M, \omega_1^M, \omega_0^M)$. Then we have the identities (2.1), (3.4) and in addition we have $[m, n] = [n, m] = 0$, if $n \in \ker \omega_0^N, m \in \ker \omega_1^M$ or $n \in \ker \omega_1^N, m \in \ker \omega_0^M$.

The definition of a split extension classifier in modified categories of interest has the following form. Consider the category of all split extensions with fixed kernel A; thus the objects are

$$0 \to A \to C \overset{s}{\underset{\curvearrowleft}{\longrightarrow}} C' \to 0$$

and the arrows are the triples of morphisms $(1_A, \gamma, \gamma'))$ between the extensions, which commute with the section homomorphisms as well. By the definition, an object $[A]$ is a split extension classifier for A if there exists a derived action of $[A]$ on A, such that the corresponding extension

$$0 \to A \to A \rtimes [A] \overset{s}{\underset{\curvearrowleft}{\longrightarrow}} [A] \to 0$$

is a terminal object in the above defined category.

Proposition 2.21 ([5]). *Let $\mathbb{C}$ be a modified category of interest and A be an object in $\mathbb{C}$. An object $B \in \mathbb{C}$ is a split extension classifier for A in the sense of [2] if and only if it satisfies the following condition: B has a derived action on A such that for all C in $\mathbb{C}$ and a derived action of C on A there is a unique morphism $\varphi : C \longrightarrow B$, with $c \cdot a = \varphi(c) \cdot a$, $c * a = \varphi(c) * a$, for all $* \in \Omega_2{}'$, $a \in A$ and $c \in C$.*

The object B in $\mathbb{C}$ satisfying the above stated condition is called an actor of A and denoted by Act(A). The corresponding universal acting object, which represents actions in the sense of [2, 3], in the categories equivalent to modified categories of interest is called a split extension classifier and denoted by $[A]$, as it is in semi-abelian categories.

Remark 2.22. As a consequence of this proposition, an actor of an object is unique up to an isomorphism.

Definition 2.23. Let $A, B \in \mathbb{C}$. A set of actions of B on A is strict if for any two elements $b, b' \in B$, from the conditions $b \cdot a = b' \cdot a$, $\omega(b) \cdot a = \omega(b') \cdot a$, $b * a = b' * a$ and $\omega(b) * a = \omega(b') * a$, for all $a \in A$, $\omega \in \Omega_1'$ and $* \in \Omega_2'$, it follows that $b = b'$.

Definition 2.24. A general actor $\mathrm{GA}(A)$ of an object A in $\mathbb{C}$, is an object of $\mathbb{C}_G$, having a set of actions on A, which is a set of derived actions in $\mathbb{C}_G$ and for any object $C \in \mathbb{C}$ and a derived action of C on A in $\mathbb{C}$, there exists in $\mathbb{C}_G$ a unique morphism $\varphi : C \longrightarrow \mathrm{GA}(A)$ such that $c \,\dot{*}\, a = \varphi(c) \,\dot{*}\, a$, for all $c \in C$, $a \in A$ and $* \in \Omega_2'$.

Definition 2.25. If the action of a general actor $\mathrm{GA}(A)$ on A is strict, then it is said that $\mathrm{GA}(A)$ is a strict general actor of A and denoted by $\mathrm{SGA}(A)$.

Condition 2.26. *Let $A \in \mathbb{C}$ and $\{B_j\}_{j\in J}$ denote the set of all objects of $\mathbb{C}$ which have derived actions on A. Let $\varphi_j : B_j \longrightarrow \mathrm{GA}(A)$, $j \in J$, denote the corresponding unique morphism such that $b_j \,\dot{*}\, a = \varphi_j(b_j) \,\dot{*}\, a$, for all $b_j \in B_j$, $a \in A$, $* \in \Omega_2'$. The elements of* $\mathrm{GA}(A)$ *satisfy the following equality:*

$$(\varphi_i(b_i) * \varphi_j(b_j))\overline{*}a = W(\varphi_i(b), \varphi_j(b'); a; *, \overline{*})$$

for any $b_i \in B_i$, $b_j \in B_j$, $* \in \Omega_2'$ and $i, j \in J$.

Definition 2.27. A universal strict general actor of an object A, denoted by $\mathrm{USGA}(A)$, is a strict general actor with Condition 2.26, such that for any strict general actor $\mathrm{SGA}(A)$ with Condition 2.26 there exists a unique morphism $\eta : \mathrm{USGA}(A) \to \mathrm{SGA}(A)$ in the category $\mathbb{C}_G$, with $\psi_j\eta = \varphi_j$, for any $j \in J$, where $\varphi_j : B_j \to \mathrm{SGA}(A)$ and $\psi_j : B_j \to \mathrm{USGA}(A)$ denote the corresponding unique morphisms with the appropriate properties from the definition of a general actor.

Proposition 2.28 ([5]). *Let $\mathbb{C}$ be a modified category of interest and $A \in \mathbb{C}$. If an actor* $\mathrm{Act}(A)$ *exists, then the unique morphism η :* $\mathrm{USGA}(A) \to \mathrm{Act}(A)$ *is an isomorphism with $x \,\dot{*}\, a = \eta(x) \,\dot{*}\, (a)$, for all $x \in \mathrm{USGA}(A)$, $a \in A$.*

Theorem 2.29 ([5]). *Let $\mathbb{C}$ be a modified category of interest and $A \in \mathbb{C}$. A has an actor if and only if the semidirect product $A \rtimes$* $\mathrm{USGA}(A)$ *is an object of $\mathbb{C}$. If it is the case, then* $\mathrm{Act}(A) \cong \mathrm{USGA}(A)$.

Let (M, w_0^M, w_1^M) be a precat^1-Leibniz. Consider all split extensions of M in $\mathbf{Precat}^1\mathbf{-Lbnz}$

$$E_j : 0 \longrightarrow (M, w_0^M, w_1^M) \longrightarrow (K_j, w_0^{K_j}, w_1^{K_j}) \xrightarrow{\curvearrowleft} (L_j, w_0^{L_j}, w_1^{L_j}) \longrightarrow 0, \qquad j \in \mathbf{J}$$

where $(L_j, w_0^{L_j}, w_1^{L_j}) = (L_k, w_0^{L_k}, w_1^{L_k}) = (L, w_0^L, w_1^L)$, for $j \neq k$ in this case the corresponding extensions derive different actions of (L, w_0^L, w_1^L) on (M, w_0^M, w_1^M). Let $\{l_j\cdot, [l_j,\text{-}], [\text{-}, l_j]\}$ be the set of functions defined by the action of $(L_j, w_0^{L_j}, w_1^{L_j})$ on (M, w_0^M, w_1^M). For any element $l_j \in L_j$ denote $\mathbf{l}_j = \{l_j\cdot, [l_j,\text{-}], [\text{-}, l_j]\}$. Let $\mathbb{L} = \{\mathbf{l}_j, l_j \in L_j, j \in \mathbb{J}\}$. Thus each element $\mathbf{l}_j \in \mathbb{L}$, $j \in \mathbb{J}$ is a special type of a function $\mathbf{l}_j = \{+, [\,,\,], [\,,\,]^{op}\} \longrightarrow$ *Maps*, $(M, w_0^M, w_1^M) \longrightarrow (M, w_0^M, w_1^M)$ defined by $\mathbf{l}_j([\text{-},\text{-}]) = [l_j,\text{-}] : M \longrightarrow M$, $\mathbf{l}_j([\text{-},\text{-}]^{op}) = [\text{-}, l_j] : M \longrightarrow M$, $\mathbf{l}_j(+) = (l_j + (\text{-})) : M \longrightarrow M$. The bracket on $\mathbb{L}$ is defined by

$$\begin{aligned}
&[[\mathbf{l}_i, \mathbf{l}_k], m] = [\mathbf{l}_i, [\mathbf{l}_k, m]] + [[\mathbf{l}_i, m], \mathbf{l}_k],\\
&[m, [\mathbf{l}_i, \mathbf{l}_k]] = [[m, \mathbf{l}_i], \mathbf{l}_k] - [[m, \mathbf{l}_k], \mathbf{l}_i],\\
&[\mathbf{l}_i, \mathbf{l}_k] \cdot (m) = m.
\end{aligned}$$

Additionally, we define

$$
\begin{aligned}
&(\mathbf{l}_i+\mathbf{l}_k)\cdot(m)=\mathbf{l}_i\cdot(\mathbf{l}_k\cdot m),\\
&[(\mathbf{l}_i+\mathbf{l}_k),m]=[\mathbf{l}_i,m]+[m,\mathbf{l}_k],\\
&w_i(\mathbf{l}_k)\cdot(m)=w_i^{L_k}(l_k)\cdot m,\quad [w_i(\mathbf{l}_k),m]=[w_i^{L_k}(l_k),m],\quad [m,w_i(\mathbf{l}_k)]=[m,w_i^{L_k}(l_k)],\quad i=0,1,\\
&w_i[\mathbf{l}_i,\mathbf{l}_k]=[w_i^{L_i}(l_i),w_i^{L_k}(l_k)],\qquad i=0,1,\\
&w_1(l_1+l_2)=w_i^{L_1}(l_1)+w_i^{L_2}(l_2),\qquad i=0,1,\\
&(-\mathbf{l}_i)\cdot(m)=(-l_i)\cdot m,\qquad (-l)\cdot m=m,\\
&[(-\mathbf{l}_i),m]=-[l_i,(m)],\qquad [(-l),(m)]=-[l,(m)],\\
&[m,(-\mathbf{l}_i)]=-[(m),l_i],\qquad [(m),(-l)]=-[(m),l],\\
&-(l_1+l_2)=-l_2-l_1
\end{aligned}
$$

where l,l_1,l_2 are certain combinations of the bracket of the elements of $\mathbb{L}$.

Denote by $\mathfrak{L}'(M)$ the set of all functions $(\Omega_2\to Maps(M,w_0^M,w_1^M)\longrightarrow(M,w_0^M,w_1^M))$ obtained by performing all kind of operations defined above on the elements of $\mathbb{L}$ and on the new obtained elements as the results of operations. Note that if may happen that $[\mathbf{l},m]=\left[\mathbf{l}',m\right]$ or $[m,\mathbf{l}]=\left[m,\mathbf{l}'\right]$, for any $m\in M$, but we do not have the equalities $[w_i(\mathbf{l}),m]=\left[w_i(\mathbf{l}'),m\right]$ or $[m,w_i(\mathbf{l})]=\left[m,w_i(\mathbf{l}')\right]$, $i=0,1$, for any $m\in M$, respectively where w is a finite combinations of w_0 and w_1. Define a relation on $\mathfrak{L}'(M)$ by "$l\sim l'$ if and only if $[l,m]=\left[l',m\right]$, $[m,l]=\left[m,l'\right]$, $[w_i(l),m]=\left[w_i(l'),m\right]$, $[m,w_i(l)]=\left[m,w_i(l')\right]$" for any $l,l'\in\mathfrak{L}'(M)$, $m\in M$ and $i=0,1$. This is a congruence relation on $\mathfrak{L}'(M)$. Denote $\mathfrak{L}'(M)/\sim$ by $\mathfrak{L}(M)$. The operations defined on $\mathfrak{L}'(M)$ define the corresponding operations on $\mathfrak{L}(M)$.

Theorem 2.30 ([5]). *Let $A\in\mathbb{C}$. Then we have $\mathfrak{B}(A)\cong\mathrm{USGA}(A)$.*

Corollary 2.31. *Let $(M,w_0,w_1)\in\mathbf{Precat}^1\!-\!\mathbf{Lbnz}$. Then $(\mathfrak{L}(M),w_0^{\mathfrak{L}(M)},w_1^{\mathfrak{L}(M)})$ is a universal strict general actor of (M,w_0^M,w_1^M).*

Proof. Follows from Theorem 2.30. □

3. Actor of an object in Precat^1-Lbnz

In this section, according to a given object (M,w_1,w_0) in **Precat^1-Lbnz**, we will construct an object $(\mathfrak{A}(M),\overline{w_0},\overline{w_1})$ and prove that it is an actor of (M,w_1,w_0) under certain conditions. The construction deduced from the interpretation of $\mathfrak{L}(M)$ constructed in Section 2.

Let (M,w_1,w_0) be a precat^1-Leibniz algebra. Consider the triples $(\varphi,\varphi^0,\varphi^1)$ of biderivations of M such that

M1) $\varphi_{l,r}^i w_i=w_i\varphi_{l,r}$, for $i=0,1$,

M2) $\varphi_{l,r}^j w_i=w_i\varphi_{l,r}^j$, for $i=0,1,j=0,1$.

We will denote the set of all these kinds of triples by $\mathfrak{A}(M)$.

$\mathfrak{A}(M)$ is a Leibniz algebra with componentwise addition, scalar multiplication and the bracket defined by

$$[(\varphi,\varphi^0,\varphi^1),(\psi,\psi^0,\psi^1)]=([\varphi,\psi],[\varphi^0,\psi^0],[\varphi^1,\psi^1]),$$

for all $(\varphi,\varphi^0,\varphi^1),(\psi,\psi^0,\psi^1)\in\mathfrak{A}(M)$. The zero element is the triple $(0,0,0)$ of zero maps. Also, $(\mathfrak{A}(M),\overline{\omega_0},\overline{\omega_1})$ is a precat^1-Leibniz algebra with the unary operations $\overline{\omega_0}:\mathfrak{A}(M)\longrightarrow\mathfrak{A}(M)$, $\overline{\omega_1}:\mathfrak{A}(M)\longrightarrow\mathfrak{A}(M)$ defined by $\overline{\omega_0}(\varphi,\varphi^0,\varphi^1)=(\varphi^0,\varphi^0,\varphi^0)$, $\overline{\omega_1}(\varphi,\varphi^0,\varphi^1)=(\varphi^1,\varphi^1,\varphi^1)$, respectively.

There is an action of $(\mathfrak{A}(M), \overline{\omega_0}, \overline{\omega_1})$ on (M, w_1, w_0) defined by the maps

$$\begin{aligned} (\mathfrak{A}(M), \overline{w_0}, \overline{w_1}) \times (M, w_1, w_0) &\longrightarrow (M, w_1, w_0) \\ ((\varphi, \varphi^0, \varphi^1), m) &\longmapsto \varphi_l(m) \end{aligned}$$

and

$$\begin{aligned} (M, w_1, w_0) \times (\mathfrak{A}(M), \overline{w_0}, \overline{w_1}) &\longrightarrow (M, w_1, w_0) \\ (m, (\varphi, \varphi^0, \varphi^1)) &\longmapsto \varphi_r(m), \end{aligned}$$

for all $m \in M$ and $(\varphi, \varphi^0, \varphi^1) \in \mathfrak{A}(M)$.

Condition A For a Leibniz algebra M, $Ann(M) = 0$ or $[M, M] = M$.

Proposition 3.1. *If M satisfies Condition **A**, then $(\mathfrak{A}(M), \overline{\omega_0}, \overline{\omega_1})$ and $(\mathfrak{L}(M), w_0^{\mathfrak{L}(M)}, w_1^{\mathfrak{L}(M)})$ are isomorphic.*

Proof. Under Condition **A** the action of $(\mathfrak{A}(M), \overline{w_0}, \overline{w_1})$ on (M, w_1, w_0) defined by us is a derived action in **Precat¹-Lbnz**. Therefore, from Definition 2.27, we have the unique morphism $\eta : (\mathfrak{A}(M)) \longrightarrow (\mathfrak{L}(M))$ such that $\eta([(f, f^0, f^1), m]) = [(f, f^0, f^1), m]$, $\eta\left([m, (f, f^0, f^1)]\right) = [m, (f, f^0, f^1)]$, for all $m \in M$ and $(f, f^0, f^1) \in \mathfrak{A}(M)$. By the constructions of $(\mathfrak{A}(M), \overline{\omega_0}, \overline{\omega_1})$ and $(\mathfrak{L}(M), w_0^{\mathfrak{L}(M)}, w_1^{\mathfrak{L}(M)})$, we find that η is an isomorphism. □

Corollary 3.2. *If M satisfies Condition **A**, then $(\mathfrak{A}(M), \overline{\omega_0}, \overline{\omega_1})$ is an actor of (M, ω_0, ω_1).*

Proof. Since $(\mathfrak{A}(M), \overline{\omega_0}, \overline{\omega_1}) \in$ **Precat¹-Lbnz** and its action on M is a derived action, then the result follows from Theorem 2.29, Corollary 2.31 and Proposition 3.1. □

3.1. Actor of a precat¹-Leibniz algebra corresponding to a given precrossed module

Let M_1, M_0 be Leibniz algebras with an action of M_0 on M_1. Let $\psi = (\psi_l, \psi_r) \in Bider(M_1 \rtimes M_0)$. Then $\psi_l : M_1 \rtimes M_0 \longrightarrow M_1 \rtimes M_0$ can be represented by four k-linear maps

$$\alpha_l : M_1 \longrightarrow M_1, \delta_l : M_1 \longrightarrow M_0, \beta_l : M_0 \longrightarrow M_0 \text{ and } \partial_l : M_0 \longrightarrow M_1$$

such that

$$\psi_l(m_1, m_0) = (\alpha_l(m_1) + \partial_l(m_0), \beta_l(m_0) + \delta_l(m_1)),$$

for all $m_1 \in M_1, m_0 \in M_0$. Similarly, $\psi_r : M_1 \rtimes M_0 \longrightarrow M_1 \rtimes M_0$ can be represented by four k-linear maps

$$\alpha_r : M_1 \longrightarrow M_1, \delta_r : M_1 \longrightarrow M_0, \beta_r : M_0 \longrightarrow M_0 \text{ and } \partial_r : M_0 \longrightarrow M_1$$

such that

$$\psi_r(m_1, m_0) = (\alpha_r(m_1) + \partial_r(m_0), \beta_r(m_0) + \delta_r(m_1)),$$

for all $m_1 \in M_1$, $m_0 \in M_0$. Let $\mathcal{M} : M_0 \xrightarrow{d} M_1$ be a precrossed module and $(M_1 \rtimes M_0, \omega_0, \omega_1)$ be the corresponding precat¹-Leibniz algebra. Suppose $\psi = (\psi_l, \psi_r)$ satisfy the Condition M2, for $j = 0$ or $j = 1$. Then by a direct checking, we find that $\delta_r = \delta_l = 0$. So, any biderivation $\psi = (\psi_l, \psi_r) \in Bider(M_1 \rtimes M_0)$ can be represented by the triple $(\alpha, \partial, \beta)$.

Let $m := (m_1, m_0), m' := (m_1', m_0') \in M_1 \times M_0$. Since $\psi = (\psi_m, \psi_r)$ is a biderivation, we obtain

$$\begin{aligned} \psi_r([m, m']) &= \psi_r\left([m_1, m_1'] + [m_0, m_1'] + [m_1, m_0'], [m_0, m_0']\right) \\ &= \left(\alpha_r[m_1, m_1'] + \alpha_r[m_0, m_1'] + \alpha_r[m_1, m_0'] + \partial_r[m_0, m_0'], \beta_r[m_0, m_0']\right) \end{aligned}$$

and

$$\begin{aligned}\psi_r([m,m']) &= [m,\psi_r(m')]+[\psi_r(m),m'] \\ &= [(m_1,m_0),\psi_r(m_1',m_0')]+[\psi_r(m_1,m_0),(m_1',m_0')] \\ &= [(m_1,m_0),\left(\alpha_r(m_1')+\partial_r(m_0'),\beta_r(m_0')\right)] \\ &\quad +[(\alpha_r(m_1)+\partial_r(m_0),\beta_r(m_0)),(m_1',m_0')] \\ &:= I_1+I_0\end{aligned}$$

where

$$I_1=\left([m_1,\alpha_r(m_1')+\partial_r(m_0')]+[m_1,\beta_r(m_0')]+[m_0,\alpha_r(m_1')+\partial_r(m_0')],[m_0,\beta_r(m_0')]\right)$$

and

$$I_0=\left([\alpha_r(m_1)+\partial_r(m_0),m_1']+[\alpha_r(m_1)+\partial_r(m_0),m_0']+[\beta_r(m_0),m_1'],[\beta_r(m_0),m_0']\right).$$

So we get:
(1) $\beta_r[m_0,m_0']=[m_0,\beta_r(m_0')]+[\beta_r(m_0),m_0']$
(2) If we take $m_0=m_0'=0$, then

$$\alpha_r[m_1,m_1']=[m_1,\alpha_r(m_1')]+[\alpha_r(m_1),m_1'].$$

(3) If we take $m_0'=0$, $m_1=0$, then

$$\alpha_r[m_0,m_1']=[m_0,\alpha_r(m_1')]+[\partial_r(m_0),m_1']+[\beta_r(m_0),m_1'].$$

(4) If we take $m_1'=0$, $m_0=0$, then

$$\alpha_r[m_1,m_0']=[m_1,\partial_r(m_0')]+[m_1,\beta_r(m_0')]+[\alpha_r(m_1),m_0'].$$

(5) If we take $m_1'=0=m_1$, then

$$\partial_r[m_0,m_0']=[m_0,\partial_r(m_0')]+[\partial_r(m_0),m_0'].$$

By similar calculations we have $\alpha\in Bider(M_1)$, $\beta\in Bider(M_0)$ and
(L1.) $\partial_l[m_0,m_0']=[\partial_l(m_0),m_0']-[\partial_l(m_0'),m_0]$,
(L2.) $\partial_r[m_0,m_0']=[\partial_r(m_0),m_0']+[m_0,\partial_r(m_0')]$,
(L3.) $[m_0,\partial_r(m_0')]=-[m_0,\partial_l(m_0')]$,
(L4.)

$$\begin{aligned}\alpha_l[m_1,m_0] &= -\alpha_l[m_0,m_1] \\ &= [\alpha_l(m_1),m_0]-[\partial_l(m_0),m_1]-[\beta_l(m_0),m_1],\end{aligned}$$

(L5.) $\alpha_r([m_0,m_1])=[\partial_r(m_0),m_1]+[\beta_r(m_0),m_1]+[m_0,\alpha_r(m_1)]$,
(L6.) $\alpha_r([m_1,m_0])=[\alpha_r(m_1),m_0]+[m_1,\partial_r(m_0)]+[m_1,\beta_r(m_0)]$,
(L7.) $[m_0,\alpha_r(m_1)]=-[m_0,\alpha_l(m_1)]$,
(L8.) $[m_1,\beta_r(m_0)]=-[m_1,\beta_l(m_0)]$,
(L9.) $[m_1,\partial_l(m_0)]=[m_1,\beta_l(m_0)]-[m_1,\partial_r(m_0)]-[m_1,\beta_r(m_0)]$,
for all $m_1\in M_1$, $m_0,m_0'\in M_0$.

Proposition 3.3. *Let $\mathcal{M}:M_0\xrightarrow{d}M_1$ be a precrossed module and $(M_1\rtimes M_0,\omega_0,\omega_1)$ be the corresponding precat^1-Leibniz algebra. Let $\varphi,\varphi^0,\varphi^1\in Bider(M_1\rtimes M_0)$ and denote $\varphi,\varphi^0,\varphi^1$ by the triples (α,∂,β), $(\alpha^0,\partial^0,\beta^0)$, $(\alpha^1,\partial^1,\beta^1)$, respectively. Then $(\varphi,\varphi^0,\varphi^1)\in\mathfrak{A}(M_1\rtimes M_0)$ if and only if $(\varphi,\varphi^0,\varphi^1)$ satisfy the following identities:*
(1.) $\beta_l(m_0)=\beta_l^0(m_0)$, $\beta_r(m_0)=\beta_r^0(m_0)$,
(2.) $\partial_l^i(m_0)=0$, $\partial_r^i(m_0)=0$, $i=0,1$,
(3.) $\beta_l^1(m_0)=\beta_l(m_0)+d\partial_l(m_0)$, $\beta_r^1(m_0)=\beta_r(m_0)+d\partial_r(m_0)$,

(4.) $\beta_l^1 d(m_1) = d\alpha_l(m_1)$, $\beta_r^1 d(m_1) = d\alpha_r(m_1)$,
(5.) $\beta_l^i d(m_1) = d\alpha_l^i(m_1)$, $\beta_r^i d(m_1) = d\alpha_r^i(m_1)$, $i = 0, 1$,
for all $m_0 \in M_0$, $m_1 \in M_1$.

Proof. Follows from related definitions and equalities L1–L9. □

Proposition 3.4. *Let* $\mathcal{M} : M_1 \xrightarrow{d} M_0$ *be a precrossed module,* M_0, M_1 *satisfy Condition* **A** *and* $(\mathfrak{A}(M_1 \rtimes M_0), \overline{w_1}, \overline{w_0})$ *be the actor of* $(M_1 \rtimes M_0, w_0, w_1)$*. Then,* $\ker \overline{w_0} = \{(\varphi, 0, \varphi^1) \in \mathfrak{A}(M_1 \rtimes M_0)\}$.

Proof. Follows from the definition of $\overline{w_0}$ and Proposition 3.3. □

Proposition 3.5. $\ker \overline{w_0} \cong Gbider(M_0, M_1)$

Proof. Let $(\varphi, 0, \varphi^1) \in \ker \overline{w_0}$. It follows from Propositions 3.3 and 3.4 that $(\varphi, 0, \varphi^1) = ((\alpha, \partial, 0), (0, 0, 0), (\alpha^1, 0, \beta^1))$ and the resulting triple $(\alpha, \partial, \alpha^1)$ is a generalized crossed biderivation. Conversely, for any $(\gamma, \lambda, \gamma^1) \in Bider(M_0, M_1)$ we have the triples $\theta = (\gamma, \lambda, 0)$, $\theta^1 = \left(\gamma^1, 0, d\lambda\right)$ such that $(\theta, 0, \theta^1) \in \ker \overline{w_0}$. □

Proposition 3.6. $\mathrm{Im}(\overline{w_0}) = Bider(\mathcal{M})$.

Proof. Direct checking. □

4. Split extension classifier of a precrossed module

According to the definition of action in semi-abelian categories [3], it is natural to define an action in **PXLbnz** in analogous way as it is defined in a modified category of interest. In this section we will construct a precrossed module Δ for a given precrossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ and prove, that if M_0 and M_1 satisfy Condition **A**, then Δ is isomorphic to $P(\mathrm{Act}(C(\mathcal{M})))$. Consequently, Δ is the split extension classifier of $\mathcal{M}$.

Let, $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a precrossed module, $(M_1 \rtimes M_0, w_0, w_1)$ be the corresponding precat^1-algebra and $(\mathfrak{A}(M_1 \rtimes M_0), \overline{w_0}, \overline{w_1})$ be its actor in **Precat**^{1}**-Lbnz**.

Proposition 4.1. *The bilinear maps*

$$\begin{aligned} Bider(\mathcal{M}) \times Gbider(M_0, M_1) &\longrightarrow Gbider(M_0, M_1) \\ ((f, g), (\alpha, \partial, \alpha^1)) &\longmapsto ([f, \alpha], \overline{\partial}, [f, \alpha^1]) \end{aligned}$$

and

$$\begin{aligned} Gbider(M_0, M_1) \times Bider(\mathcal{M}) &\longrightarrow Gbider(M_0, M_1) \\ ((\alpha, \partial, \alpha^1), (f, g)) &\longmapsto ([\alpha, f], \overline{\overline{\partial}}, \left[\alpha^1, f\right]) \end{aligned}$$

define an action of $Bider(\mathcal{M})$ *on* $Gbider(M_0, M_1)$ *where* $[\alpha, f]$, $[f, \alpha]$, $[\alpha^1, f]$, $[f, \alpha^1]$, *are brackets of biderivations, and*

$$\begin{aligned} \overline{\partial}_l &= f_l \partial_l + \partial_r g_l \\ \overline{\partial}_r &= -f_r \partial_r + \partial_r g_r \\ \overline{\overline{\partial}}_l &= f_r \partial_l + \partial_l g_l \\ \overline{\overline{\partial}}_r &= f_r \partial_r - \partial_r g_r \end{aligned}$$

Proof. Direct checking by using the definitions. □

Define a map $\Delta : Gbider(M_0, M_1) \longrightarrow Bider(\mathcal{M})$ by $(\alpha, \partial, \alpha^1) \longmapsto (\alpha^1, \beta^1)$ where $\beta^1_{l,r} = d\partial_{l,r}$.

Proposition 4.2. *$\Delta : Gbider(M_0, M_1) \longrightarrow Bider(\mathcal{M})$ is a precrossed module with the action defined in Proposition 4.1.*

Proof. The proof is direct consequence of the definitions. □

Proposition 4.3. $\Delta \cong P(\mathfrak{A}(C(\mathcal{M})))$.

Proof. Follows from Propositions 3.5, 3.6 and 4.2, since Δ is isomorphic to the restriction of $\overline{\omega_1}$. □

Theorem 4.4. *If M_1 and M_0 satisfy Condition **A**, then the precrossed module $\Delta : Gbider(M_0, M_1) \longrightarrow Bider(\mathcal{M})$ defined in Proposition 4.2 is the split extension classifier of $\mathcal{M}$.*

Proof. If M_0 and M_1 satisfy Condition **A**, then the semidirect product $M_1 \rtimes M_0$ also satisfies this condition. So the result direct consequence of Corollary 3.2, Proposition 4.3 and the fact that P and S define an equivalence between the categories **PXLbnz** and **Precat**^{1}**-Lbnz**. □

The split extension classifier of a precrossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ will be denoted here by $[\mathcal{M}]_{\mathbf{PXLbnz}}$.

Example 4.5. Let M be a Leibniz algebra. Consider the precrossed module $\mathcal{M} : M \xrightarrow{id} M$. Then the actor of $\mathcal{M}$ is the precrossed module $(Bider(\mathcal{M}), Bider(\mathcal{M}), id)$.

5. Split extension classifier of a crossed module

In this section, by a similar discussion given in Sections 3 and 4, we will define an actor of an object in **Cat**^{1}**-Lbnz** and the split extension classifier of an object in **XLbnz**. We omit the proofs since the results are similar to those given in Sections 3 and 4 with additional modifications.

5.1. *Actor of an object in* Cat^1-Lbnz

Let (M, w_1, w_0) be a cat^1-Leibniz algebra. Consider the subset $(\overline{\mathfrak{A}(M)}, \overline{w_0}, \overline{w_1})$ of $(\mathfrak{A}(M), \overline{w_0}, \overline{w_1})$ whose elements satisfy the following additional conditions:
M3) $\varphi_{l,r}(x) = \varphi^1_{l,r}(x)$, for all $x \in \ker w_0$,
M4) $\varphi_{l,r} = \varphi^0_{l,r}(x)$, for all $x \in \ker w_1$.

Proposition 5.1. *$(\overline{\mathfrak{A}(M)}, \overline{w_0}, \overline{w_1})$ is a subobject of $(\mathfrak{A}(M), \overline{w_0}, \overline{w_1})$.*

Proof. Direct checking. □

By a similar discussion given in Section 3, if M satisfies Condition **A**, then $(\overline{\mathcal{A}(M)}, \overline{w_0}, \overline{w_1})$ is an actor of (M, w_1, w_0) in **Cat**^{1}**-Lbnz**.

Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a crossed module. Any biderivation $\varphi = (\varphi_l, \varphi_r) \in Bider(M_1 \rtimes M_0)$ can be represented by a triple $(\alpha, \partial, \beta)$ as it was for the precrossed module case.

Proposition 5.2. *Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a crossed module and $(M_1 \rtimes M_0, w_1, w_0)$ is the corresponding cat^1-Leibniz algebra. Let $\varphi, \varphi^0, \varphi^1 \in Bider(M_1 \rtimes M_0)$ and denote $\varphi, \varphi^0, \varphi^1$ by the triples $(\alpha, \partial, \beta), (\alpha^0, \partial^0, \beta^0), (\alpha^1, \partial^1, \beta^1)$, respectively. $(\varphi, \varphi^0, \varphi^1) \in \overline{\mathfrak{A}}(M_1 \rtimes M_0)$ if and only if the triple $(\varphi, \varphi^0, \varphi^1)$ satisfy the following identities:*
1. $\beta_l(m_0) = \beta_l^0(m_0), \beta_r(m_0) = \beta_r^0(m_0)$,
2. $\partial_l^i(m_0) = 0, \partial_r^i(m_0) = 0, i = 0, 1$,
3. $\beta_l(m_0) + d\partial_l(m_0) = \beta_l^1(m_0), \beta_r(m_0) + d\partial_r(m_0) = \beta_r^1(m_0)$,
4. $\beta_l^1 d(m_1) = d\alpha_l(m_1), \beta_r^1 d(m_1) = d\alpha_r(m_1)$,
5. $\beta_l^i d(m_1) = d\alpha_l^i(m_1), \beta_r^i d(m_1) = d\alpha_r^i(m_1), i = 0, 1$,
6. $\alpha_{l,r}(m_1) = \alpha_{l,r}^1(m_1)$,
7. $\alpha_{l,r}(m_1) = \alpha_{l,r}^0(m_1) + \partial_{l,r} d(m_1)$,

for all $m_0 \in M_0, m_1 \in M_1$.

Proof. Let $(\varphi, \varphi^0, \varphi^1) \in \overline{\mathfrak{A}}(M_1 \rtimes M_0)$. All equalities 1–7 are direct consequences of M1–M4. We will demonstrate the proofs of properties 6 and 7. By the definition of w_0, we have $(m_1, 0) \in \ker w_0$, for all $m_1 \in M_1$. From M3, we have $\varphi_{(l,r)}(a_1, 0) = \varphi^1_{(l,r)}(a_1, 0)$, which means $\alpha_{l,r}(m_1) = \alpha^1_{l,r}(m_1)$, for all $m_1 \in M_1$.

Also, by the definition of w_1, we have $(m_1,\text{-}d(m_1)) \in \ker w_1$, for all $m_1 \in M_1$. Since $\partial^i_{l,r}(m_0) = 0$, $i = 0, 1$, we have $\alpha_{l,r}(m_1) = \alpha^0_{l,r}(m_1) + \partial_{l,r} d(m_1)$, for all $m_1 \in M_1$.

The converse statement can be proved by a direct checking. □

Remark 5.3. Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a crossed module and let $(M_1 \rtimes M_0, w_1, w_0)$ be the corresponding cat^1-Leibniz algebra. From the definition of $\overline{w_0}$ we have $\ker \overline{w_0} = \left\{(\varphi, 0, \varphi^1) \in \overline{\mathcal{A}}(M_1 \rtimes M_0)\right\}$, and from Proposition 5.2, any element $(\varphi, 0, \varphi^1) \in \ker \overline{w_0}$ can be represented by $((\alpha, \partial, 0), (0, 0, 0), (\alpha, 0, \beta^1))$. Also, from Definition 2.7, for any $((\alpha, \partial, 0), (0, 0, 0), (\alpha, 0, \beta^1))$ we have that, (α, β^1) is a biderivation of $\mathcal{M}$ and $d\partial_{l,r} = \beta^1_{l,r}$. Consequently, we get $\ker \overline{w_0} \cong Bider(M_0, M_1)$, $\mathrm{Im}\overline{w_0} \cong Bider(\mathcal{M})$.

5.2. Split extension classifier of a crossed module

Proposition 5.4. *The bilinear maps $Bider(\mathcal{M}) \times Bider(M_0, M_1) \longrightarrow Bider(M_0, M_1)$, $((f, g), \partial) \longmapsto \overline{\partial}$, and $Bider(M_0, M_1) \times Bider(\mathcal{M}) \longrightarrow Bider(M_0, M_1)$, $(\partial, (f, g)) \longmapsto \overline{\overline{\partial}}$, define a derived action of $Bider(\mathcal{M})$ on $Bider(M_0, M_1)$ where*

$$\overline{\partial}_l = f_l\partial_l + \partial_r g_l,$$
$$\overline{\partial}_r = -f_r\partial_r + \partial_r g_r,$$
$$\overline{\overline{\partial}}_l = f_r\partial_l + \partial_l g_l,$$
$$\overline{\overline{\partial}}_r = f_r\partial_r - \partial_r g_r.$$

Proof. Direct consequence of the definitions. □

Proposition 5.5. *Define the map $\overline{\Delta} : Bider(M_0, M_1) \longrightarrow Bider(\mathcal{M})$ by $\partial \longmapsto (\partial d, d\partial)$, for all $\partial \in Bider(M_0, M_1)$. Then $Bider(M_0, M_1) \xrightarrow{\overline{\Delta}} Bider(\mathcal{M})$ is a crossed module with the action defined in Proposition 5.4.*

Proof. Direct checking by using the definitions. □

Proposition 5.6. *Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a crossed module. Then $\overline{\Delta} \cong P(\mathfrak{A}(C(\mathcal{M})))$.*

Proof. Follows from Remark 5.3 and Proposition 5.5. □

Theorem 5.7. *If M_1 and M_0 satisfy Condition* **A**, *then $\overline{\Delta} : Bider(M_0, M_1) \longrightarrow Bider(\mathcal{M})$ is the split extension classifier of the crossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$.*

The split extension classifier of a crossed module $\mathcal{M} : M_1 \xrightarrow{d} M_0$ will be denoted by $[\mathcal{M}]_{\mathbf{XLbnz}}$.

6. Comparison

In this section, we will give the relation between the split extension classifiers in the categories **PXLbnz**, **XLbnz**, **PXLie** of precrossed modules of Lie algebras and **XLie** of crossed modules of Lie algebras. The construction of split extension classifiers in **PXLie** and **XLie** can be found in [8, 10]. In [8] and [10], split extension classifiers were called actors.

Let M_1, M_0 be Leibniz algebras satisfying Condition **A.**

Proposition 6.1. *Let $\mathcal{M} : M_1 \xrightarrow{d} M_0$ be a crossed module of Leibniz algebras. Then $[\mathcal{M}]_{\mathbf{XLbnz}}$ is a subobject of $[\mathcal{M}]_{\mathbf{PXLbnz}}$, in* **PXLbnz**.

Proof. Every crossed biderivation $\partial \in Bider(M_0, M_1)$ gives rise to a triple $(\alpha, \partial, \alpha^1) \in Gbider(M_0, M_1)$ where $\alpha = \alpha^1 = \partial d$. Then $Bider(M_0, M_1)$ is a Leibniz subalgebra of $Gbider(M_0, M_1)$. Consequently, we get

$$[\mathcal{M}]_{\mathbf{XLbnz}} \leq [\mathcal{M}]_{\mathbf{PXLbnz}}\,.$$ □

For a given precrossed module $\mathcal{L} : L_1 \xrightarrow{d} L_0$ in Lie algebras, denote its split extension classifier by $[\mathcal{L}]_{\mathbf{PXLie}}$ in **PXLie** and for a given crossed module $\mathcal{L}' : L'_1 \longrightarrow L'_0$ of Lie algebras, denote its split extension classifier by $[\mathcal{L}']_{\mathbf{XLie}}$ in **XLie**. As indicated in [8], $[\mathcal{L}']_{\mathbf{XLie}}$ is a subobject of $[\mathcal{L}']_{\mathbf{PXLie}}$.

Proposition 6.2. *Let $\mathcal{L} : L_1 \xrightarrow{d} L_0$ be a precrossed module in the category of Lie algebras. Then $[\mathcal{L}]_{\mathbf{PXLie}}$ is a subobject of $[\mathcal{L}]_{\mathbf{PXLbnz}}$ in* **PXLbnz**.

Proof. Direct checking. □

Proposition 6.3. *Let $\mathcal{L}' : L'_1 \xrightarrow{d'} L'_0$ be a crossed module in the category of Lie algebras. Then. $[\mathcal{L}']_{\mathbf{XLie}}$ is a subobject of $[\mathcal{L}']_{\mathbf{XLbnz}}$ in* **XLbnz**.

Proof. Direct checking. □

Corollary 6.4. *Let $\mathcal{L}' : L'_1 \xrightarrow{d'} L'_0$ be a crossed module in the category of Lie algebras. We have the following:*

$$[\mathcal{L}']_{\mathbf{XLie}} \leq [\mathcal{L}']_{\mathbf{XLbnz}} \leq [\mathcal{L}']_{\mathbf{PXLbnz}}$$

and

$$[\mathcal{L}']_{\mathbf{XLie}} \leq [\mathcal{L}']_{\mathbf{PXLie}} \leq [\mathcal{L}']_{\mathbf{PXLbnz}}$$

in **PXLbnz**.

Proof. Follows from Propositions 6.2 and 6.3. □

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