For Assigment 2 (p. 36 #9)

isolated if there is a neighborhood about p which contains no other point of S. Note that any interior point of a set is also a cluster point, since a ball always contains infinitely many points. Every point in the closure of a set is either an isolated member of the set or a cluster point for the set. In the example in Fig. 1-22, (0, 2) is an isolated point, but (0, 0) is a cluster point.

The subject of this book is analysis and not elementary topology; the topological terminology and concepts appear in this chapter because they will be useful tools in the study of functions on $\mathbf{R}^n$ that will follow in subsequent chapters. Our purpose is therefore not to study all the interconnections and implications of the various definitions, but to explain them and state certain useful relations. The following list summarizes a number of these basic properties, to be used whenever they are found helpful.

(1-29)

(i) If A and B are open sets, so are $A \cup B$ and $A \cap B$.
(ii) The union of any collection of open sets is open, but the intersection of an infinite number of open sets need not be open.
(iii) If A and B are closed sets, so are $A \cup B$ and $A \cap B$.
(iv) The intersection of any collection of closed sets is a closed set, but the union of an infinite number of closed sets need not be closed.
(v) A set is open if and only if its complement is closed.
(vi) The interior of a set S is the largest open set that is contained in S.
(vii) The closure of a set S is the smallest closed set that contains S.
(viii) The boundary of a set S is always a closed set and is the intersection of the closure of S and the closure of the complement of S.
(ix) A set S is closed if and only if every cluster point for S belongs to S.
(x) The interior of a set S is obtained by deleting every point in S that is on the boundary of S.

Each of these can be verified by a proof based on the definitions given above. We present this only for the first two assertions in the list, to show the nature of the proofs.

PROOF OF (i) Suppose that A and B are open sets. To show that $A \cup B$ is open, suppose that p_0 belongs to $A \cup B$. Then p_0 is in A or it is in B. In either case, p_0 is the center of an open ball that is a subset of A or a subset of B, since A and B are themselves open and p_0 must be interior to one of them. This open ball is then a subset of $A \cup B$, and p_0 is therefore interior to $A \cup B$. Every point of $A \cup B$ is therefore interior to $A \cup B$, and $A \cup B$ is open. The proof that $A \cap B$ is open is slightly different. Let p_0 be any point in $A \cap B$; we must show that p_0 is interior to $A \cap B$.

For Assignment 9 (p. 80 #1)

2.2 BASIC DEFINITIONS

Let f be a numerical-valued function, defined on a region D in the plane. Suppose that we interpret $f(p)$ to be the temperature at the point p. Then, the intuitive notion of continuity can be described by saying that the temperature on a small neighborhood of any point p_0 in D will vary only slightly from that at p_0; moreover, we feel that these variations can be made as small as we like by decreasing the size of the neighborhood. This behavior can be shown on a graph of f. In Fig. 2-1, we have shown the range of variation in the values of a function of one variable when x is confined to a neighborhood of a point x_0; note also that the size of neighborhood needed to attain the same limitation of variation may be smaller at another point.

Formally, we are led to the following.

Definition 1 *A numerical-valued function f, defined on a set D, is said to be* **continuous** *at a point $p_0 \in D$ if, given any number $\varepsilon > 0$, there is a neighborhood U about p_0 such that $|f(p) - f(p_0)| < \varepsilon$ for every point $p \in U \cap D$. The function f is said to be continuous on D if it is continuous at each point of D.*

The work of checking from this definition that a specific function is continuous can be easy or difficult, depending upon how the function has been described and how simple it is. As a start, let us show that the function F given by $F(x, y) = x^2 + 3y$ is continuous on the unit square S consisting

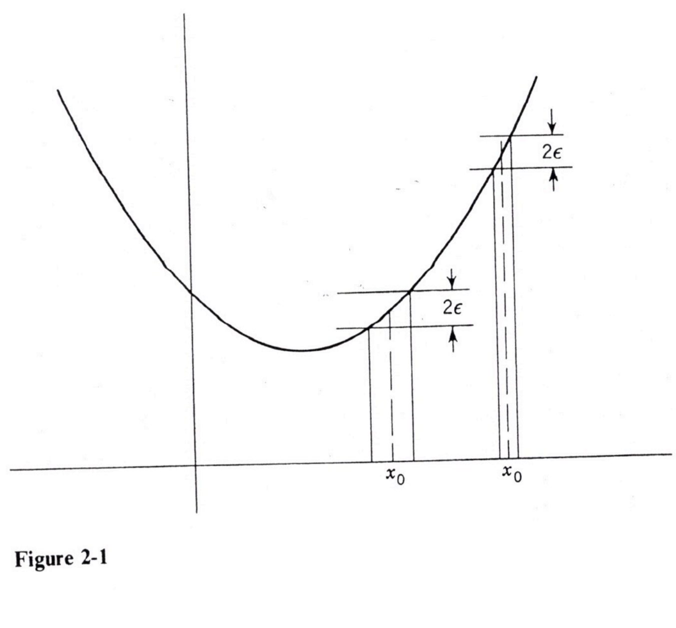

Figure 2-1

For Assignment 9 (p. 80 #3)

It is also easy to see that "convergence preserving" is a characteristic property of continuous functions.

Theorem 2 *If a function f defined on D has the property that, whenever $p_n \in D$ and $\lim p_n = p_0 \in D$, then it follows that $\lim_{n\to\infty} f(p_n) = f(p_0)$, then f is continuous at p_0.*

It is easiest to prove this by an indirect argument. Assume that f is convergence preserving but that f is not continuous at p_0. If the definition of continuity (Definition 1) is read carefully, one sees that, in order for f *not* to be continuous at p_0, there must exist a particular value of $\varepsilon > 0$ such that no neighborhood U can be found to satisfy the required condition. (There is a routine for carrying out the logical maneuver of constructing the denial of a mathematical statement in a semimechanical fashion; those who have difficulty reasoning verbally will find it explained in Appendix 1.) Thus if f is not continuous at p_0, and we think of trying a specific neighborhood U, then it must fail because there is a point p of D in U with $|f(p) - f(p_0)| \geq \varepsilon$. If we try U in turn to be a spherical neighborhood of radius $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and let U_n therefore be

$$U_n = \left\{\text{all } p \text{ with } |p - p_0| < \frac{1}{n}\right\}$$

then there must be a point $p_n \in U_n \cap D$ with $|f(p_n) - f(p)| > \varepsilon$. Since $p_n \in U_n$, $|p_n - p_0| < 1/n$ and $\lim p_n = p_0$. We have therefore produced a sequence $\{p_n\}$ in D that converges to p_0, but such that $f(p_n)$ does not converge to $f(p_0)$. This contradicts the assumed convergence-preserving nature of f and forces us to conclude that f was continuous at p_0. ▌

Theorem 2 is more useful as a way to show that a specific function is *not* continuous than it is as a way to show that a function *is* continuous. To use it for the latter purpose, one would have to prove something about $\{f(p_n)\}$ for *every* sequence $\{p_n\}$ converging to p_0, and there are infinitely many such sequences. However, if there is *one* sequence $\{p_n\}$ in D which converges to p_0, but for which $\{f(p_n)\}$ is divergent, then we know at once that f is not continuous at p_0. For example, let f be defined on the plane by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & (x, y) \neq (0, 0) \\ 0 & x = y = 0 \end{cases} \tag{2-2}$$

We want to see if f is continuous at $(0, 0)$. Among the sequences that approach the origin, look at those of the form $p_n = (1/n, c/n)$. As c takes on different values, the sequence p_n will approach $(0, 0)$ along different lines, taking

For Assignment 9 (p. 80 #17)

into $\mathbf{R}^2$,

erefore
tion F

This describes a mapping F from a square in the UV plane into 3-space. As the point $p = (u, v)$ moves throughout the square, the image point $F(p) = (x, y, z)$ moves in space, tracing out the shape shown in Fig. 1-16. Thus, $F(1, 0) = (1, 2, 1)$ and $F(0, 1) = (1, 0, 0)$. Other similar pictures will be found in Chap. 8, especially Fig. 8-17, 8-18, and 8-20, which show more complicated examples. The study of curves and surfaces is one of the more difficult areas of analysis, and some aspects of this are treated there as an application of the tools to be developed.

We will also study functions that map a portion of $\mathbf{R}^n$ into $\mathbf{R}^n$. An illustration is the function F from 3-space into 3-space described by the formula $F(x, y, z) = (u, v, w)$ where

$$\begin{cases} u = x - y \\ v = y^2 + 2z \\ w = yz + 3x^2 \end{cases}$$

For example, we have $F(1, 2, 1) = (-1, 6, 5)$ and $F(1, -1, 3) = (2, 7, 0)$.

All these cases can be subsumed under one general formula. A mapping F from $\mathbf{R}^n$ into $\mathbf{R}^m$ has the form $y = F(x)$, where we write $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$ and where

$$\begin{cases} y_1 = f(x_1, x_2, \ldots, x_n) \\ y_2 = g(x_1, x_2, \ldots, x_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_m = k(x_1, x_2, \ldots, x_n) \end{cases} \tag{1-26}$$

Here, $f, g, h, \ldots, k$ are m specific real-valued functions of the n real variables $x_1, x_2, \ldots, x_n$. Such functions as F are often called **transformations** to emphasize their nature; the study of their properties is one of the central topics of later chapters in this book.

Side by side with the view of a function as a mapping $A \to B$, there is also the equally important and useful idea of its **graph.** If f is a function of one variable, with domain $D \subset \mathbf{R}^1$, then the graph of f is the set of all points (x, y) in the plane, with $x \in D$ and $y = f(x)$. The graph of a function of two variables is the set of points (x, y, z), with (x, y) in the domain of the function f and with $z = f(x, y)$. Generalizing this, if f is a function on A into B,

$$f: A \to B$$

then the graph of f is the set of all ordered pairs (a, b), with $a \in A$ and $b = f(a)$. It is customary to use the term **cartesian product** of A and B, written as $A \times B$, to denote the class of all possible ordered (a, b), with $a \in A$ and $b \in B$. The graph of f is therefore a special subset of $A \times B$. By analogy, (a, b) is often called a point in $A \times B$, and the graph of f can be visualized as something like a curve in the space $A \times B$ (see Fig. 1-17).

Let us apply this to a numerical-valued function f of three real variables. Suppose that the domain of f is a set $D \subset \mathbf{R}^3$. Since f is numerical-valued, its

For Assignment 16

A linear transformation is continuous everywhere, since this is clearly true for a linear function. More than this is true; any linear transformation from $\mathbf{R}^n$ into $\mathbf{R}^m$ is everywhere uniformly continuous. This is equivalent to the next theorem.

Theorem 8 *Let L be a linear transformation from $\mathbf{R}^n$ into $\mathbf{R}^m$ represented by the matrix $[a_{ij}]$. Then, there is a constant B such that $|L(p)| \le B|p|$ for all points p.*

We shall find that the number $\left\{\sum\sum |a_{ij}|^2\right\}^{1/2}$ will serve for B. Put $p = (x_1, x_2, \ldots, x_n)$ and $q = L(p) = (y_1, y_2, \ldots, y_m)$, so that

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \qquad i = 1, 2, \ldots, m$$

We have $|p|^2 = \sum_{j=1}^{n} |x_j|^2$ and $|q|^2 = \sum_{i=1}^{m} |y_i|^2$. Accordingly,

$$|y_i|^2 \le \left\{\sum_{j=1}^{n} |a_{ij}|\,|x_j|\right\}^2 \le \sum_{j=1}^{n} |a_{ij}|^2 \sum_{j=1}^{n} |x_j|^2 = |p|^2 \sum_{j=1}^{n} |a_{ij}|^2$$

where we have used the Schwarz inequality (Sec. 1.3):

$$\left(\sum a_k b_k\right)^2 \le \sum |a_k|^2 \sum |b_k|^2$$

Adding these for $i = 1, 2, \ldots, m$, we obtain

$$|q|^2 = \sum_{i=1}^{m} |y_i|^2 \le |p|^2 \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2$$

and $|L(p)| = |q| \le B|p|$ where $B = \left\{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2\right\}^{1/2}$. ∎

It should be remarked that the number B which we have found is not the smallest number with this property. For example, the transformation L specified by the identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is such that $|L(p)| = |p|$, while the theorem provides the number

$$B = \sqrt{2} > 1$$

However, this is not the case for linear *functions*. Let L be specified by the row matrix $[c_1, c_2, \ldots, c_n]$. Then, according to the theorem,

$$|L(p)| \le \left(\sum_{i}^{n} |c_j|^2\right)^{1/2} |p|$$

For Assignment 17

or the **directional derivative** of f at p_0 in the direction β, is defined to be

$$(\mathbf{D}_\beta f)(p_0) = \lim_{t\to 0} \frac{f(p_0 + t\beta) - f(p_0)}{t} \tag{3-6}$$

As an illustration, let $f(x, y) = x^2 + 3xy$, $p_0 = (2, 0)$, and $\beta = (1/\sqrt{2}, -1/\sqrt{2})$. (Note that this specifies the direction $-45°$.)

Since $p_0 + t\beta = (2 + t/\sqrt{2}, -t/\sqrt{2})$, we have

$$\begin{aligned} f(p_0 + t\beta) &= (2 + t/\sqrt{2})^2 + 3(2 + t/\sqrt{2})(-t/\sqrt{2}) \\ &= 4 - \frac{2}{\sqrt{2}}t - t^2 \end{aligned}$$

Accordingly,

$$\begin{aligned} (\mathbf{D}_\beta f)(p_0) &= \lim_{t\to 0} \frac{\left(4 - \frac{2}{\sqrt{2}}t - t^2\right) - 4}{t} \\ &= -\frac{2}{\sqrt{2}} \end{aligned}$$

If we hold p_0 the same and vary β, the value of $(\mathbf{D}_\beta f)(p_0)$ need not remain the same. Intuitively, it is clear that reversal of the direction β ought to reverse the sign of the directional derivative. Indeed,

$$(\mathbf{D}_{-\beta} f)(p_0) = \lim_{t\to 0} \frac{f(p_0 - t\beta) - f(p_0)}{t}$$

If we put $\lambda = -t$, we have

$$\frac{f(p_0 - t\beta) - f(p_0)}{t} = -\frac{f(p_0 + \lambda\beta) - f(p_0)}{\lambda}$$

so that $(\mathbf{D}_{-\beta} f)(p_0) = -(\mathbf{D}_\beta f)(p_0)$, as conjectured.

The **partial derivatives** of a function f of n variables are the directional derivatives that are obtained by specializing β to be each of the **basic unit vectors** $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 0, 1)$ in turn. There are a variety of notations in use; depending upon the circumstances, one may be more convenient than another, and the table below gives most of the more common ones. Since the case of three variables is typical, we treat this alone.

A preliminary word of caution and explanation is needed. It is customary to use certain notations in mathematics, even when this can lead to confusion or misunderstandings. In particular, this is true of partial derivatives; the

For Assignment 22

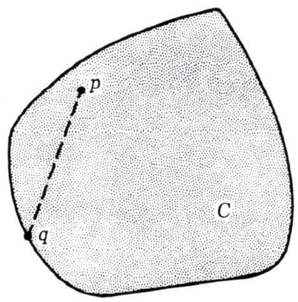

Figure 1-12 Convexity.

In a similar manner, it is possible to show that if $\lambda > 1$, the point p lies on the portion of the line L that is beyond q_2, and if $\lambda < 0$, on the portion beyond q_1. Finally, we note that formula (1-21) is in fact the same as the parametric equation (1-20), substituting q_1 for p_0, $q_2 - q_1$ for v, and λ for t.

Another important geometric concept which is conveniently described in terms of the ideas of the present section and which is suitable for n space is that of convexity. In the plane, a region C is said to be **convex** if it always contains the line segment joining any two points in the region (see Fig. 1-12). This definition is used in space as well and carries over at once to $\mathbf{R}^n$.

Definition 1 *A set C in n space is convex if it has the property that, whenever two points p and q are in C, then so are all points of the form*

$$\lambda p + (1 - \lambda)q \qquad 0 < \lambda < 1 \tag{1-22}$$

An important example of a convex set in n space is the solid spherical **ball,** which we define as follows:

$$\begin{aligned} B(p_0, r) &= \{\text{all } p \text{ with } |p - p_0| < r\} \\ &= \text{the open ball, center } p_0\text{, radius } r \end{aligned} \tag{1-23}$$

In 3-space, this is the interior of an ordinary sphere; in the plane, it is a round disc without the edge; in 1-space, $B(x_0, r)$ is the real interval consisting of the numbers x that obey $x_0 - r < x < x_0 + r$.

Let us show that the ball $B(\mathbf{0}, r)$ is convex. Suppose that p and q lie in B, so that $|p| < r$ and $|q| < r$. Choose any λ, $0 < \lambda < 1$; we must show that the point $\lambda p + (1 - \lambda)q$ lies in B. We calculate its distance from $\mathbf{0}$. Using the triangle inequality, we have

$$\begin{aligned} |\lambda p + (1 - \lambda)q - \mathbf{0}| &\le |\lambda p| + |(1 - \lambda)q| \\ &\le \lambda|p| + (1 - \lambda)|q| \\ &< \lambda r + (1 - \lambda)r = r \end{aligned}$$

EXERCI

1 For $n =$
(a) $|p|$
2 Let $A =$
(a) $|p|$
3 Sketch t
(a) $|x$
(b) $(x$
4 Show th
5 Prove tl
6 If $p = ($
(a) $|p$
(b) $|u$
7 Use th
that $p \cdot q =$
8 Show
triangle wi
9 Find
(0, 1, −2,
10 If the
hyperplan
11 Write
hyperplan
12 Wher
of Exercis
13 Give
each of th
14 Forn
*15 In th
the way
vertices c
16 Let
when $\lambda >$
17 Sho
not have

1.4 F

The n
valued
$f(p)$ c
formu

For Assignment 22

each subinterval $[x_k, x_{k+1}]$, it is clear that we have proved this uniformly throughout the entire interval $[a, b]$, and $\|F - f\| < \varepsilon$. ∎

It is interesting to note that the process given in (2-6) for obtaining the values of the approximating function F at points x intermediate between x_k and x_{k+1} is nothing more nor less then ordinary linear interpolation, as it is standardly done in mathematical tables.

For functions of two or more variables, a similar process can be used, obtaining approximate values for a function f by interpolating from the values at a discrete set of points. In two variables, for example, one can use linear interpolation in triangles to replace the formula given in (2-6) (see Exercise 10).

In order to apply these methods to construct uniform approximations to a given function f, one must know a value of δ that is appropriate for a given ε. There is one case in which this step is simple.

Definition 4 *A function f is said to obey a* **Lipschitz** *condition on the set D if there is a constant M such that*

$$|f(p) - f(q)| \le M|p - q|$$

for every choice of p and q in D.

When this happens, it is clear that f is uniformly continuous on D and that we may choose $\delta = \varepsilon/M$. For, if $|p - q| < \delta$, then $|f(p) - f(q)| \le M\delta \le \varepsilon$. In Sec. 3.2 we will see that any function of one variable that has a continuous derivative on an interval $[a, b]$ obeys a Lipschitz condition on that interval. An analogous result will also be proved later for functions of several variables.

EXERCISES

1 Show that $F(x, y) = x^2 + 3y$ is not uniformly continuous on the whole plane.

2 Prove that the function $f(x) = 1/(1 + x^2)$ is uniformly continuous on the whole line.

3 Let f and g each be uniformly continuous on a set E. Show that $f + g$ is uniformly continuous on E.

4 Let A and B be disjoint sets, and let f be continuous on A and continuous on B. When is it continuous on $A \cup B$?

5 Let A and B be disjoint closed sets and suppose that f is uniformly continuous on each.

(*a*) Show that f is necessarily uniformly continuous on $A \cup B$ if A is compact.

(*b*) Show that f need not be uniformly continuous on $A \cup B$ if neither A nor B is compact.

6 If f is uniformly continuous on D, show that it has the property that if $p_n, q_n \in D$ and $|p_n - q_n| \to 0$, then $|f(p_n) - f(q_n)| \to 0$.

7 Let D be a bounded set and let f be uniformly continuous on $D \subset \mathbf{R}^n$. Prove that f is bounded on D.

8 Let f be a function defined on a set E which is such that it can be uniformly approximated within ε on E by functions F that are uniformly continuous on E, for every $\varepsilon > 0$. Show that f must itself be uniformly continuous on E.