EXERCISE PAGES FROM BUCK - Advanced Calculus

EXERCISES†

Assignment 1 page 10

#5, 10, 23

1 How would you describe the world lines of two particles that collide and destroy each other?

2 Construct a world-line diagram for the motion of two elastic balls of different mass that move along a line toward each other, collide, and rebound.

***3** Draw a sketch to illustrate the following events: A photon vanishes, giving rise to two particles, one an electron and one a positron. The electron moves off in one direction, the positron in another. The positron strikes another electron, and the two annihilate each other, giving rise to a photon which travels off. Could this be the history of only one particle?

4 If A and B are sets and $A \subset B$, what are $A \cup B$ and $A \cap B$?

5 For any sets A and B, let $A - B$ be the set of those things which belong to A but do not belong to B. What is $A - (A - B)$? Is it true that $C \cap (A - B) = (C \cap A) - (C \cap B)$?

6 (a) Which is larger, $[\sqrt{243}/3]$ or $[12/\sqrt{5}]$?

(b) Find a rational number between $\sqrt{37}$ and $\sqrt{39}$.

7 Solve for P in each of the following equations:

(a) $(2, 1, -3) + P = (0, 2, 4)$

(b) $(1, -1, 4) + 2P = 3P + (2, 0, 5)$

8 Solve for the points P and Q if

$$2P + 3Q = (0, 1, 2)$$
$$P + 2Q = (1, -1, 3)$$

9 Solve for P and Q if

$$3P + Q = (1, 0, 1, -4)$$
$$P - Q = (2, 1, 2, 3)$$

10 Let $A = (1, 1, 3)$ and $B = (2, -1, 1)$. Can you find a point p such that $p \cdot A = 0$ and $p \cdot B = 0$?

11 Draw a diagram to illustrate that the associative law of addition, $p + (q + r) = (p + q) + r$, holds for the operation of addition of vectors.

12 The "center of gravity" of the triangle with vertices at A, B, and C is the point $\frac{1}{3}(A + B + C)$. Show that the center of gravity of a triangle is always the same as that of the triangle formed by the midpoints of its sides.

13 What is the center of gravity of the triangle whose vertices are $(1, 2, -4, 1)$, $(2, 0, 5, 2)$, $(0, 4, 2, -3)$?

14 Show that any three noncollinear points can be the midpoints of the sides of a unique triangle.

15. Using the definition given at the end of this section, give an algebraic proof that a four-sided polygon is a parallelogram if and only if the diagonals bisect each other.

16. Show that the four points A, B, C, D are the vertices of a parallelogram if and only if $A + C = B + D$, or $A + B = C + D$, or $A + D = B + C$.

17. In the rules (i), (ii), (iii) which were given for real numbers, there was no mention of subtraction. Formulate a set of rules concerning subtraction, and then check these with the development given in Appendix 2.

18. Show that the rules in (1-4) hold, by direct use of the definition of addition of points and the previously given rules about real numbers.

19 Using the order properties (iv) to (vii) for the real field, derive the following additional properties:

(a) For any $a, b \in \mathbf{R}$, if $a < b$, then $-b < -a$.

(b) For any $x \in \mathbf{R}$, $x^2 \geq 0$, with equality only if $x = 0$.

(c) For any real x and y, if $x^2 + y^2 = 0$, then $x = y = 0$. Does this extend to more terms?

20 (a) If $0 < m < n$, show that $m^2 < n^2$ and $1/n < 1/m$.

(b) If $m < n < 0$, show that $m^2 > n^2$ and $1/m > 1/n$.

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21 Suppose that a, b, A, B are all >0. Is it always true that

$$\frac{a+b}{A+B} \le \frac{a}{A} + \frac{b}{B}$$

22 Using the coordinate representation for points and the dot product, prove the identities (1-6) and (1-7).

23 Show that the set Z of all integers is countable.

Assignment 1 continued p. 19 #1, 2, 5, 6

EXERCISES

1 For $n = 1$, 2, and 3 in turn, plot the set of points p in $\mathbf{R}^n$ where
(a) $|p| < 1$ (b) $|p| \ge 1$ (c) $|p| = 1$.

2 Let $A = (4, 2)$. Graph the set of points p in the plane for which
(a) $|p| < |p - A|$ (b) $|p| + |p - A| = 6$ (c) $|p| + |p - A| \le 4$.

3 Sketch the set of points (x, y) where
(a) $|x + 2y| \le x - y$
(b) $(x^2 - y)(x - y^2) < 0$

4 Show that $|p_1 + p_2 + p_3 + \cdots + p_n| \le |p_1| + |p_2| + \cdots + |p_n|$.

5 Prove that $|p - q| \ge |p| - |q|$.

6 If $p = (u, v, w)$, show that

(a) $|p| \le |u| + |v| + |w|$
(b) $|u| \le |p|$, $|v| \le |p|$, $|w| \le |p|$

7 Use the law of cosines in the plane and the properties of the norm and scalar product to verify that $p \cdot q = |p||q| \cos \theta$.

8 Show that the three points $A = (2, -1, 3, 1)$, $B = (4, 2, 1, 4)$, and $C = (1, 3, 6, 1)$ form a triangle with two equal angles. Find its area.

9 Find the equation of the hyperplane in 4-space which goes through the point $p_0 = (0, 1, -2, 3)$ perpendicular to the vector $\mathbf{a} = (4, 3, 1, -2)$.

10 If the angle between two hyperplanes is defined as the angle between their normals, are the hyperplanes $3x + 2y + 4z - 2w = 5$ and $2x - 4y + z + w = 6$ orthogonal?

11 Write the parametric equations of the line through $(2, 3, -1, 1)$ which is perpendicular to the hyperplane $3x + 2y - 4z + w = 0$.

12 Where does the line through $q_1 = (1, 0, 1, 0)$ and $q_2 = (0, 1, 0, 1)$ intersect the two hyperplanes of Exercise 10?

13 Given a triangle with vertices at A, B, C, show that the point $R = \frac{1}{3}(A + B + C)$ lies on each of the medians (the line from a vertex to the midpoint of the opposite side).

14 Formulate and prove an analogous property for the tetrahedron with vertices at A, B, C, D.

***15** In the triangle ABC, join A to a point $\frac{1}{3}$ of the way from B toward C, join B to a point $\frac{1}{3}$ of the way from C toward A, and join C to a point $\frac{1}{3}$ of the way from A toward B. Express the vertices of the smaller triangle thus formed in terms of A, B, and C.

16 Let l be the line determined by the two points p and q. Let $P = \lambda p + (1 - \lambda)q$. Show that, when $\lambda > 1$, $|P - p| + |p - q| = |P - q|$, and interpret this geometrically.

17 Show that the intersection of two convex sets is convex but that the union of convex sets does not have to be convex.

Assignment 2 p 27 # 3, 15, 16

EXERCISES

~~1~~ Give the domain of definition of each function f defined below, and describe or sketch its graph:

(a) $f(x) = 1/(1 + x^2)$ (b) $f(x, y) = 4 - x^2 - y^2$

(c) $f(x) = x/(x - 1)$ (d) $f(x, y) = 1/(x^2 - y^2)$

(e) $f(x, y) = \begin{cases} 1 & \text{for } x < y \\ 0 & \text{for } x = y \\ \frac{1}{2} & \text{for } x > y \end{cases}$

2 Let $f(x) = x^2 + x$, $g(x, y) = xy$, and $h(x) = x + 1$. What are:

~~(a)~~ $f(g(1, 2))$ (b) $h(f(3))$

(c) $g(f(1), h(2))$ ~~(d)~~ $g(f(x), h(y))$

~~(e)~~ $g(h(x), f(x))$ (f) $f(g(x, h(y)))$

(g) $f(f(x))$

3 ~~(a)~~ If $F(x) = x^2 + x$ and $G(s) = s + s^2$, are F and G different functions?

~~(b)~~ If $F(x, y) = x^2 + y$ and $G(x, y) = x + y^2$, are F and G different functions?

4 (a) What is the natural domain of the function $g(x) = \sqrt{2 - x}$?

(b) What is the natural domain of the function $f(x) = \sqrt{x - 3} + \sqrt{2 - x}$?

~~*5~~ Sketch F for which $F(1/(x - 1)) = x/(x + 1)$, $x \neq 1$.

~~6~~ Sketch the level curves of the function described by $f(x, y) = x^2 - y^2$.

~~7~~ Sketch the level curves for f when

(a) $f(x, y) = y^2 - x$ (b) $f(p) = |p| - 1$

(c) $f(p) = \begin{cases} 1 & \text{when } |p| < 1 \\ x - y & \text{when } |p| \geq 1 \end{cases}$

~~8~~ Sketch the level surfaces for the function $f(x, y, z) = x^2 + y^2 - z^2$.

9 Let $F(x, y, z, t) = (x - t)^2 + y^2 + z^2$. By interpreting this as the temperature at the point (x, y, z) at time t, see if you can get a feeling for the behavior of the function.

10 If f is a function of two variables, show that if the set of points above the graph of f is a convex set, then f satisfies (1-27).

11 What can you say about the problem of solving for y in the equation $x^3 - y^3 + x - y = 0$? How many real functions on $-\infty < x < \infty$ does this equation define?

*12 Given $E(x, y) = x^2 - y^2$, how many different functions f are there that are "defined by the equation $E(x, y) = 0$ so that $y = f(x)$"?

13. Find the first six terms of the sequence defined by $a_n = (-2)^{n+1} + (-3)^n$

14 Given $x_n = 3n + (-1)^n(n - 5) + 7$,

(a) Calculate $x_1, x_2, \ldots, x_{10}$.

(b) Find all the numbers that ever appear twice in the entire sequence.

*(c) Do any terms appear three times?

15 What is $\bigcup_1^\infty D_n$ where $D_n = \{\text{all points } p \text{ with } |p| \leq n\}$?

*16 Show that the collection of all functions defined on a set D, with values in $\mathbf{R}^3$, is a vector

EXERCISES Assignment 2 continued p. 36 #1,5,9,13
Assignment 7 p. 36 #2,6,10,11

1 By quoting appropriate definitions and statements, verify the following assertions. Sketches may be helpful.

(i) The set $W = \{\text{all } p = (x, y) \text{ with } 1 \le |p| \le 2\}$ is closed, bounded, connected, and bdy(W) is the disconnected set, consisting of two circles of radius 1 and 2.

(ii) The set $R = \{\text{all } (x, y) \text{ with } x \ge 0\}$ is closed, unbounded, connected, and its boundary is the vertical axis.

(iii) The set $T = \{\text{all } (x, y) \text{ with } |x| = |y|\}$ is closed, connected, unbounded, has empty interior, and bdy(T) = T.

(iv) The set $Q = \{\text{all } (x, y) \text{ with } x \text{ and } y \text{ integers}\}$ is closed, unbounded, infinite, countable, disconnected, and its boundary is itself.

2 Tell which of the properties described in this section apply to the set of points (x, y) such that:

(a) $x^2 + y^2 = 1$ (b) $x^2 + y \ge 0$
(c) $x = y$ (d) $x > 1$
(e) $xy > 0$ (f) $y = |x - 1| + 2 - x$

3 The same as Exercise 2 for the set of points (x, y, z) such that:

(a) $x^2 + y^2 + z^2 > 4$ (b) $x^2 + y^2 \le 4$
(c) $xy > z$ (d) $(x - y)^2 = z^2$

4 The same as Exercise 2 for the set of points x on the line such that:

(a) $x(x-1)^2 > 0$ (b) $x(x-1)(x+1)^2 \le 0$

5 Let $S = \{\text{all } (x, y) \text{ with } x \text{ and } y \text{ rational numbers}\}$.

(a) What is the interior of S?
(b) What is the boundary of S?

6 What are the cluster points for the set

$$S = \left\{\text{all } \left(\frac{1}{n}, \frac{1}{m}\right) \text{ with } n = 1, 2, \ldots, m = 1, 2, \ldots\right\}$$

7 Produce an unbounded infinite set with no cluster points.

8 By constructing an example, show that the union of an infinite collection of closed sets does not have to be closed.

9 Let C be a closed set and V an open set. Then $C - V$ is closed and $V - C$ is open. Verify this statement, using statements in (1-29).

10 Construct pictures to show that each of the following is false.

(i) If $A \subset B$, then bdy(A) $\subset$ bdy(B).
(ii) bdy(S) is the same as the boundary of the closure of S.
(iii) bdy(S) is the same as the boundary of the interior of S.
(iv) The interior of S is the same as the interior of the closure of S.

11 Let S be any bounded set in n space. Is the closure of S a bounded set?

12 What is the relationship of bdy($A \cap B$) to bdy(A) and bdy(B)?

13 (a) Why should the empty set $\varnothing$ be called both open and closed?
(b) Why, in the plane, is the set $\{\text{all } (x, y), x^2 + y^2 \ge 0\}$ both open and closed?

14 Let $U_n = \{\text{all } p = (x, y) \text{ with } |p - (0, n)| < n\}$. Show that the union of all the open sets U_n, for $n = 1, 2, 3, \ldots$, is the open upper half plane.

15 Let A and B be connected sets in the plane which are not disjoint. Is $A \cap B$ necessarily connected? Is $A \cup B$ necessarily connected?

16 Show that the complement of the set in Exercise 5 is a polygon connected set. Is it an open set?

17 (a) Is the interior of a connected set necessarily connected?
*(b) Is the closure of a connected set necessarily connected?

18 (a) Show that any two disjoint nonempty open sets are mutually separated.
(b) Show that any two disjoint nonempty closed sets are mutually separated.

***19** When a set is not connected, the separate pieces that it comprises are called **components.** How many components are there to the set $S = \{\text{all } (x, y) \text{ with } x^2 - 3xy + y^3 < 0\}$?

Assignment 8 p. 69 #1, 2

EXERCISES

1 Show that any compact set in n space must be bounded.

2 Show that any compact set in n space must be closed.

3 Show that every closed subset of a compact set is compact.

4 Fill in the details of the proof of Corollary 1 of Theorem 30.

5 Prove that a set S in n space is compact if and only if every sequence in S has a limit point that belongs to S.

6 Let A and B be compact sets on the line. Use Exercise 5 to show that their cartesian product $A \times B$ is a compact set in the plane.

7 (*a*) Prove Theorem 29, assuming that the set S is both closed and bounded.

(*b*) Prove Theorem 29, assuming only that S is bounded. [The difficulty lies in showing that the points p_i can be chosen in S itself.]

8 Must every bounded nonempty set in n space have a nonempty boundary?

9 For any set S in the plane, let

$$X(S) = \{\text{all } x \text{ for which } (x, y) \in S \text{ for some } y\}$$

This is called the projection of S into the X axis.

(*a*) If S is bounded, is $X(S)$ necessarily bounded?

(*b*) If S is closed, is $X(S)$ necessarily closed?

(*c*) If S is compact, show that $X(S)$ is necessarily compact.

10 (*a*) Show that the open unit disc in the plane can be expressed as the union of a collection of closed squares.

*(*b*) Can this be done with a countable collection of closed squares?

Assignment 9 p. 80 #1, 2, 3, 4, 7, 8, 12, 13, 14, 17

EXERCISES

1 Prove directly from Definition 1 that the function $M(x, y) = xy$ is continuous on any disk: $|(x, y)| \le r$.

2 Prove that $Q(x, y) = x/y$ is continuous everywhere except on the line $y = 0$.

3 Check that the function defined in (2-2) is such that it is convergence preserving for all sequences of the form $p_n = (a/n^2, b/n^2)$.

4 Let f be defined by $f(x, y) = x^2y^2/(x^2 + y^2)$, with $f(0, 0) = 0$. By checking various sequences, test this for continuity at $(0, 0)$. Can you tell whether or not it is continuous there?

5 Show that a real-valued function f is continuous in D if the set $S = \{\text{all } p \in D \text{ with } b < f(p) < c\}$ is open, relative to D, for every choice of the numbers b and c.

6 Using Theorem 4, where can you be sure that the function given by $F(x, y) = (x + y)/(x^2 - xy - 2y^2)$ is continuous?

7 Use the example $f(x, y) = x^2$ to show that a continuous function does not have to map an open set onto an open set.

***8** Use the example $f(x) = x^2/(1 + x^2)$ to show that a continuous function does not always have to map a closed set onto a closed set.

9 Using the assumed continuity properties of the sine function, what can you say about the set on which the function $g(x) = \csc(\sin(1/x))$ is continuous?

10 Show that f is continuous if and only if the inverse images of closed sets are closed sets relative to D.

11 Prove Theorem 5 using Theorems 1 and 2.

12 How are $f^{-1}(A \cap B)$ and $f^{-1}(A \cup B)$ related to $f^{-1}(A)$ and $f^{-1}(B)$?

13 Let $f: X \to Y$ be a function, and S and T arbitrary sets. Show that:

(*a*) $ff^{-1}(S) \subset S$ (*b*) $T \subset f^{-1}f(T)$

14 Let $F(x, y)$ be continuous on the square $|x| \le 1$, $|y| \le 1$. Where is the function $f(x) = F(x, c)$ continuous?

15 Formulate the definition of continuity for a complex-valued function f. Show that f is continuous if and only if its real and imaginary parts are continuous functions.

16 (*a*) Setting $z = x + iy = (x, y)$, consider the function f defined from complex numbers to complex numbers ($\mathbf{R}^2$ to $\mathbf{R}^2$) by $f(z) = z^2 + (1 - i)z + 2$, and show that it is continuous everywhere.

(*b*) What can you say about the continuity of the function f where:

(i) $f(z) = \dfrac{1}{z}$ (ii) $f(z) = \dfrac{z}{z^2 + 1}$

17 Show that a mapping $y = F(x)$ on a set D in $\mathbf{R}^n$ to $\mathbf{R}^m$ as given in (1-26) is continuous if and only if the component functions $f, g, h, \ldots, k$ are continuous on D.

Assignment 10 p. 54 # 1, 2, 3, 4, 32, 35

EXERCISES

1 Show that the sequence defined by $p_n = (n, 1/n)$ does not converge.

2 Show that the sequence described by $p_n = \left(\dfrac{n+1}{n}, \dfrac{(-1)^n}{n}\right)$ converges.

3 Let $|p_{n+1} - q| \le c|p_n - q|$ for all n, where $c < 1$. Show that $\lim_{n\to\infty} p_n = q$.

4 Let $\{p_n\}$ and $\{q_n\}$ be sequences in 3-space with $p_n \to p$ and $q_n \to q$. Prove that $\lim_{n\to\infty} p_n \cdot q_n = p \cdot q$.

***5** Starting at the origin in the plane, draw a polygonal line as follows: Go 1 unit east, 2 units north, 3 units west, 4 units south, 5 units east, 6 units north, and so on. Find a formula for the nth vertex of this polygon.

~~6~~ Among the following sequences, some are subsequences of others; determine all those which are so related.

(*a*) $1, -1, 1, -1, \ldots$
(*b*) $1, 1, -1, 1, 1, -1, \ldots$
(*c*) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
(*d*) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$
(*e*) $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots$

7 Exhibit a sequence having exactly three limit points. Can a sequence have an infinite number of limit points? No limit points? Could a divergent sequence have exactly one limit point?

8 Discuss the behavior of the sequence $\{a_n\}$, where

$$a_n = n + 1 + 1/n + (-1)^n n$$? 0

9 If $a_n \le x_n \le b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, show that $\lim_{n\to\infty} x_n = L$. (This is sometimes called the "sandwich" property.)

10 Find a formula for the sequence that begins with 1/2, 1/5, 1/10, 1/17, 1/26, ..., and show that it converges to 0. $\frac{1}{n^2+1}$. then $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ //.

11 Prove that $\lim_{n\to\infty} 1/\sqrt{n} = 0$.

12 What are the correct hypotheses for the truth of the following assertion? If $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{A}$.

13 Prove that $\lim_{n\to\infty} c^{1/n} = 1$ for any $c > 1$ by setting $a_n = c^{1/n} - 1$, and then deriving the estimate $0 \le a_n \le (c-1)/n$.

14. Investigate the convergence of the sequence

$$a_n = \sqrt{n^2 + n} - n$$

15 Show that the sequence $\{x_n\}$ defined by the recursive formula:

$$x_1 = 1, \qquad x_{n+1} = x_n + 1/x_n \qquad \text{for } n > 1$$

obeys the inequality $x_n > \sqrt{n}$ for all $n \ge 2$.

16 Define a sequence $\{x_n\}$ by

$$x_1 = 1, \qquad x_{n+1} = x_n + \sqrt{x_n} \qquad \text{for } n > 1$$

(a) Prove that $\{x_n\}$ is unbounded.
(b) Prove that if, for some N, $x_N \le N^2/4$, then $x_{N+1} \le (N+1)^2/4$.
(c) Is there a value of n for which $x_n \le n^2/4$?

(You will have to do some calculation. Use a pocket calculator; that's what they are for!)

(d) Show that $x_n \ge n^2/9$ for all $n \ge 1$.
*(e) Show that $\lim_{n\to\infty} x_n/n^2 = \frac{1}{4}$.

17 Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$ *

(a) Prove that $\{a_n\}$ is convergent.
*(b) Can you determine $\lim_{n\to\infty} a_n$?

18 Let $x_1 = 1$, $x_2 = 3$, and define all later terms recursively by $x_n = (x_{n-1} + x_{n-2})/2$. Thus, $x_3 = 2$, $x_4 = 5/2$. Is the sequence $\{x_n\}$ monotonic? Does it converge?

19 Let $a_1 = 1$, $a_2 = 2$, and $a_{n+2} = (4a_{n+1} - a_n)/3$. Show that $\{a_n\}$ converges.

20 Define the sequence $\{x_n\}$ by $x_1 = a$, $x_2 = b$, and $x_{n+2} = (1 + x_{n+1})/x_n$. Investigate the convergence of $\{x_n\}$. [*Hint:* It may help to try some numerical values.]

21 Let $a_1 = 0$, $a_2 = 1$, and $\qquad a_{n+2} = \dfrac{na_{n+1} + a_n}{n+1}$

(a) Calculate the value of a_6 and a_7.
(b) Prove that $\{a_n\}$ converges.
*(c) Show that $\lim_{n\to\infty} a_n = 1 - e^{-1}$.

22 Show that $x_n = \sqrt{\log \log n}$, for $n \ge 3$ defines a divergent increasing sequence such that

$$\lim_{n\to\infty} (x_{n^2} - x_n) = 0$$

23 Let $\{x_n\}$ be a bounded real sequence and set $\beta = \limsup_{n\to\infty} x_n$. Show that for any $\varepsilon > 0$, $x_n \le \beta + \varepsilon$ holds for all but a finite number of n, and $x_n \ge \beta - \varepsilon$ holds for infinitely many n. What are the analogous statements about $\liminf_{n\to\infty} x_n$?

24 If $b \le x_n \le c$ for all but a finite number of n, show that $b \le \liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n \le c$.

25 If $\{x_n\}$ is a bounded sequence with $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, show that $\{x_n\}$ converges.

26 Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ when:

(*a*) $a_n = (-1)^n$ inf = -1 sup = 1 (*b*) $a_n = (-1)^n\left(2 + \dfrac{3}{n}\right)$

(*c*) $a_n = \dfrac{n + (-1)^n(2n+1)}{n}$ (*d*) $a_n = \sin\left(n\dfrac{\pi}{3}\right)$? 0

27 If $\limsup_{n\to\infty} a_n = A$ and $\limsup_{n\to\infty} b_n = B$, must it be true that

$$\limsup_{n\to\infty} (a_n + b_n) = A + B \quad ?$$

***28** Show that, for any bounded sequences a_n and b_n,

$$\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \le \liminf_{n\to\infty} (a_n + b_n)$$

and that

$$\limsup_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

***29** Let $\{a_n\}$ be any sequence of numbers converging to 0, and let σ_n be the sequence of **arithmetic means** (averages),

$$\sigma_n = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

Prove that $\lim_{n\to\infty} \sigma_n = 0$.

30 If we start with $x_1 = 2$, how far must we go with the square root algorithm to get $\sqrt{2}$ accurate to 10^{-50}? 10^{-100}?

31 Define a sequence of points thus: Starting at the origin, move 1 unit east, then $\frac{1}{2}$ unit north, then $\frac{1}{4}$ unit east, then $\frac{1}{8}$ unit north, then $\frac{1}{16}$ unit east, and so on. (*a*) Does the sequence of vertices converge? (*b*) Can you find the "end" of this polygon?

32 Show that a convergent sequence $\{p_n\}$ must be a Cauchy sequence.

***33** Let $A = (0, 1)$ and $B = (1, 0)$. Let P_1 be any point in the plane, and construct a sequence $\{P_n\}$, with P_1 as its first term, as follows: Let Q_1 = midpoint of AP_1 and P_2 = midpoint of BQ_1; then, let Q_2 = midpoint of AP_2 and P_3 = midpoint of BQ_2, and so on. Prove that $\{P_n\}$ converges.

34 Show that every Cauchy sequence $\{p_n\}$ is bounded.

35 Let $p_n = (x_n, y_n, z_n)$. Show that if $\{p_n\}$ is Cauchy, so are $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$.

36 (*a*) Explain why $\lim_{m,n\to\infty} \dfrac{3n + 5m}{4n^2 + 7m^2} = 0$.

(*b*) Is it true that

$$\lim_{m,n\to\infty} \frac{2n + m}{3n + 5m^2} = \frac{2}{3}$$

37 Let $a_k > 0$ for all k and suppose that

$$\lim_{m,n\to\infty} a_m/a_n = 1$$

Prove that $\{a_k\}$ converges.

38 Show that Theorems 10 and 12 still hold if the sequences $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers.

assignment 11 p. 88 #1,2,3,4,5,6,7

EXERCISES

1 Show that $F(x, y) = x^2 + 3y$ is not uniformly continuous on the whole plane.

2 Prove that the function $f(x) = 1/(1 + x^2)$ is uniformly continuous on the whole line.

3 Let f and g each be uniformly continuous on a set E. Show that $f + g$ is uniformly continuous on E.

4 Let A and B be disjoint sets, and let f be continuous on A and continuous on B. When is it continuous on $A \cup B$?

5 Let A and B be disjoint closed sets and suppose that f is uniformly continuous on each.

(*a*) Show that f is necessarily uniformly continuous on $A \cup B$ if A is compact.

(*b*) Show that f need not be uniformly continuous on $A \cup B$ if neither A nor B is compact.

6 If f is uniformly continuous on D, show that it has the property that if $p_n, q_n \in D$ and $|p_n - q_n| \to 0$, then $|f(p_n) - f(q_n)| \to 0$.

7 Let D be a bounded set and let f be uniformly continuous on $D \subset \mathbf{R}^n$. Prove that f is bounded on D.

8 Let f be a function defined on a set E which is such that it can be uniformly approximated within ε on E by functions F that are uniformly continuous on E, for every $\varepsilon > 0$. Show that f must itself be uniformly continuous on E.

9 Using the special notation explained in (2-5), prove that, if f and g are defined on a set E, then

$$\|f + g\|_E \le \|f\|_E + \|g\|_E$$

10 Let p_1, p_2, p_3 be the vertices of a triangle D in the plane.

(*a*) Show that any point p in D can be expressed as

$$p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \qquad \text{where } \alpha_i \ge 0 \text{ and } \alpha_1 + \alpha_2 + \alpha_3 = 1$$

(*b*) If f is a function defined on D with Lipschitz constant M, use part (*a*) to define a function F on D by

$$F(p) = \alpha_1 f(p_1) + \alpha_2 f(p_2) + \alpha_3 f(p_3)$$

and show that $\|F - f\|_D \le M \operatorname{diam}(D)$.

Exercises 11 *and* 12 *constitute a different proof of Theorem* 6.

11 Let f be continuous on the interval $[a, b]$. Given $\varepsilon > 0$ and a point t in the interval, choose $\rho = \rho(t)$ so that, if $|x - t| < \rho$, then $|f(x) - f(t)| < \varepsilon$. Let U_t be the symmetric interval centered on t of radius $\frac{1}{2}\rho(t)$. Show that there are points $t_1, t_2, \cdots, t_m$ such that the sets U_{t_j} together cover the interval $[a, b]$.

12 (*Continuation of Exercise* 11.) Let ρ_0 be the smallest of the numbers $\rho(t_j)$, and let x' and x'' be any two points of the interval $[a, b]$ with $|x' - x''| < \frac{1}{2}\rho_0$. Show that there must be one of the points t_j such that $|x' - t_j| < \rho(t_j)$ and $|x'' - t_j| < \rho(t_j)$.

Conclude that $|f(x') - f(x'')| < 2\varepsilon$ and hence that f is uniformly continuous on $[a, b]$.

Assignment 12 p. 134 #4,5

EXERCISES

1 Find $f_1(x, y), f_2(x, y), f_{12}(x, y)$ if

(*a*) $f(x, y) = x^2 \log (x^2 + y^2)$

(*b*) $f(x, y) = x^y$.

2 With $f(x, y) = x^2y^3 - 2y$, find $f_1(x, y), f_2(x, y), f_2(2, 3)$, and $f_2(y, x)$.

3 Compute $\mathbf{D}f$ for each of the following functions at the given point:
(a) $f(x, y) = 3x^2y - xy^3 + 2$ at $(1, 2)$
(b) $f(u, v) = u \sin(uv)$ at $(\pi/4, 2)$
(c) $f(x, y, z) = x^2yz + 3xz^2$ at $(1, 2, -1)$.

4 (a) Let $f(x, y) = xy/(x^2 + y^2)$, with $f(0, 0) = 0$. Show that f_1 and f_2 exist everywhere, but that f is not of class C'.
(b) Does f have directional derivatives at the origin?
(c) Is f continuous at the origin?

5 Let a function f be defined in an open set D of the plane, and suppose that f_1 and f_2 are defined and bounded everywhere in D. Show that f is continuous in D.

6 Can you formulate and prove an analog for Rolle's theorem, for functions of two real variables?

7 Let f and g be of class C' in a compact set S, and let $f = g$ on bdy (S). Show that there must exist a point $p_0 \in S$ where $\mathbf{D}f(p_0) = \mathbf{D}g(p_0)$.

8 Find the derivative of $f(x, y, z) = xy^2 + yz$ at the point $(1, 1, 2)$ in the direction $(2/3, -1/3, 2/3)$.

9 Let $f(x, y) = xy$. Show that the direction of the gradient of f is always perpendicular to the level lines of f.

10 Show that each of the following obeys $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0$:
(a) $u = e^x \cos y$ (b) $u = \exp(x^2 - y^2) \sin(2xy)$

11 Let $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ with $f(0, 0) = 0$. Show that f is continuous everywhere, that f_1, f_2, f_{12}, and f_{21} exist everywhere, but $f_{12}(0, 0) \neq f_{21}(0, 0)$.

12 Find the directional derivative of $F(x, y, z) = xyz$ at $(1, 2, 3)$ in the direction from this point toward the point $(3, 1, 5)$.

13 If $F(x, y, z, w) = x^2y + xz - 2yw^2$, find the derivative of F at $(1, 1, -1, 1)$ in the direction $\beta = (4/7, -4/7, 1/7, -4/7)$.

14 Some economics students have been quoted as saying the following: $F(x_1, x_2, \ldots, x_n)$ is such that it does not change if you change only one variable, leaving the rest alone, but it does change if you make changes in two of them. What reaction would you give to such a statement?

Assignment 18 p. 351 #1,2,7,8

EXERCISES

1 Compute the differentials of the following transformations at the indicated points.

(a) $\begin{cases} u = xy^2 - 3x^3 \\ v = 3x - 5y^2 \end{cases}$ at $(1, -1)$ and $(1, 3)$

(b) $\begin{cases} u = xyz^2 - 4y^2 \\ v = 3xy^2 - y^2z \end{cases}$ at $(1, -2, 3)$

(c) $\begin{cases} u = x + 6y \\ v = 3xy \\ w = x^2 - 3y^2 \end{cases}$ at $(1, 1)$

2 If L is a linear transformation, show that $dL|_p = L$ at every point p.

3 Using the methods of Sec. 3.4, verify Theorem 11 for the following transformations:

(a) $S: u = F(x, y) \qquad T: \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}$

(b) $S: \begin{cases} u = F(x, y) \\ v = G(x, y) \end{cases} \qquad T: \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases}$

4 Let $w = F(x, y, t)$, $x = \phi(t)$, $y = \psi(t)$. Show that Theorem 11 can be applied to yield Eq. (3-22) for dw/dt.

5 Let T be a transformation from $\mathbf{R}^2$ into $\mathbf{R}^2$ given by $u = f(x, y)$, $v = g(x, y)$. Let L be a linear transformation

$$L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

such that $T(p_0 + \Delta p) - T(p_0) = L(\Delta p) + R(\Delta p)$, where $\lim_{\Delta p \to 0} R(\Delta p)/|\Delta p| = \mathbf{0}$. Prove that $L = dT|_{p_0}$.

6 Prove: If T is of class C'' in an open connected set D, and $dT = 0$ at each point of D, then T is constant in D.

7 Is there a transformation T of the plane into itself whose differential at (x, y) is given by

$$\begin{bmatrix} 3x^2y & x^3 \\ y & x \end{bmatrix}$$

8 Is there a transformation whose differential is

$$\begin{bmatrix} y & x \\ xy & x + y \end{bmatrix}$$

9 Explain the difference between (7-24) as applied to a scalar function of one variable and (7-25).

10 Show that a differentiable transformation (Definition 3) cannot have two different differentials at the same point.

11 Given the transformation

$$T: \begin{cases} u = 3x^2y^3 \\ v = x^3 - y^2 \end{cases}$$

on E, the unit square with (0, 0) and (1, 1) as diagonal corners, show that an estimate for the Lipschitz constant M for E in (7-31) is $\sqrt{130}$.

12 Consider the transformation $T(x, y) = (u, v)$ defined on the unit square $E: 0 \le x \le 1, 0 \le y \le 1$, where

$$u = 2x^2 + 6xy - 4x^3/3 - 3xy^2 \qquad \text{and} \qquad v = x^3 - y^2$$

Show that an estimate for the Lipschitz constant M for E in (7-31) is $M = \sqrt{65}$.

1 Construct schematic diagrams to show the following functional relationships, and find the indicated derivatives: Assignment 21 p. 145 #1, 2

(a) $w = f(x, y, z)$, $x = \phi(t)$, $y = \psi(t)$, $z = \theta(t)$. Find dw/dt.

(b) $w = F(x, u, t)$, $u = f(x, t)$, $x = \phi(t)$. Find dw/dt.

(c) $w = F(x, u, v)$, $u = f(x, y)$, $v = g(x, z)$. Find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

2 If $w = f(x, y)$ and $y = F(x)$, find dw/dx and d^2w/dx^2.

3 When x, y, and z are related by the equation $x^2 + yz^2 + y^2x + 1 = 0$, find $\partial y/\partial x$ and $\partial y/\partial z$ when $x = -1$ and $z = 1$.

4 Let x, y, u, v be related by the equations $xy + x^2u = vy^2$, $3x - 4uy = x^2v$. Find $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ first by implicit differentiation, and then by solving the equations explicitly for u and v.

5 Let $F(x, y, z) = 0$. Assuming that this can be solved for z in terms of (x, y), find $\partial z/\partial x$ and $\partial z/\partial y$.

6 Under the same assumptions as Exercise 5, find expressions for $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial x\, \partial y$ in terms of F and its derivatives.

7. Let $F(x,y,z)=0$. Prove that $\left.\frac{\partial z}{\partial y}\right|_x \left.\frac{\partial y}{\partial x}\right|_z \left.\frac{\partial x}{\partial z}\right|_y = -1$

8 Let $F(x, y, t) = 0$ and $G(x, y, t) = 0$ be used to express x and y in terms of t. Find general formulas for dx/dt and dy/dt.

9 Let $z = f(xy)$. Show that this obeys the differential relation

$$x\left(\frac{\partial z}{\partial x}\right) - y\left(\frac{\partial z}{\partial y}\right) = 0$$

10 Let $w = F(xz, yz)$. Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}$$

11 A function f is said to be homogeneous of degree k in a neighborhood $\mathcal{N}$ of the origin if $f(tx, ty) = t^k f(x, y)$ for all points $(x, y) \in \mathcal{N}$ and all t, $0 \leq t \leq 1$. Assuming appropriate continuity conditions, prove that f satisfies in $\mathcal{N}$ the differential equation

$$xf_1(x, y) + yf_2(x, y) = kf(x, y)$$

12 Setting $z = f(x, y)$, Exercise 11 shows that $x(\partial z/\partial x) + y(\partial z/\partial y) = 0$ whenever f is homogeneous of degree $k = 0$. Show that in polar coordinates this differential equation becomes simply $r(\partial z/\partial r) = 0$, and from this deduce that the general homogeneous function of degree 0 is of the form $f(x, y) = F(y/x)$.

13 If $z = F(ax + by)$, then $b(\partial z/\partial x) - a(\partial z/\partial y) = 0$.

14 If $u = F(x - ct) + G(x + ct)$, then

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

15 If $z = \phi(x, y)$ is a solution of $F(x + y + z, Ax + By) = 0$, show that $A(\partial z/\partial y) - B(\partial z/\partial x)$ is constant.

16 Show that the substitution $x = e^s$, $y = e^t$ converts the equation

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + y^2\left(\frac{\partial^2 u}{\partial y^2}\right) + x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 0$$

into the equation $\partial^2 u/\partial s^2 + \partial^2 u/\partial t^2 = 0$.

17 Show that the substitution $u = x^2 - y^2$, $v = 2xy$ converts the equation $\partial^2 W/\partial x^2 + \partial^2 W/\partial y^2 = 0$ into $\partial^2 W/\partial u^2 + \partial^2 W/\partial v^2 = 0$.

18 Show that if p and E are regarded as independent, the differential equation (3-32) takes the form

$$\frac{\partial T}{\partial p} - T\frac{\partial V}{\partial E} + p\frac{\partial(V, T)}{\partial(E, p)} = 0$$

19 Let f be of class C'' in the plane, and let S be a closed and bounded set such that $f_1(p) = 0$ and $f_2(p) = 0$ for all $p \in S$. Show that there is a constant M such that $|f(p) - f(q)| \leq M|p - q|^2$ for all points p and q lying in S.

***20** (*Continuation of Exercise* 19) Show that if S is the set of points on an arc given by the equations $x = \phi(t)$, $y = \psi(t)$, where ϕ and ψ are of class C', then the function f is constant-valued on S.

21 Let f be a function of class C' with $f(1, 1) = 1$, $f_1(1, 1) = a$, and $f_2(1, 1) = b$. Let $\phi(x) = f(x, f(x, x))$. Find $\phi(1)$ and $\phi'(1)$.

Assignment 22 P. 153 # 18

1 Show that $\sin x$ can be approximated by $x - x^3/6$ within .01 on the interval $[-1, 1]$.

2 Determine the accuracy of the approximation

$$\cos x \sim 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

on the interval $[-1, 1]$.

3 Determine the accuracy of the approximation

$$\log(1 + x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

4 Determine the accuracy of the approximation

$$\sqrt{x} \sim 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8}$$

on the interval $[\frac{1}{2}, \frac{3}{2}]$.

5 How many terms of the Taylor expansion for $\sin x$ about a conveniently chosen point are needed to obtain a polynomial approximation accurate to .01 on the interval $[0, \pi]$? On the interval $[0, 2\pi]$?

6 Suppose that $f(0) = f(-1) = 0$, $f(1) = 1$, and $f'(0) = 0$. Assuming that f is of class C^3, show that there is a point c in $[-1, 1]$ where $f'''(c) \geq 3$.

7 Assume that $f \in C''$, that $|f''(x)| \leq M$, and that $f(x) \to 0$ as $x \uparrow \infty$. Prove that $f'(x) \to 0$ as $x \uparrow \infty$.

8 Let

$$P(x) = f(c) + f'(c)(x - c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$$

be a Taylor polynomial at c of degree 3. Let $g(x) = f(x) - P(x) - A(x - c)^4/4!$ Suppose that A is chosen so that $g(\bar{x}) = 0$. Prove that there is a point τ between c and $\bar{x}$ with $A = f^{(4)}(\tau)$. (*Hint:* Use Exercise 2, Sec. 3.2.)

9 Show that for all $x \geq 1$, $\log x \leq \sqrt{x} - 1/\sqrt{x}$.

10 Show that for all $x \geq 0$, $e^x \geq \frac{3}{2}x^2$. Can you replace $\frac{3}{2}$ by a larger number?

11 Let f obey the condition $|f^{(n)}(x)| \leq B^n$ for all x in an open interval I, and all n. Show that f is analytic on I.

***12** Let f be of class C'' on $[0, 1]$ with $f(0) = f(1) = 0$, and suppose that $|f''(x)| \leq A$ for all x, $0 < x < 1$. Show that $|f'(\frac{1}{2})| \leq A/4$ and that $|f'(x)| \leq A/2$ for $0 < x \leq 1$.

***13** If $f(0) = 0$ and $|f'(x)| \leq M|f(x)|$ for $0 \leq x \leq L$, show that on that interval $f(x) \equiv 0$.

***14** Say that f is locally a polynomial on $-\infty, \infty$ if, given x_0, there is a neighborhood $\mathcal{N}_{x_0}$ and a polynomial $P(x)$, and on $\mathcal{N}_{x_0}$, $f(x) = P(x)$. Prove that f is a polynomial.

15 Sketch the graph of $y = 1/(1 + x^2)$ and then, for comparison, the graph of the polynomials

$$P_2(x) = 1 - x^2 \quad \text{and} \quad P_4(x) = 1 - x^2 + x^4$$

16 Sketch the graph of the function $f(x) = e^x$, and then the graph of its Taylor polynomials

$$P_2(x) = 1 + x + \tfrac{1}{2}x^2 \quad \text{and} \quad P_3(x) = 1 + x + x^2/2 + x^3/6$$

What is the contrast in behavior between this and the results in the preceding exercise?

17 Use (3-43) to obtain the coordinate form for Taylor's theorem in three variables, with a remainder of total degree 3.

18 Let $f \in C'$ in an open convex set S in $\mathbf{R}^n$. Show that f obeys a Lipschitz condition on any compact subset $E \subset S$.

Assigment 23 p. 361 #11

1 Find the Jacobians of each of the transformations described in Exercise 10, Sec. 7.2.

2 Compute the Jacobians of the following transformations:

(a) $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$ (b) $\begin{cases} u = x^2 \\ v = y/x \end{cases}$

(c) $\begin{cases} u = x^2 + 2xy + y^2 \\ v = 2x + 2y \end{cases}$ *(d) $\begin{cases} u = x + y \\ v = 2xy^2 \end{cases}$

3 Discuss the local behavior of the transformations in Exercise 2.

4 Where it is possible, find formulas for the local inverses of the transformations in Exercise 2.

5 Find the image under each of the transformations of Exercise 2 of the open set $D = [\text{all } (x, y), 0 < x < 1, 0 < y < 1\}$. For which is the image an open set?

6 The second half of the proof of Theorem 15 assumed that T was a transformation from 2-space into 2-space. Carry out the corresponding discussion with $n = 3$.

7 Let T be the transformation sending (x, y) into $(2x + 4y, x - 3y)$. Find T^{-1} and verify directly that the differential of T^{-1} is $(dT)^{-1}$.

8 Let T be a transformation from $\mathbf{R}^3$ into $\mathbf{R}^3$ which is of class C' in an open set D, and let $J(p) = \det (dT|_p)$. Show that J is continuous throughout D. Is the rank of dT a continuous function of p?

9 Let $T(x, y) = (u, v)$. Show that

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$$

10 (a) Find an example of a function f that is infinitely differentiable on an interval I, maps I onto itself 1-to-1, has a continuous inverse f^{-1}, but for which f^{-1} fails to be differentiable at some point of I.

(b) Show by an example that a transformation of the plane can be 1-to-1, have an everywhere-defined inverse, and still have the Jacobian vanishing somewhere.

11 Let T be defined by

$$T: \begin{cases} u = \sin x \cos y + \sin y \cos x \\ v = \cos x \cos y - \sin x \sin y \end{cases}$$

Find the Jacobian of T. Is there anywhere that T is locally 1-to-1?

12 The following transformation is continuous everywhere in the plane and differentiable there except on the lines $y = \pm x$.

$$T: \begin{cases} u = x^2 + y^2 - |x^2 - y^2| \\ v = x^2 + y^2 + |x^2 - y^2| \end{cases}$$

(a) Find dT where it exists.

(b) Discuss the local and global mapping behavior of T.

(c) Is T differentiable at $(0, 0)$ according to definition 3?

13 Let T be the transformation sending (x, y) into $(2xy, x^2 + y^2)$ and S the transformation sending (x, y) into $(x - y, x + y)$.

(a) Using the Jacobians, discuss the local and global mapping behavior of T and S.

(b) Obtain formulas for the two product transformations, ST and TS, and then repeat part (a) for these new transformations.

14 Prove Theorem 13 by filling in the details in the following argument: T^{-1} is continuous if its inverse carries closed sets into closed sets; however, the inverse of T^{-1} is T and any closed subset of D is compact.

15 Let f be a real-valued continuous function defined for $-\infty < x < \infty$. Suppose that f is locally 1-to-1 everywhere. Prove that f is globally 1-to-1. [*Note*: Do not assume that f' exists.]

16 Let $J_T(p)$ denote the Jacobian of a transformation T at the point p. Show that if S and T are transformations from n space into itself, and $T(p) = q$, then $J_{ST}(p) = J_S(q)J_T(p)$.

17 Let D be the unit disk, $x^2 + y^2 \le 1$. Consider a transformation T of class C' on an open set containing D,

$$T: \begin{cases} u = f(x, y) \\ v = g(x, y) \end{cases}$$

whose Jacobian is never 0 in D. Suppose that T is near the identity map in the sense that $|T(p) - p| \le \frac{1}{3}$ for all $p \in D$. Prove that there is a point p_0 with $T(p_0) = (0, 0)$.

Assignment 25 p. 366 # 2, 5, 9, 11

1 Can the curve whose equation is $x^2 + y + \sin(xy) = 0$ be described by an equation of the form $y = f(x)$ in a neighborhood of the point $(0, 0)$? Can it be described by an equation of the form $x = g(y)$?

2 Can the surface whose equation is $xy - z \log y + e^{xz} = 1$ be represented in the form $z = f(x, y)$ in a neighborhood of $(0, 1, 1)$? In the form $y = g(x, z)$?

3 The point $(1, -1, 2)$ lies on both of the surfaces described by the equations $x^2(y^2 + z^2) = 5$ and $(x - z)^2 + y^2 = 2$. Show that in a neighborhood of this point, the curve of intersection of the surfaces can be described by a pair of equations of the form $z = f(x)$, $y = g(x)$.

4 Study the corresponding question for the surfaces with equations $x^2 + y^2 = 4$ and $2x^2 + y^2 - 8z^2 = 8$ and the point $(2, 0, 0)$ which lies on both.

5 The pair of equations

$$\begin{cases} xy + 2yz = 3xz \\ xyz + x - y = 1 \end{cases}$$

is satisfied by the choice $x = y = z = 1$. Study the problem of solving (either in theory or in practice) this pair of equations for two of the unknowns as a function of the third, in the vicinity of the $(1, 1, 1)$ solution.

6 (*a*) Let f be a function of one variable for which $f(1) = 0$. What additional conditions on f will allow the equation

$$2f(xy) = f(x) + f(y)$$

to be solved for y in a neighborhood of $(1, 1)$?

(*b*) Obtain the explicit solution for the choice $f(t) = t^2 - 1$.

***7** With f again a function of one variable obeying $f(1) = 0$, discuss the problem of solving the equation $f(xy) = f(x) + f(y)$ for y near the point $(1, 1)$.

8 Using the method of Theorem 18, state and prove a theorem which gives sufficient conditions for the equations

$$F(x, y, z, t) = 0 \qquad G(x, y, z, t) = 0 \qquad \text{and} \qquad H(x, y, z, t) = 0$$

to be solvable for x, y, and z as functions of t.

9 Apply Theorem 18 to decide if it is possible to solve the equations

$$xy^2 + xzu + yv^2 = 3 \qquad \text{and} \qquad u^3yz + 2xv - u^2v^2 = 2$$

for u and v as functions of (x, y, z) in a neighborhood of the points $(x, y, z) = (1, 1, 1)$, $(u, v) = (1, 1)$.

10 Find the conditions on the function F which allow you to solve the equation

$$F(F(x, y), y) = 0$$

for y as a function of x near $(0, 0)$. Assume $F(0, 0) = 0$.

11 Find conditions on the functions f and g which permit you to solve the equations

$$f(xy) + g(yz) = 0 \qquad \text{and} \qquad g(xy) + f(yz) = 0$$

for y and z as functions of x, near the point where $x = y = z = 1$; assume that $f(1) = g(1) = 0$.