

# REGULAR EVOLUTION ALGEBRAS ARE UNIVERSALLY FINITE

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**ABSTRACT.** In this paper we show that evolution algebras over any given field  $\mathbb{k}$  are universally finite. In other words, given any finite group  $G$ , there exist infinitely many regular evolution algebras  $X$  such that  $\text{Aut}(X) \cong G$ . The proof is built upon the construction of a covariant faithful functor from the category of finite simple (non oriented) graphs to the category of (finite dimensional) regular evolution algebras. Finally, we show that any constant finite algebraic affine group scheme  $\mathbf{G}$  over  $\mathbb{k}$  is isomorphic to the algebraic affine group scheme of automorphisms of a regular evolution algebra.

## 1. INTRODUCTION

Given a category  $\mathcal{C}$ , we say that  $\mathcal{C}$  is finitely universal [1, Section 4.1] if every finite group can be represented as the full automorphism group of an object in  $\mathcal{C}$ , that is, if given  $G$  a finite group, there exists  $X \in \text{Obj}(\mathcal{C})$  such that  $\text{Aut}_{\mathcal{C}}(X) \cong G$ .

Deciding whether a given category is finitely universal is a hard problem. From the celebrated result of Frucht [14], who considered finite graphs, to the recent results in [9, 10] (resp. [7, 8]), where the homotopy category of spaces (resp. the category of differential graded algebras) is considered, mathematicians have been addressing the identification of finitely universal categories for almost a century. The reader is encouraged to consult [16, Introduction] for a (non exhaustive) account of milestones in this question.

In this manuscript we consider the property of being finitely universal for evolution algebras over an arbitrary field  $\mathbb{k}$  (see Section 2 for precise definitions). Evolution algebras are, non necessarily associative, commutative algebras that were originally introduced by Tian [20] (see also [21]) as a mathematical model of self-reproduction in non-Mendelian genetics. Despite of the lack of nice structural properties, for example they are not power associative nor closed under subalgebras, evolution algebras have been a very active research topic the last decade (see [4, 22, 5, 18] and the references provided there).

Within this context, our main result is:

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**Theorem 1.1.** *Let  $\mathcal{E}_{\mathbb{k}}$  be the category of evolution algebras over the field  $\mathbb{k}$ , and  $G$  be a finite group. Then there are infinitely many non isomorphic  $X \in \text{Obj}(\mathcal{E}_{\mathbb{k}})$  such that*

- (1)  $X^2 = X$ , that is,  $X$  is regular, and
- (2)  $\text{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \cong G$ .

*In particular the category  $\mathcal{E}_{\mathbb{k}}$  is finitely universal.*

*Proof.* Given a finite group  $G$ , there exist infinitely many connected simple graphs  $\mathcal{G}$  such that  $\text{Aut}_{\text{graphs}}(\mathcal{G}) \cong G$  [14] (see also [11] and [19]). Therefore, according to Theorem 3.1, there are infinitely many regular evolution algebras  $X := \mathcal{X}(\mathcal{G}) \in \text{Obj}(\mathcal{E}_{\mathbb{k}})$  such that  $\text{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \cong \text{Aut}_{\text{graphs}}(\mathcal{G}) \cong G$ .  $\square$

Since the automorphism group of a regular evolution algebra is always finite, [12, Theorem 4.8], we obtain the following:

**Corollary 1.2.** *Let  $G$  be an abstract group. Then  $G$  is finite if and only if there exists a regular evolution algebra  $X$  such that  $\text{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \cong G$ .*

Given a field  $\mathbb{k}$ , let  $\text{Alg}_{\mathbb{k}}$  denote the category of unital commutative, associative  $\mathbb{k}$ -algebras. Following [13], any evolution algebra  $X$  over  $\mathbb{k}$  defines an algebraic affine group scheme over  $\mathbb{k}$  (see [23] or [17, Chapter VI])

$$\mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X): \text{Alg}_{\mathbb{k}} \rightsquigarrow \text{Groups}$$

that takes any  $R \in \text{Obj}(\text{Alg}_{\mathbb{k}})$  to  $\text{Aut}_R(X_R)$ , the group of  $R$ -algebra automorphisms of  $X_R = X \otimes_{\mathbb{k}} R$ . Within this framework, we prove:

**Theorem 1.3.** *Given any field  $\mathbb{k}$ , and any constant finite algebraic affine group scheme  $\mathbf{G}$  over  $\mathbb{k}$ , there are infinitely many non isomorphic  $X \in \text{Obj}(\mathcal{E}_{\mathbb{k}})$  such that the group scheme  $\mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X)$  is isomorphic to  $\mathbf{G}$ .*

*Proof.* Let  $G = \mathbf{G}(\mathbb{k})$ . Then there exist infinitely many connected simple non oriented graphs  $\mathcal{G}$  such that  $\text{Aut}_{\text{graphs}}(\mathcal{G}) \cong G$  [11, 14]. For any such graph  $\mathcal{G}$ , let  $X = \mathcal{X}(\mathcal{G}) \in \text{Obj}(\mathcal{E}_{\mathbb{k}})$  be the evolution algebra with natural basis  $B$  given in Definition 3.2. According to [13, Theorem 3.2] there exists an exact sequence of algebraic affine group schemes

$$1 \rightarrow \mathbf{Diag}(\Gamma) \rightarrow \mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \rightarrow \text{Aut}(\Gamma)$$

where  $\Gamma = \Gamma(X, B)$  is the oriented graph attached to the evolution algebra  $X$  relative to the natural basis  $B$ , Definition 2.5,  $\mathbf{Diag}(\Gamma)$  is the diagonal group scheme of  $\Gamma$  [13, Definition 2.1], and  $\text{Aut}(\Gamma)$  is the constant group scheme given by the group of oriented graph automorphisms of  $\Gamma$ .

Since the structure constants  $\omega_{ii}$  equal 1 for every  $b_i \in B$ , then the oriented graph  $\Gamma$  has a loop at every vertex, and therefore no sources, which implies  $\mathbf{Diag}(\Gamma) = 1$  [13, Corollary 2.8]. Hence  $\mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X) \leq \text{Aut}(\Gamma)$ .

An easy calculation shows that  $\text{Aut}(\Gamma) \cong \text{Aut}_{\text{graphs}}(\mathcal{G})$  while the functor  $\mathcal{X}(-) \otimes_{\mathbb{k}} R$  implies that  $\text{Aut}_{\text{graphs}}(\mathcal{G}) \leq \mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X)(R)$  for every  $R \in \text{Obj}(\text{Alg}_{\mathbb{k}})$ . Therefore the group scheme  $\mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X)$  is isomorphic to the constant group scheme given by  $\text{Aut}_{\text{graphs}}(\mathcal{G})$ , that is, to  $\mathbf{G}$ .  $\square$

Translating Gordeev-Popov's work [15] to the language of algebraic affine group schemes, Theorem 1.3 may be seen as a strengthening of [15, Corollary 2], and it is natural to consider the following question (compare to [15, Theorem 1] and the results in [13]):

**Question 1.4.** *Let  $\mathbb{k}$  be a field, and  $\mathbf{G}$  be an algebraic affine group scheme over  $\mathbb{k}$ . Does there exist an evolution algebra defined over  $\mathbb{k}$ ,  $X$ , such that the algebraic affine group scheme  $\mathbf{Aut}_{\mathcal{E}_{\mathbb{k}}}(X)$  is isomorphic to  $\mathbf{G}$ ?*

*Remark 1.5.* Notice that in contrast to [15, Theorem 1], the algebras in Question 1.4 are not required to be simple since the algebraic affine group scheme  $\mathbf{G}$  is not assumed to be finite at  $\mathbb{k}$ . Indeed, simple evolution algebras are regular [3, Corollary 4.6], and therefore as we have mentioned above, their automorphism groups are finite.

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## 2. BASICS ON EVOLUTION ALGEBRAS

In this section we briefly introduce the basics on evolution algebras that we need in the forthcoming sections. While the standard reference on the subject is [20], our notation is closer to the more recent work [2].

Given a field  $\mathbb{k}$ , we say that  $A$  is an algebra over  $\mathbb{k}$ , or a  $\mathbb{k}$ -algebra, if  $A$  is a  $\mathbb{k}$ -vector space furnished with an inner product, a  $\mathbb{k}$ -bilinear map  $A \times A \rightarrow A$ . An algebra morphism is then a linear map which commutes with inner products. The algebras considered in this manuscript are all finite dimensional vector spaces unless otherwise is stated.

We now recall the definition of an evolution algebra [20, Definition 1], [2, Definitions 1.2.1]:

**Definition 2.1.** An *evolution algebra* over a field  $\mathbb{k}$  is a  $\mathbb{k}$ -algebra  $X$  provided with a basis  $B = \{b_i \mid i \in \Lambda\}$  such that  $b_i b_j = 0$  whenever  $i \neq j$ . Such a basis  $B$  is called a *natural basis*.

Every  $\mathbb{k}$ -vector space  $X$  gives rise to a degenerated evolution algebra by defining  $xy = 0$  for every  $x, y \in X$ . More generally, given a  $\mathbb{k}$ -vector space  $X$  spanned by a basis  $B = \{b_i \mid i \in \Lambda\}$ , an evolution algebra with natural basis  $B$  is completely determined by just giving the linear expression of  $b_i^2 := b_i b_i \in X$  for every  $i \in \Lambda$ . This motivates the following definition [2, Definitions 1.2.1]:

**Definition 2.2.** Let  $X$  be an evolution algebra, and fix a natural basis  $B$  in  $X$ . The scalars  $\omega_{ki} \in \mathbb{k}$  such that  $b_i^2 := b_i b_i = \sum_{k \in \Lambda} \omega_{ki} b_k$  are called the *structure constants* of  $X$  relative to  $B$ , and the matrix  $M_B := (\omega_{ki})$  is said to be the *structure matrix* of  $X$  relative to  $B$ .

Notice that given an evolution algebra  $X$ , with natural basis  $B$ , the column vectors of  $M_B$  span  $X^2$ , the algebra generated by twofold products of elements in  $X$ . Therefore, if  $X = X^2$ , then  $M_B$  must be of maximal rank, that is,  $M_B$  must be a regular matrix (see also [12, Remark 4.2]). This justifies the following definition:

**Definition 2.3.** We say that an evolution algebra  $X$  is *regular* if and only if  $X = X^2$ , or equivalently, if and only if for any natural basis  $B$ , the structure matrix  $M_B$  is a regular matrix.

The following result shows that regular evolution algebras have essentially just one natural basis, which plays a key role in our arguments.

**Proposition 2.4.** ([12, Theorem 4.4], [6, Proposition 1]) *Let  $X$  be a regular evolution algebra over  $\mathbb{k}$  provided with a natural basis  $B = \{b_i \mid i \in \Lambda\}$ . Given any other natural basis  $\hat{B} = \{\hat{b}_i \mid i \in \Lambda\}$ , there exists a permutation  $\sigma \in \text{Perm}(\Lambda)$ , and scalars  $\lambda_i \in \mathbb{k}$  such that  $\hat{b}_i = \lambda_i b_{\sigma(i)}$ .*

Every finite dimensional evolution algebra has attached a directed graph [20, Definition 15], [12, Definition 2.2]:

**Definition 2.5.** Let  $X$  be a finite dimensional evolution algebra over  $\mathbb{k}$  provided with a natural basis  $B = \{b_1, \dots, b_n\}$ , and structure matrix  $M_B := (\omega_{ij})$ . The directed graph  $\Gamma(X, B) = (V, E)$  with  $V = \{1, \dots, n\}$  and  $E = \{(i, j) \in V \times V : \omega_{ij} \neq 0\}$  is called the oriented graph attached to the evolution algebra  $X$  relative to the natural basis  $B$ .

Notice that assigning to an evolution algebra  $X$  with natural basis  $B$  the directed graph  $\Gamma(X, B)$  is not functorial. The isomorphism type of  $\Gamma(X, B)$  depends on the choice of the natural basis [12, Example 2.4], and not every algebra morphism can be described in terms of a directed graph morphism.

### 3. FROM SIMPLE GRAPHS TO EVOLUTION ALGEBRAS

In this section we show that simple (non oriented) graphs give rise to evolution algebras in a functorial way.

Let  $\mathcal{G}raphs$  denote the category of *finite simple graphs*. Objects in  $\mathcal{G}raphs$  are pairs  $\mathcal{G} = (V, E)$  where  $V$  is a finite set, the set of vertices of  $\mathcal{G}$ , and  $E$  is a set of unordered pairs of different vertices (subsets of  $V$  of cardinality 2), so we shall frequently denote elements  $e \in E$  as  $e = \{v, w\} \subset V$ . Notice that by definition, two vertices in a simple graph can be connected by at most one edge, and loops are not allowed.

Given two simple graphs  $\mathcal{G}_i = (V_i, E_i)$ ,  $i = 1, 2$ , a graph morphism  $f: \mathcal{G}_1 \rightarrow \mathcal{G}_2$  is a set morphism  $f: V_1 \rightarrow V_2$  such that if  $e = \{v, w\} \in E_1$ , then  $f(e) = \{f(v), f(w)\} \in E_2$ .

Our main result in this section is the following:

**Theorem 3.1.** *Given any field  $\mathbb{k}$ , there exists a covariant faithful functor*

$$\mathcal{X}: \mathcal{G}raphs \rightsquigarrow \mathcal{E}_{\mathbb{k}}$$

*such that for every  $\mathcal{G} \in \mathcal{G}raphs$  the following hold:*

- (1)  $\mathcal{X}(\mathcal{G})^2 = \mathcal{X}(\mathcal{G})$ , that is,  $\mathcal{X}(\mathcal{G})$  is regular, and
- (2)  $\text{Aut}_{\mathcal{E}_{\mathbb{k}}}(\mathcal{X}(\mathcal{G})) \cong \text{Aut}_{\mathcal{G}raphs}(\mathcal{G})$ .

For convenience of the reader, the proof of Theorem 3.1 has been split in several lemmas. Henceforth,  $\mathbb{k}$  is a fixed field.

We first define the functor  $\mathcal{X}$  on objects:

**Definition 3.2.** Given a simple graph  $\mathcal{G} = (V, E)$ , we define  $\mathcal{X}(\mathcal{G})$  to be the evolution algebra over  $\mathbb{k}$ , with natural basis  $B = \{b_v : v \in V\} \cup \{b_e : e \in E\}$  and multiplication given by

- $b_v^2 = b_v$  for all  $v \in V$ .
- $b_e^2 = b_e + \sum_{v \in e} b_v$  for all  $e \in E$ .

We now prove that  $\mathcal{X}$  actually gives rise to regular evolution algebras.

**Lemma 3.3.** *For any simple graph  $\mathcal{G}$ , the evolution algebra  $\mathcal{X}(\mathcal{G})$  is regular.*

*Proof.* Let  $M_B$  be the structure matrix of  $\mathcal{X}(\mathcal{G})$  relative to the natural basis  $B$  given in Definition 3.2. Then  $M_B$  is upper triangular, and every input on the diagonal is equal to 1. Therefore  $\det(M_B) = 1$  and  $M_B$  is regular, hence so is  $\mathcal{X}(\mathcal{G})$ .  $\square$

We now define the functor  $\mathcal{X}$  on morphisms:

**Definition 3.4.** Given a graph morphism  $f: \mathcal{G}_1 \rightarrow \mathcal{G}_2$  where  $\mathcal{G}_i = (V_i, E_i)$ ,  $i = 1, 2$ , define  $\mathcal{X}(f): \mathcal{X}(\mathcal{G}_1) \rightarrow \mathcal{X}(\mathcal{G}_2)$  to be the algebra morphism given by  $\mathcal{X}(f)(b_v) = b_{f(v)}$  and  $\mathcal{X}(f)(b_e) = b_{f(e)}$ .

*Remark 3.5.* The map  $\mathcal{X}(f)$  above is a well defined algebra morphism. Indeed, if  $f$  is a graph morphism,  $f$  maps vertices to vertices, and edges to edges while preserving adjacency. Therefore for any vertex  $v \in V_1$ ,  $f(v) \in V_2$  and

$$\mathcal{X}(f)(b_v)^2 = b_{f(v)}^2 = b_{f(v)} = \mathcal{X}(f)(b_v),$$

while for any edge  $e = \{v, w\} \in E_1$ ,  $f(e) = \{f(v), f(w)\} \in E_2$  and

$$\mathcal{X}(f)(b_e)^2 = b_{f(e)}^2 = b_{f(e)} = b_{f(v)} + b_{f(w)} = \mathcal{X}(f)(b_e + b_v + b_w) = \mathcal{X}(f)(b_e^2).$$

Finally, it is straightforward to check that  $\mathcal{X}(f) = \mathcal{X}(g)$  if and only if  $f = g$ , and  $\mathcal{X}(f) \circ \mathcal{X}(g) = \mathcal{X}(f \circ g)$ .

The functor  $\mathcal{X}$  is also faithful on objects.

**Lemma 3.6.** *Let  $\mathcal{G}_i = (V_i, E_i)$ ,  $i = 1, 2$ , be simple graphs. Then  $\mathcal{X}(\mathcal{G}_1) \cong \mathcal{X}(\mathcal{G}_2)$  if and only if  $\mathcal{G}_1 \cong \mathcal{G}_2$ .*

*Proof.* If  $\mathcal{G}_1 \cong \mathcal{G}_2$ , functoriality provides the algebra isomorphism  $\mathcal{X}(\mathcal{G}_1) \cong \mathcal{X}(\mathcal{G}_2)$ .

Assume now that  $\psi: \mathcal{X}(\mathcal{G}_1) \cong \mathcal{X}(\mathcal{G}_2)$  is an isomorphism of regular evolution algebras. Let  $B_i$  be the natural basis of  $\mathcal{X}(\mathcal{G}_i)$ ,  $i = 1, 2$ , given in Definition 3.2. Then  $\psi(B_1)$  is a natural basis of  $\mathcal{X}(\mathcal{G}_2)$ , and according to Proposition 2.4, there exist scalars  $\lambda_v, \lambda_e \in \mathbb{k}$ , for  $v \in V_1$  and  $e \in E_1$ , such that  $\lambda_v \psi(b_v), \lambda_e \psi(b_e) \in B_2$ . Moreover, the number of non trivial structure constants must be preserved for each element in the basis, hence for  $v \in V_1$  and  $e \in E_1$  there exist  $\sigma(v) \in V_2$  and  $\tau(e) \in E_2$  such that  $\lambda_v \psi(b_v) = b_{\sigma(v)}$  and  $\lambda_e \psi(b_e) = b_{\tau(e)}$ .

Comparing

$$\begin{aligned} b_{\sigma(v)}^2 &= b_{\sigma(v)}, \text{ and} \\ (\lambda_v \psi(b_v))^2 &= \lambda_v^2 \psi(b_v)^2 = \lambda_v^2 \psi(b_v^2) = \lambda_v^2 \psi(b_v) = \lambda_v (\lambda_v \psi(b_v)) \\ &= \lambda_v b_{\sigma(v)}, \end{aligned}$$

we obtain that  $\lambda_v = 1$  for all  $v \in V_1$ .

Now, if  $e = \{v, w\} \in E_1$  and  $\tau(e) = \{\bar{v}, \bar{w}\} \in E_2$ , comparing

$$\begin{aligned} b_{\tau(e)}^2 &= b_{\tau(e)} + b_{\bar{v}} + b_{\bar{w}}, \text{ and} \\ (\lambda_e \psi(b_e))^2 &= \lambda_e^2 \psi(b_e)^2 = \lambda_e^2 \psi(b_e + b_v + b_w) = \lambda_e^2 (\psi(b_e) + \psi(b_v) + \psi(b_w)) \\ &= \lambda_e b_{\tau(e)} + \lambda_e^2 (b_{\sigma(v)} + b_{\sigma(w)}), \end{aligned}$$

we get that  $\lambda_e = 1$  and  $\tau(e) = \{\sigma(v), \sigma(w)\}$ .

In other words, the set morphism  $\sigma: V_1 \rightarrow V_2$ , which has to be bijective since  $\psi$  is an isomorphism, maps edges in  $E_1$  to edges in  $E_2$ . That is  $\sigma: \mathcal{G}_1 \cong \mathcal{G}_2$  and moreover  $\psi = \mathcal{X}(\sigma)$  as in Definition 3.4.  $\square$

Finally, we check that  $\mathcal{X}$  is full on the automorphism group of any object in the image. The proof follows similar arguments to those in the previous one:

**Lemma 3.7.** *Given a simple graph  $\mathcal{G} = (V, E)$ , and  $g \in \text{Aut}_{\mathcal{E}_k}(\mathcal{X}(\mathcal{G}))$ , there exists  $f \in \text{Aut}_{\text{graphs}}(\mathcal{G})$  such that  $g = \mathcal{X}(f)$ .*

*Proof.* Let  $g \in \text{Aut}_{\mathcal{E}_k}(\mathcal{X}(\mathcal{G}))$ . Recall that  $\mathcal{X}(\mathcal{G})$  is regular, and therefore  $B$ , the natural basis of  $\mathcal{X}(\mathcal{G})$  given in Definition 3.2, is unique up to permutation and scalar multiplication by Proposition 2.4. Since  $g(B)$  is another natural basis of  $\mathcal{X}(\mathcal{G})$ , and  $g$  has to preserve the structure constants, then  $g(b_v) = \lambda(v)b_{\sigma(v)}$ , and  $g(b_e) = \lambda(e)b_{\tau(e)}$  for some permutations  $\sigma \in \text{Perm}(V)$ , and  $\tau \in \text{Perm}(E)$ , and scalars  $\lambda(v), \lambda(e) \in \mathbb{k}$ .

Comparing

$$\begin{aligned} g(b_v^2) &= g(b_v) = \lambda(v)b_{\sigma(v)}, \text{ and} \\ (g(b_v))^2 &= (\lambda(v)b_{\sigma(v)})^2 = \lambda(v)^2 b_{\sigma(v)}, \end{aligned}$$

we obtain  $\lambda(v)^2 = \lambda(v)$  and therefore  $\lambda(v) = 1$ .

If  $e = \{v, w\} \in E$ , and  $\tau(e) = \{\bar{v}, \bar{w}\}$ , comparing

$$\begin{aligned} g(b_e^2) &= g(b_e + b_v + b_w) = \lambda(e)b_{\tau(e)} + \lambda(v)b_{\sigma(v)} + \lambda(w)b_{\sigma(w)}, \text{ and} \\ (g(b_e))^2 &= (\lambda(e)b_{\tau(e)})^2 = \lambda(e)^2 (b_{\tau(e)} + b_{\bar{v}} + b_{\bar{w}}), \end{aligned}$$

we obtain  $\lambda(e)^2 = \lambda(e)$ , thus  $\lambda(e) = 1$ , and  $\tau(e) = \{\sigma(v), \sigma(w)\}$ . Therefore  $\sigma$  is a permutation of the vertices of  $\mathcal{G}$  that maps edges to edges. In other words,  $\sigma \in \text{Aut}_{\text{graphs}}(\mathcal{G})$  and  $g = \mathcal{X}(\sigma)$  as in Definition 3.4.  $\square$

## REFERENCES

1. László Babai, *Automorphism groups, isomorphism, reconstruction*, Handbook of combinatorics, Vol. 1, 2, Elsevier Sci. B. V., Amsterdam, 1995, pp. 1447–1540. MR 1373683
2. Yolanda Cabrera Casado, *Evolution algebras*, Ph.D. thesis, University of Málaga, Spain, December 2016.
3. Yolanda Cabrera Casado, Mercedes Siles Molina, and M. Victoria Velasco, *Evolution algebras of arbitrary dimension and their decompositions*, Linear Algebra Appl. **495** (2016), 122–162. MR 3462991
4. L. M. Camacho, J. R. Gómez, B. A. Omirov, and R. M. Turdibaev, *Some properties of evolution algebras*, Bull. Korean Math. Soc. **50** (2013), no. 5, 1481–1494. MR 3116885
5. J. M. Casas, M. Ladra, B. A. Omirov, and U. A. Rozikov, *On nilpotent index and dibaricity of evolution algebras*, Linear Algebra Appl. **439** (2013), no. 1, 90–105. MR 3045225
6. M. Eugenia Celorrio and M. Victoria Velasco, *Classifying evolution algebras of dimensions two and three*, Mathematics **7** (2019), no. 12, 1236.
7. Cristina Costoya, David Méndez, and Antonio Viruel, *Homotopically rigid Sullivan algebras and their applications*, An alpine bouquet of algebraic topology, Contemp. Math., vol. 708, Amer. Math. Soc., Providence, RI, 2018, pp. 103–121. MR 3807752
8. ———, *Realisability problem in arrow categories*, Collect. Math. **71** (2020), no. 3, 383–405. MR 4129534
9. Cristina Costoya and Antonio Viruel, *Every finite group is the group of self-homotopy equivalences of an elliptic space*, Acta Math. **213** (2014), no. 1, 49–62. MR 3261010
10. ———, *On the realizability of group actions*, Adv. Math. **336** (2018), 299–315. MR 3846153
11. J. de Groot, *Groups represented by homeomorphism groups*, Math. Ann. **138** (1959), 80–102. MR 119193
12. Alberto Elduque and Alicia Labra, *Evolution algebras and graphs*, J. Algebra Appl. **14** (2015), no. 7, 1550103, 10. MR 3339402
13. Alberto Elduque and Alicia Labra, *Evolution algebras, automorphisms, and graphs*, to appear in Linear Multilinear Algebra (2019).

14. R. Frucht, *Herstellung von Graphen mit vorgegebener abstrakter Gruppe*, Compositio Math. **6** (1939), 239–250. MR 1557026
15. Nikolai L. Gordeev and Vladimir L. Popov, *Automorphism groups of finite dimensional simple algebras*, Ann. of Math. (2) **158** (2003), no. 3, 1041–1065. MR 2031860
16. Gareth A. Jones, *Realisation of groups as automorphism groups in categories*, to appear in Ars Math. Contemp.
17. Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998, With a preface in French by J. Tits. MR 1632779
18. U. A. Rozikov and Sh. N. Murodov, *Dynamics of two-dimensional evolution algebras*, Lobachevskii J. Math. **34** (2013), no. 4, 344–358. MR 3146637
19. Gert Sabidussi, *Graphs with given infinite group*, Monatsh. Math. **64** (1960), 64–67. MR 115935
20. Jianjun Paul Tian, *Evolution algebras and their applications*, Lecture Notes in Mathematics, vol. 1921, Springer, Berlin, 2008. MR 2361578
21. Jianjun Paul Tian and Petr Vojtěchovský, *Mathematical concepts of evolution algebras in non-Mendelian genetics*, Quasigroups Related Systems **14** (2006), no. 1, 111–122. MR 2268830
22. Jianjun Paul Tian and Yi Ming Zou, *Finitely generated nil but not nilpotent evolution algebras*, J. Algebra Appl. **13** (2014), no. 1, 1350070, 10. MR 3096851
23. William C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979. MR 547117

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