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Split 3-Leibniz algebras

Antonio J. Calderón Martín [a,*], Juana Sánchez-Ortega [b]

[a] Department of Mathematics, University of Cádiz, Spain

[b] Department of Mathematics and Applied Mathematics, University of Cape Town, South Africa

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ABSTRACT

We introduce and describe the class of split 3-Leibniz algebras as the natural extension of the class of split Lie algebras, split Leibniz algebras, split Lie triple systems and split 3-Lie algebras. More precisely, we show that any of such split 3-Leibniz algebras T is of the form $T = \mathcal{U} + \sum_j I_j$, with $\mathcal{U}$ a subspace of the 0-root space T_0, and I_j an ideal of T satisfying

$$[T, I_j, I_k] + [I_j, T, I_k] + [I_j, I_k, T] = 0$$

for $j \neq k$. Moreover, if T is of maximal length, we characterize the simplicity of T in terms of a connectivity property in its set of non-zero roots.

1. Introduction

Leibniz algebras, a "non-commutative" version of Lie algebras, were originally introduced in the mid-1960s by Bloh [1,2] under the name "D-algebras". These D-algebras became much popular in the 1990s after Loday's work [3], where the terminology of Leibniz algebras was introduced. Since then a significant number of researchers have been attracted by this kind of algebras, and as a result their theory has rapidly developed.

We recall that a Leibniz algebra is just an algebra in which the left multiplication operators act as derivations. That is:

Definition 1.1. A **(left) Leibniz algebra** L is a vector space over a base field $\mathbb{K}$ endowed with a bilinear product $[\cdot, \cdot]$ satisfying the *(left) Leibniz identity:*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]],$$

for all $x, y, z \in L$.

In presence of anti-commutativity, the Jacobi identity becomes the Leibniz identity, and so Lie algebras are examples of Leibniz algebras.

Leibniz n-algebras were introduced in [4], as a non-antisymmetric version of Nambu algebras (also known as n-Lie algebras or Filippov algebras). Leibniz n-algebras have been shown of utility in Mathematical Physics, in the study of the

* Corresponding author.
E-mail addresses: ajesus.calderon@uca.es (A.J. Calderón Martín), juana.sanchez-ortega@uct.ac.za (J. Sánchez-Ortega).

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supersymmetry Bagger–Lambert theory or Nambu mechanics. It has been the motivation to many authors to study them in the last few years.

In fact, n-ary algebras have been considered in physics in the context of Nambu mechanics and in the search for the effective action of coincident M2-branes in M-theory initiated by the Bagger–Lambert–Gustavsson model [5–7]. An interesting and complete review of this matter can be found in [8].

Here we focus our attention on 3-Leibniz algebras, that is, $n = 3$. 3-Leibniz algebras generalize both Leibniz and 3-Lie algebras.

In the present paper we study the structure of split 3-Leibniz algebras of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. Split structures appeared first in the classical theory of (finite dimensional) Lie algebras but have been extended to more general settings like, for example, Leibniz algebras, Poisson algebras or Lie and Leibniz superalgebras among many others. In the framework of ternary structures see [9–11] for Lie triple systems, twisted inner derivation triple systems and 3-Lie algebras, and [12,13] for Leibniz triple systems.

At this point it is worth paying some attention to the difference between 3-Leibniz algebras and Leibniz triple systems. While 3-Leibniz algebras are naturally defined as a vector space with a trilinear operation $[\cdot,\cdot,\cdot]$ in which the left multiplication operators $[x, y, \cdot]$ are derivations (see Definitions 2.1), Leibniz triple systems are vector spaces with a trilinear operation which satisfies two different identities. (See [12–15] for further details.) Leibniz triple systems were introduced in a functorial way using the Kolesnikov–Pozhidaev algorithm for dialgebras.

The paper is organized as follows: in Section 2 we recall the main definitions and results related to the split 3-Leibniz algebras theory. In Section 3 we develop connections of root techniques in this framework, which become the main tool in our study of split 3- Leibniz algebras. In Section 4 we apply all of the machinery introduced in the previous section to describe 3-Leibniz algebras. We will show that an arbitrary 3-Leibniz algebra T is of the form $T = \mathcal{U} + \sum_j I_j$, with $\mathcal{U}$ a subspace of the 0-root space T_0 and I_j an ideal of T, satisfying

$$[T, I_j, I_k] + [I_j, T, I_k] + [I_j, I_k, T] = 0, \text{ for } j \neq k.$$

Finally, in Section 5 we focus our attention on split 3-Leibniz algebras of maximal length. We will characterize the simplicity of this family of 3-Leibniz algebras in terms of connectivity properties among the non-zero roots. Our results extend the ones for split Leibniz algebras [16], and for split 3-Lie algebras [11]. They provide a common structure theory for split 3-Leibniz algebras and for split Leibniz triple systems, as established in [12,15].

2. Preliminaries

2.1. 3-Leibniz algebras, subalgebras and ideals

Definitions 2.1. A **3-Leibniz algebra** is a vector space T, over an arbitrary base field $\mathbb{K}$, with a trilinear operation $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, called the **triple product** of T, satisfying the following identity:

$$[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]], \tag{L}$$

for any $x, y, a, b, c \in T$.

Let T be a 3-Leibniz algebra. A linear subspace of T closed under the triple product is called a **3-Leibniz subalgebra** of T. An **ideal** of T is a linear subspace I which satisfies $[I, T, T] + [T, I, T] + [T, T, I] \subseteq I$. The **annihilator** of T is the set

$$\mathrm{Ann}(T) = \{x \in T : [x, T, T] + [T, x, T] + [T, T, x] = 0\}.$$

It is straightforward to check that $\mathrm{Ann}(T)$ is an ideal of T.

The ideal J generated by the set

$$\big\{[x, x, y], [x, y, x], [y, x, x] : x, y \in T\big\}$$

plays an important role when dealing with 3-Leibniz algebras, since it determines whether a 3-Leibniz algebra is a 3-Lie algebra. Clearly, T is a 3-Lie algebra if and only if $J = \{0\}$.

Note that the usual definition of simplicity lacks of interest in our case. It would imply that $J = T$ or $J = \{0\}$, which yields that T is an abelian or a 3-Lie algebra, respectively. Following the ideas of Abdykassymova and Dzhumaldil'daev [17] for Leibniz algebras, and of Cao and Chen [12,15] for Leibniz triple systems, we introduce the following notion.

Definition 2.2. We will say that T is a **simple 3-Leibniz algebra** if its triple product is nonzero and its only ideals are $\{0\}, J$ and T.

Note that this definition is consistent with the notion of a simple 3-Lie algebra since $J = \{0\}$ in that case.

2.2. Ternary derivations

Let T be a 3-Leibniz algebra and $D : T \to T$ a linear map. We call D a **ternary derivation** if it satisfies

$$D\big([a, b, c]\big) = [D(a), b, c] + [a, D(b), c] + [a, b, D(c)].$$

Using this terminology, the identity (L) says that the left multiplication operators $\mathrm{ad}(x, y) : T \to T$ given by $\mathrm{ad}(x, y)(z) := [x, y, z]$ are ternary derivations.

The identity (L) applies to get that $\mathfrak{L} := span_{\mathbb{K}}\{\mathrm{ad}(x, y) : x, y \in T\}$ with the bracket

$$\big[\mathrm{ad}(a, b),\ \mathrm{ad}(c, d)\big] = \mathrm{ad}\big([a, b, c], d\big) + \mathrm{ad}\big(c, [a, b, d]\big). \tag{1}$$

is a Lie algebra. Note that since every element of $\mathfrak{L}$ is of the form $\sum \mathrm{ad}(x_i, y_i)$, we have

$$[\ell, \mathrm{ad}(a, b)] = \mathrm{ad}(\ell a, b) + \mathrm{ad}(a, \ell b) \tag{2}$$

for every $a, b \in T$ and $\ell \in \mathfrak{L}$.

2.3. Split structures

An algebra A over $\mathbb{K}$ is said to be **2-graded** if there exist two linear subspaces A^0 and A^1 of A, called the **even** and the **odd part** respectively, such that $A = A^0 \oplus A^1$ and $A^\alpha A^\beta \subset A^{\alpha+\beta}$ for every $\alpha,\ \beta \in \mathbb{Z}_2$.

Definition 2.3. The **standard embedding** of a 3-Leibniz algebra T is the 2-graded algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, where $\mathcal{A}^0 := \mathfrak{L}$, $\mathcal{A}^1 := T$, and product given by

$$(\mathrm{ad}(x, y), z) \cdot (\mathrm{ad}(u, v), w) :== (\mathrm{ad}([x, y, u], v) + \mathrm{ad}(u, [x, y, v]) + \mathrm{ad}(z, w), [x, y, w] - [u, v, z]).$$

Although $\mathcal{A}^0$ is a Lie algebra, $\mathcal{A}$ is not, in general, a (2-graded) Lie algebra.

Next, we introduce the class of split algebras in the framework of 3-Leibniz algebras. Observe that the product in $\mathcal{A}$ gives us a natural action:

$$\mathcal{A}^0 \times \mathcal{A}^1 \to \mathcal{A}^1$$
$$(x, y) \mapsto xy.$$

Since $\mathcal{A}^0$ is a Lie algebra, the identity (L) allows us to conclude that the action above endows $\mathcal{A}^1$ with an $\mathcal{A}^0$-module structure. Thus, we can introduce the concept of split 3-Leibniz algebra in a similar spirit to the ones of split Lie triple system and split 3-Lie algebra [9,11].

We begin by recalling the notion of a split Lie algebra. Let $(L,\ [\cdot, \cdot])$ be a Lie algebra and $x \in L$. We write $\mathrm{ad}(x)$ to denote the **adjoint operator** $\mathrm{ad} : L \to L$, $\mathrm{ad}(x)(y) = [x, y]$, $y \in L$.

A **splitting Cartan subalgebra** H of L is a maximal abelian subalgebra (MASA for short) of L, which satisfies that the adjoint operators $\mathrm{ad}(h)$ for $h \in H$ are simultaneously diagonalizable. We say that L is a **split Lie algebra** if it contains a splitting Cartan subalgebra H of L. (See [16,18] for further details.) It means that we have a **root spaces decomposition** $L = H \oplus \big(\bigoplus_{\alpha\in\Lambda} L_\alpha\big)$, where

$$L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)v_\alpha \text{ for any } h \in H\}$$

for a linear functional $\alpha \in H^*$, (where H^* is the dual space of H), and $\Lambda := \{\alpha \in H^* \setminus \{0\} : L_\alpha \neq 0\}$. The subspaces L_α ($\alpha \in H^*$) are called **root spaces** of L (respect to H), and the elements $\alpha \in \Lambda \cup \{0\}$ are called **roots** of L respect to H. Clearly, $L_0 = H$.

In the past few years the first author jointly with M. Forero have extended the notion of a split Lie algebra to many ternary algebraic structures like, for example, Lie triple systems [9], twisted inner derivation triple systems [10] and 3-Lie algebras [11]. Also Y. Cao and L. Chen have carried out this extension in the case of Leibniz triple systems, see [12,13]. The present paper deals with the class of 3-Leibniz algebras.

Definition 2.4. Let T be a 3-Leibniz algebra, $\mathcal{A} = \mathfrak{L} \oplus T$ its standard embedding and H a MASA of $\mathfrak{L}$. The **root space of** T (respect to H) associated to a linear functional $\alpha \in H^*$ is the subspace $T_\alpha := \{v_\alpha \in T : hv_\alpha = \alpha(h)v_\alpha \text{ for any } h \in H\}$. The elements $\alpha \in H^*$ satisfying $T_\alpha \neq 0$ are called **roots** of T (respect to H), and we write $\Lambda^T := \{\alpha \in H^* \setminus \{0\} : T_\alpha \neq 0\}$. Note that

$$T_0 = \{v_0 \in T : hv_0 = 0 \text{ for any } h \in H\}.$$

In the sequel, we denote by $\Lambda^{\mathfrak{L}}$ the set of all nonzero $\alpha \in H^*$ such that $\mathfrak{L}_\alpha \neq 0$, where $\mathfrak{L}_\alpha := \{e_\alpha \in \mathfrak{L} : [h, e_\alpha] = \alpha(h)e_\alpha \text{ for any } h \in H\}$.

The following result collects some basic properties of the subspaces T_α and $\mathfrak{L}_\alpha$. The proof is based on the Jacobi identity (which is satisfied by the Lie algebra $\mathfrak{L}$) and on the identity (L). We omit the details here since it is similar to the proof of [11, Lemma 2.1].

Lemma 2.5. *Let T be a 3-Leibniz algebra, $\mathcal{A} = \mathfrak{L} \oplus T$ its standard embedding and H a MASA of $\mathfrak{L}$. If $\alpha, \beta, \gamma \in \Lambda^T \cup \{0\}$ and $\delta, \epsilon \in \Lambda^{\mathfrak{L}} \cup \{0\}$, the following assertions hold.*

(i) *If* $\mathrm{ad}(T_\alpha, T_\beta) \neq 0$ *then* $\alpha + \beta \in \Lambda^{\mathfrak{L}} \cup \{0\}$, *and* $\mathrm{ad}(T_\alpha, T_\beta) \subset \mathfrak{L}_{\alpha+\beta}$.
(ii) *If* $\mathfrak{L}_\delta T_\alpha \neq 0$ *then* $\delta + \alpha \in \Lambda^T \cup \{0\}$ *and* $\mathfrak{L}_\delta T_\alpha \subset T_{\delta+\alpha}$.
(iii) *If* $T_\alpha \mathfrak{L}_\delta \neq 0$ *then* $\alpha + \delta \in \Lambda^T \cup \{0\}$ *and* $T_\alpha \mathfrak{L}_\delta \subset T_{\alpha+\delta}$.
(iv) *If* $[\mathfrak{L}_\delta, \mathfrak{L}_\epsilon] \neq 0$ *then* $\delta + \epsilon \in \Lambda^{\mathfrak{L}} \cup \{0\}$ *and* $[\mathfrak{L}_\delta, \mathfrak{L}_\epsilon] \subset \mathfrak{L}_{\delta+\epsilon}$.
(v) *If* $[T_\alpha, T_\beta, T_\gamma] \neq 0$ *then* $\alpha + \beta + \gamma \in \Lambda^T \cup \{0\}$ *and* $[T_\alpha, T_\beta, T_\gamma] \subset T_{\alpha+\beta+\gamma}$.

Definition 2.6. A 3-Leibniz algebra T is said to be a **split 3-Leibniz algebra** if

$$T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha \right). \tag{3}$$

The set Λ^T is called the **root system** of T. We refer to (3) as the **root spaces decomposition** of T.

Remark 2.7. Let $T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha \right)$ be a split 3-Leibniz algebra. Applying Lemma 2.5(i), taking into account that $H \subset \mathfrak{L} = \mathrm{ad}(T, T)$, we have

$$H = \mathrm{ad}(T_0, T_0) + \sum_{\alpha \in \Lambda^T} \mathrm{ad}(T_\alpha, T_{-\alpha}). \tag{4}$$

On the other hand, from $\mathrm{ad}(T_0, T_0) \subset \mathfrak{L}_0 \subset H$ we get

$$[T_0, T_0, T_0] = 0. \tag{5}$$

3. Connections of roots

In order to study the character split of an algebraic structure, the main tool is the so-called connections of roots. This section is devoted to the development of connections of roots for 3-Leibniz algebras. In what follows, T denotes a split 3-Leibniz algebra and $T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha \right)$ is its corresponding root spaces decomposition.

Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ $(h \in H)$. Let $-\Upsilon = \{-\alpha : \alpha \in \Upsilon\}$ where $\emptyset \neq \Upsilon \subset H^*$, and

$$\Omega = \{\alpha \in \pm\Lambda^T : \mathrm{ad}(T_\alpha, T_{-\alpha}) + \mathrm{ad}(T_{-\alpha}, T_\alpha) \neq 0\} \cup \{\alpha \in \pm\Lambda^T : \alpha(\mathrm{ad}(T_\beta, T_{-\beta})) \neq 0 \text{ for some } \beta \in \Lambda^T\}.$$

Note that

$$\alpha \in \Omega \Rightarrow -\alpha \in \Omega. \tag{6}$$

For each $\alpha \in \Omega$, we introduce a new variable θ_α, and consider $\Theta_\Omega = \{\theta_\alpha : \alpha \in \Omega\}$, the set consisting of all these new symbols. Next, by denoting as above by H^* the dual space of H, we consider the operation

$$\dotplus : (\pm\Lambda^T \cup \pm\Lambda^{\mathfrak{L}} \cup \Theta_\Omega) \times (\pm\Lambda^T \cup \{0\}) \to H^* \cup \Theta_\Omega,$$

defined as follows:

- For $\alpha \in \pm\Lambda^T$,

  $$\alpha \dotplus (-\alpha) = \begin{cases} \theta_\alpha, & \text{if } \alpha \in \Omega, \\ 0, & \text{if } \alpha \notin \Omega, \end{cases}$$

  where $0 : H \to \mathbb{K}$ denotes the zero linear functional.
- For $\alpha \in \pm\Lambda^T$ and $\beta \in \pm\Lambda^T \cup \{0\}$ with $\beta \neq -\alpha$, the new operation $\alpha \dotplus \beta \in H^*$ coincides with the usual sum of linear functionals, that is,

  $$(\alpha \dotplus \beta)(h) = (\alpha + \beta)(h) = \alpha(h) + \beta(h),$$

  for any $h \in H$.
- For $\alpha \in \pm\Lambda^{\mathfrak{L}} \setminus \pm\Lambda^T$ and $\beta \in \pm\Lambda^T \cup \{0\}$, we also have that $\alpha \dotplus \beta \in H^*$ equals to the usual sum of linear functionals.
- For $\theta_\alpha \in \Theta_\Omega$ and $\beta \in \pm\Lambda^T$,

  $$\theta_\alpha \dotplus \beta = \begin{cases} \beta, & \text{if either } \alpha(\mathrm{ad}(T_\beta, T_{-\beta}) + \mathrm{ad}(T_{-\beta}, T_\beta)) \neq 0 \text{ or } \beta(\mathrm{ad}(T_\alpha, T_{-\alpha}) + \mathrm{ad}(T_{-\alpha}, T_\alpha)) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

  where 0 denotes, as above, the zero linear functional.
- For $\theta_\alpha \in \Theta_\Omega$ and $0 : H \to \mathbb{K}$, it is defined $\theta_\alpha \dotplus 0 = 0$.

Note that given $\alpha \in \Omega$, Eq. (6) applies to get

$$\alpha \dotplus (-\alpha) = \theta_\alpha \Rightarrow -\alpha \dotplus \alpha = \theta_{-\alpha}. \tag{7}$$

Remark 3.1. In the rest of the paper, we will consider two different operations involving elements in $\pm\Lambda^T \cup \pm\Lambda^{\mathfrak{L}} \cup \{0\}$, the usual sum of linear functionals $+$, and the new operation $\dotplus$. However, observe that $\alpha + \beta$ coincides with $\alpha \dotplus \beta$ for any $\alpha \in \pm\Lambda^T \cup \pm\Lambda^{\mathfrak{L}}$ and $\beta \in \pm\Lambda^T \cup \{0\}$, except for the case $\beta = -\alpha$, where we have that $\alpha + \beta = 0$ if and only if $\alpha \dotplus \beta \in \{0, \theta_\alpha\}$.

The following results are direct consequences of the definition of $\dotplus$. See also [11, Lemma 3.1] and [11, Lemmas 3.2 and 3.3].

Lemma 3.2. *For any $\alpha \in \Omega$ and $\beta \in \pm\Lambda^T$ such that $\theta_\alpha \dotplus \beta = \beta$ the following assertions hold.*

(i) $\beta \in \Omega$.
(ii) $\beta \dotplus (-\beta) = \theta_\beta$.
(iii) $\theta_\beta \dotplus \alpha = \alpha$.
(iv) $\theta_{-\alpha} \dotplus \beta = \beta$.
(v) $\theta_{-\alpha} \dotplus (-\beta) = -\beta$.

Lemma 3.3. *The following assertions hold.*

(i) *Let $\alpha, \delta \in \pm\Lambda^T \cup \pm\Lambda^{\mathfrak{L}}$ and $\beta \in \pm\Lambda^T \cup \{0\}$. If $\alpha \dotplus \beta = \delta$ then $\delta \dotplus (-\beta) = \alpha$ and $-\alpha \dotplus (-\beta) = -\delta$.*
(ii) *Let $\alpha, \beta, \gamma, \delta \in \pm\Lambda^T$. If $(\alpha \dotplus \beta) \dotplus \gamma = \delta$ with $\alpha \dotplus \beta \in \Theta_\Omega$, then $\beta = -\alpha$, $\delta = \gamma$, $\delta \dotplus (-\gamma) = \theta_\gamma$, $-\alpha \dotplus (-\beta) = \theta_{-\alpha}$, $(\delta \dotplus (-\gamma)) \dotplus (-\beta) = \alpha$, and $(-\alpha \dotplus (-\beta)) \dotplus (-\gamma) = -\delta$.*

We are now ready to introduce the key tool in our study.

Definition 3.4. Let α and β be two nonzero roots of T, we say that α **is connected to** β if there exists a family $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\} \subset \pm\Lambda^T \cup \{0\}$ satisfying the following conditions:

1. $\alpha_1 = \alpha$.

2. An odd number of factors operated under $\dotplus$ belongs to $\pm\Lambda^T$. More precisely,

$$\left\{\alpha_1, (\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3, (((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \alpha_4) \dotplus \alpha_5, \dots, ((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_{2n}) \dotplus \alpha_{2n+1}\right\} \subset \pm\Lambda^T.$$

3. The result of the operation of an even number of factors under $\dotplus$ either belongs to $\pm\Lambda^{\mathfrak{L}}$ or Θ_Ω, that is,

$$\left\{(\alpha_1 \dotplus \alpha_2), ((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \alpha_4, \dots, ((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_2 n-1) \dotplus \alpha_{2n}\right\} \subset \pm\Lambda^{\mathfrak{L}} \cup \Theta_\Omega.$$

4. $((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_{2n}) \dotplus \alpha_{2n+1} \in \pm\beta$.

The family $\{\alpha_1, \alpha_2, \dots \alpha_{2n}, \alpha_{2n+1}\}$ is called a **connection from** α to β.

For any $\alpha \in \Lambda^T$ the set $\{\alpha\}$ is trivially a connection from α to itself. Moreover, if $-\alpha$ also belongs to Λ^T, then the same set is a connection from α to $-\alpha$.

Given α and β two elements of Λ^T, we write $\alpha \sim \beta$ if α is connected to β. Note that $\alpha \sim \alpha$ for any $\alpha \in \Lambda^T$. That is, the relation $\sim$ is reflexive. If $\alpha \sim \beta$, then there exists a connection $\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{2n-1}, \alpha_{2n}, \alpha_{2n+1}\} \subset \pm\Lambda^{\mathfrak{T}} \cup \{0\}$ from α to β satisfying, in particular, that

$$((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_{2n}) \dotplus \alpha_{2n+1} = \varepsilon\beta \text{ for some } \varepsilon \in \{\pm\}.$$

If $n = 0$, then $\alpha_1 = \alpha$ and $\beta \in \{\pm\alpha\}$. So $\{\beta\}$ is a connection from β to α. If $n \geq 1$, then Lemmas 3.2 and 3.3 yield that $\{\beta, -\varepsilon\alpha_{2n+1}, -\varepsilon\alpha_{2n}, \dots, -\varepsilon\alpha_3, -\varepsilon\alpha_2\}$, is a connection from β to α. That is, $\sim$ is symmetric. On the other hand, if $\alpha \sim \beta$ and $\beta \sim \gamma$, then there exit connections $\{\alpha_1, \alpha_2, \dots, \alpha_{2n+1}\}$ from α to β (satisfying $(\cdots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = \epsilon\beta$ for $\varepsilon \in \{\pm\}$) and $\{\beta_1, \beta_2, \dots, \beta_{2m+1}\}$ for a connection from β to γ. If $m = 0$, then $\gamma = \pm\beta$ and so $\{\alpha_1, \alpha_2, \dots, \alpha_{2n+1}\}$ is a connection from α to γ. If $m \geq 1$, then it is easy to show that $\{\alpha_1, \alpha_2, \dots, \alpha_{2n+1}, \epsilon\beta_2, \dots, \epsilon\beta_{2m+1}\}$ is a connection from α to γ. That is, $\sim$ is transitive. We have indeed proved the following:

Proposition 3.5. *The relation $\sim$ in Λ^T, defined by $\alpha \sim \beta$, if and only if α is connected to β is an equivalence relation.*

4. Decompositions

Let T be a split 3-Leibniz algebra and $T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha\right)$ its corresponding root spaces decomposition. By Proposition 3.5 the connection relation is an equivalence relation in Λ^T and so we can consider the quotient set

$$\Lambda^T/\sim = \{[\alpha] : \alpha \in \Lambda^T\},$$

where $[\alpha]$ is the set of nonzero roots of T which are connected to α. By the definition of $\sim$, it is clear that if $\beta \in [\alpha]$ and $-\beta \in \Lambda^T$, then $-\beta \in [\alpha]$.

Our goal in this section is to associate an ideal $I_{[\alpha]}$ of T to each $[\alpha]$. Given $\alpha \in \Lambda^T$, we start by defining a set $T_{0,[\alpha]} \subset T_0$ as follows:

$$T_{0,[\alpha]} := \mathrm{span}_{\mathbb{K}}\left\{[T_\beta, T_\gamma, T_\delta] : \beta, \gamma, \delta \in [\alpha] \cup \{0\}\right\} \cap T_0.$$

Applying Lemma 2.5(v), we obtain that

$$T_{0,[\alpha]} = \mathrm{span}_{\mathbb{K}}\left\{[T_\beta, T_\gamma, T_\delta] : \beta, \gamma, \delta \in [\alpha] \cup \{0\} \text{ and } \beta + \gamma + \delta = 0\right\}.$$

Next, we consider $V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} T_\beta$, and $T_{[\alpha]} := T_{0,[\alpha]} \oplus V_{[\alpha]}$.

Proposition 4.1. *The subspace $T_{[\alpha]}$ is a 3-Leibniz subalgebra of T, for all $\alpha \in \Lambda^T$. We will refer to $T_{[\alpha]}$ as the 3-Leibniz subalgebra of T associated to $[\alpha]$.*

Proof. We have to check that $T_{[\alpha]}$ satisfies $\left[T_{[\alpha]}, T_{[\alpha]}, T_{[\alpha]}\right] \subset T_{[\alpha]}$. Applying (5) and Lemma 2.5(v), we have that

$$\left[T_{0,[\alpha]}, T_{0,[\alpha]}, T_{[\alpha]}\right] + \left[T_{0,[\alpha]}, T_{[\alpha]}, T_{0,[\alpha]}\right] + \left[T_{[\alpha]}, T_{0,[\alpha]}, T_{0,[\alpha]}\right] \subset T_{[\alpha]}.$$

Next, we claim that

$$\left[T_{0,[\alpha]}, V_{[\alpha]}, V_{[\alpha]}\right] + \left[V_{[\alpha]}, T_{0,[\alpha]}, V_{[\alpha]}\right] + \left[V_{[\alpha]}, V_{[\alpha]}, T_{0,[\alpha]}\right] \subset T_{[\alpha]}.$$

In fact, given $\beta, \gamma \in [\alpha]$ we consider three cases:

(1) $\left[T_0, T_\beta, T_\gamma\right] \neq 0$.

If $\gamma = -\beta$ then $\left[T_0, T_\beta, T_\gamma\right] \subseteq T_{0,[\alpha]}$. Assume now $\gamma \neq -\beta$, and apply Lemma 2.5(i) and (v) to get $\beta \in \Lambda^{\mathfrak{L}}$ and $\beta + \gamma \in \Lambda^T$. Since $\beta \in [\alpha]$ there exists a connection $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}\}$ from α to β. Thus

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, 0, \gamma\}$$

is a connection from α to $\beta + \gamma$ if $(\cdots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = \beta$, and

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, 0, -\gamma\}$$

if $(\ldots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = -\beta$. In any case, we have proved $\beta + \gamma \in [\alpha]$ which jointly with Lemma 2.5(v) imply $\left[T_0, T_\beta, T_\gamma\right] \subset T_{\beta+\gamma} \subset V_{[\alpha]} \subset T_{[\alpha]}$, and therefore $\left[T_{0,[\alpha]}, V_{[\alpha]}, V_{[\alpha]}\right] \subset T_{[\alpha]}$.

(2) $\left[T_\beta, T_0, T_\gamma\right] \neq 0$.

Reasoning like in the previous case, we obtain $\left[V_{[\alpha]}, T_{0,[\alpha]}, V_{[\alpha]}\right] \subset T_{[\alpha]}$.

(3) $\left[T_\beta, T_\gamma, T_0\right] \neq 0$.

If $\gamma = -\beta$, then $\left[T_\beta, T_\gamma, T_0\right] \subseteq HT_0 = 0$. Suppose that $\gamma \neq -\beta$ and apply Lemma 2.5(i) and (v) to get that $\beta + \gamma \in \Lambda^{\mathfrak{L}} \cap \Lambda^T$. If $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}\}$ is a connection from α to β, then we can construct a connection from α to $\beta + \gamma$. More concretely, $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, \gamma, 0\}$ is a connection from α to $\beta + \gamma$, provided $(\cdots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = \beta$. On the other hand, if $(\cdots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = -\beta$, then $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, -\gamma, 0\}$ is a connection from α to $\beta + \gamma$. In any case, we have shown that $\beta + \gamma \in [\alpha]$, and by Lemma 2.5 $\left[T_\beta, T_\gamma, T_0\right] \subset T_{\beta+\gamma} \subset V_{[\alpha]} \subset T_{[\alpha]}$, which yields $\left[V_{[\alpha]}, V_{[\alpha]}, T_{0,[\alpha]}\right] \subset T_{[\alpha]}$.

It remains to check that $\left[V_{[\alpha]}, V_{[\alpha]}, V_{[\alpha]}\right] \subset T_{[\alpha]}$. Given $\beta, \gamma, \delta \in [\alpha]$, assume that $\left[T_\beta, T_\gamma, T_\delta\right] \neq 0$. If $\beta + \gamma + \delta = 0$, then by the definition of $T_{0,[\alpha]}$, we have that $\left[T_\beta, T_\gamma, T_\delta\right] \subset T_{0,[\alpha]} \subset T_{[\alpha]}$. Suppose now that $\beta + \gamma + \delta \neq 0$ and consider two cases:

(1) $\beta + \gamma = 0$.

Applying Lemma 2.5(v) we obtain that $\left[T_\beta, T_\gamma, T_\delta\right] \subset T_\delta \subset V_{[\alpha]} \subset T_{[\alpha]}$, and we are done with this case.

(2) $\beta + \gamma \neq 0$.

From Lemma 2.5(i) and (v), we get that $\beta + \gamma \in \Lambda^{\mathfrak{L}}$ and $\beta + \gamma + \delta \in \Lambda^T$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}\}$ be a connection from α to β. Then Remark 3.1 and the previous considerations allow us to conclude that $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, \gamma, \delta\}$ is a connection from α to $\beta + \gamma + \delta$, provided $(\cdots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = \beta$, and $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, -\gamma, -\delta\}$ is a connection from α to $\beta + \gamma + \delta$, provided $(\cdots(\alpha_1 \dotplus \alpha_2) \dotplus \cdots) \dotplus \alpha_{2n+1} = -\beta$. Thus, $\beta + \gamma + \delta \in [\alpha]$ which implies $\left[T_\beta, T_\gamma, T_\delta\right] \subset T_{\beta+\gamma+\delta} \subset V_{[\alpha]} \subset T_{[\alpha]}$.

Therefore $\left[V_{[\alpha]}, V_{[\alpha]}, V_{[\alpha]}\right] \subset T_{[\alpha]}$, which concludes the proof. □

Our aim is to show that $T_{[\alpha]}$ is indeed an ideal of T. We first need some preliminary results.

Lemma 4.2 *([11, Lemma 4.2]). If $\alpha, \beta \in \Lambda^T$ and $\alpha + \beta \in \Lambda^{\mathfrak{L}} \cup \{0\}$, then α is connected to β.*

Lemma 4.3 *([11, Lemma 4.3]). Assume that $\alpha, \overline{\beta} \in \Lambda^T$ are not connected, then $\overline{\beta}(\mathrm{ad}(T_\alpha, T_{-\alpha})) = 0$.*

Lemma 4.4. *Fix $\alpha_0 \in \Lambda^T$. The following assertions hold for $\alpha \in [\alpha_0]$ and $\beta, \gamma \in \Lambda^T \cup \{0\}$.*

(i) *If $\left[T_\alpha, T_\beta, T_\gamma\right] \neq 0$, then $\beta, \gamma, \alpha + \beta + \gamma \in [\alpha_0] \cup \{0\}$.*
(ii) *If $\left[T_\beta, T_\alpha, T_\gamma\right] \neq 0$, then $\beta, \gamma, \beta + \alpha + \gamma \in [\alpha_0] \cup \{0\}$.*
(iii) *If $\left[T_\beta, T_\gamma, T_\alpha\right] \neq 0$, then $\beta, \gamma, \beta + \gamma + \alpha \in [\alpha_0] \cup \{0\}$.*

Proof. (i) Apply Lemma 2.5(i) to get that $\alpha + \beta \in \Lambda^{\mathfrak{L}} \cup \{0\}$. If $\beta \neq 0$, then Lemma 4.2 says that α is connected to β, and therefore $\beta \in [\alpha_0] \cup \{0\}$. Next, we consider two cases:

(1) $\alpha+\beta+\gamma=0$.
It remains to check that $\gamma\in[\alpha_0]\cup\{0\}$. If $\gamma\neq 0$ since $-\gamma=\alpha+\beta\in\Lambda^{\mathfrak{L}}$, keeping in mind Remark 3.1, one can deduce that the family $\{\alpha,\beta,0\}$ is a connection from α to γ. Thus, $\gamma\in[\alpha_0]$.

(2) $\alpha+\beta+\gamma\neq 0$.
Assume first that $\alpha+\beta\neq 0$, then $\alpha+\beta\in\Lambda^{\mathfrak{L}}$. Remark 3.1 allows us to conclude that $\{\alpha,\beta,\gamma\}$ is a connection from α to $\alpha+\beta+\gamma$. Therefore, $\alpha+\beta+\gamma\in[\alpha_0]$. If $\gamma\neq 0$, then the family $\{\alpha,\beta,-\alpha-\beta-\gamma\}$ provides us a connection from α to γ. Hence, $\gamma\in[\alpha_0]$.
Finally, if $\alpha+\beta=0$ then necessarily $\gamma\neq 0$. We claim that α is connected to γ (which implies $\gamma\in[\alpha_0]$). Suppose on the contrary that α is not connected to γ; then from Lemma 4.3 we obtain that $\gamma(\mathrm{ad}(T_\alpha,T_{-\alpha}))=0$. In particular, $0=\gamma(\mathrm{ad}(T_\alpha,T_{-\alpha}))T_\gamma=\left[T_\alpha,T_{-\alpha},T_\gamma\right]=\left[T_\alpha,T_\beta,T_\gamma\right]$, which contradicts our hypothesis.

(ii) It can be proved similarly.

(iii) Lemma 2.5(i) and (v) imply $\beta+\gamma\in\Lambda^{\mathfrak{L}}\cup\{0\}$ and $\alpha+\beta+\gamma\in\Lambda^T\cup\{0\}$. If $\beta\neq 0$, an application of Lemma 4.2 gives that β is connected to γ, provided $\gamma\neq 0$. We then consider two cases:

(1) $\alpha+\beta+\gamma=0$.
If $\gamma=0$, then $\beta=-\alpha$ and we have done. Suppose then $\gamma\neq 0$. The set $\{\alpha,0,\beta\}$ is a connection from α to γ, since $\beta+\gamma=-\alpha\in\Lambda^{\mathfrak{L}}$. It shows that $\gamma\in[\alpha_0]$, which imply $\beta\in[\alpha_0]\cup\{0\}$.

(2) $\alpha+\beta+\gamma\neq 0$. If $\beta=\gamma=0$ there is nothing to prove. Suppose now that $\beta=0$ and $\gamma\neq 0$, which imply that $\gamma\in\Lambda^{\mathfrak{L}}\cap\Lambda^T$. Thus $\{\alpha,-(\alpha+\gamma),0\}$ is a connection from α to γ, and $\{\gamma,0,\alpha\}$ is a connection from γ to $\alpha+\gamma$ and we are done. If $\beta\neq 0$ and $\gamma=0$, then $\beta\in\Lambda^{\mathfrak{L}}$ and $\alpha+\beta\in\Lambda^T$. From here we can deduce that $\{\alpha,-(\alpha+\beta),0\}$ is a connection from α to β and $\{\beta,0,\alpha\}$ is a connection from β to $\alpha+\beta$. Hence $\beta,\alpha+\beta+\gamma\in[\alpha_0]$. Finally, if $\beta\neq 0$ and $\gamma\neq 0$, then $\{\alpha,-(\alpha+\beta+\gamma),\gamma\}$ and $\{\beta,\gamma,\alpha\}$ are connections from α to β and from α to $\alpha+\beta+\gamma$, respectively, (provided $\beta+\gamma\neq 0$), which imply $\beta,\gamma,\alpha+\beta+\gamma\in[\alpha_0]$. The case $\beta+\gamma=0$ follows from Lemma 4.3, since it implies that $\alpha\sim\beta$ whenever $\left[T_\beta,T_{-\beta},T_\alpha\right]\neq 0$. □

Remark 4.5. If $\alpha,\beta,\gamma\in\Lambda^T$ are such that $\alpha+\beta+\gamma=0$ and $[T_\alpha,T_\beta,T_\gamma]\neq 0$, then necessarily $\alpha+\beta\neq 0$ and $\gamma\neq 0$. Assume on the contrary that $\alpha+\beta=0$. Then, $[T_\alpha,T_{-\alpha},T_0]=\mathrm{ad}(T_\alpha,T_{-\alpha})T_0=0$, a contradiction. Hence, $\alpha+\beta\neq 0$ and $\gamma\neq 0$. Observe that $\alpha+\beta=0$ if and only if $\gamma=0$.

Lemma 4.6. *Fix $\alpha_0\in\Lambda^T$. If $\alpha,\beta,\gamma\in[\alpha_0]\cup\{0\}$ with $\alpha+\beta+\gamma=0$ and $\delta,\epsilon\in\Lambda^T\cup\{0\}$. Then the following assertions hold.*

(i) *If $[[T_\alpha,T_\beta,T_\gamma],T_\delta,T_\epsilon]\neq 0$ then $\delta,\epsilon,\delta+\epsilon\in[\alpha_0]\cup\{0\}$.*
(ii) *If $\left[T_\delta,[T_\alpha,T_\beta,T_\gamma],T_\epsilon\right]\neq 0$ then $\delta,\epsilon,\delta+\epsilon\in[\alpha_0]\cup\{0\}$.*
(iii) *If $\left[T_\delta,T_\epsilon,[T_\alpha,T_\beta,T_\gamma]\right]\neq 0$ then $\delta,\epsilon,\delta+\epsilon\in[\alpha_0]\cup\{0\}$.*

Proof. (i) From Remark 4.5 we have that $\alpha+\beta\neq 0$ and $\gamma\neq 0$. Since

$$0\neq[[T_\alpha,T_\beta,T_\gamma],T_\delta,T_\epsilon]\subset[T_\alpha,T_\beta,[T_\gamma,T_\delta,T_\epsilon]]+[T_\gamma,[T_\alpha,T_\beta,T_\delta],T_\epsilon]+[T_\gamma,T_\delta,[T_\alpha,T_\beta,T_\epsilon]],$$

one of the three summands above is nonzero.

(1) Suppose that $\left[T_\alpha,T_\beta,[T_\gamma,T_\delta,T_\epsilon]\right]\neq 0$, which implies $[T_\gamma,T_\delta,T_\epsilon]\neq 0$. Then Lemma 4.4(i) applies to get that $\delta,\epsilon,\gamma+\delta+\epsilon\in[\alpha_0]\cup\{0\}$, since $\gamma\neq 0$. Next Lemma 2.5(v) yields that $0\neq\left[T_\alpha,T_\beta,[T_\gamma,T_\delta,T_\epsilon]\right]\subset\left[T_\alpha,T_\beta,T_{\gamma+\delta+\epsilon}\right]$. From here and taking into account that α and β cannot be both zero, Lemma 4.4 either (i) or (ii) gives $\alpha+\beta+\gamma+\delta+\epsilon=\delta+\epsilon\in[\alpha_0]\cup\{0\}$.

(2) Assume $[T_\gamma,[T_\alpha,T_\beta,T_\delta],T_\epsilon]\neq 0$. Since $\gamma\neq 0$, applying Lemma 4.4(i) to $[T_\gamma,T_{\alpha+\beta+\delta},T_\epsilon]\neq 0$, we get that

$$\alpha+\beta+\delta,\ \ \epsilon,\ \ \delta+\epsilon=\delta+\alpha+\beta+\gamma+\epsilon\in[\alpha_0]\cup\{0\}.$$

Now, Lemma 4.4 either (i) or (ii) ($\alpha+\beta\neq 0$) applied to $[T_\alpha,T_\beta,T_\delta]\neq 0$ gives $\delta\in[\alpha_0]\cup\{0\}$, which concludes the proof in that case.

(3) To finish, let us assume that $[T_\gamma,T_\delta,[T_\alpha,T_\beta,T_\epsilon]]\neq 0$. It implies that $[T_\alpha,T_\beta,T_\epsilon]\neq 0$. Since either $\alpha\neq 0$ or $\beta\neq 0$, Lemma 4.4 either (i) or (ii) yields

$$\beta,\ \ \epsilon,\ \ \alpha+\beta+\epsilon\in[\alpha_0]\cup\{0\}.$$

On the other hand, taking into account that $\gamma\neq 0$, and $[T_\gamma,T_\delta,T_{\alpha+\beta+\epsilon}]\neq 0$, a second use of Lemma 4.4(i) gives

$$\delta,\ \ \alpha+\beta+\epsilon,\ \ \delta+\epsilon=\gamma+\delta+\alpha+\beta+\epsilon\in[\alpha_0]\cup\{0\},$$

as desired.

Statements (ii) and (iii) can be proved in a similar way. □

The following result is an immediate consequence of Lemma 5.2.

Lemma 4.7. *Fix $\alpha_0\in\Lambda^T$. If $\alpha,\beta,\gamma\in[\alpha_0]\cup\{0\}$ are such that $\alpha+\beta+\gamma=0$, and $\delta\in\Lambda^T\setminus[\alpha_0]$. Then $\delta(\mathrm{ad}([T_\alpha,T_\beta,T_\gamma],T_0))=0$.*

Theorem 4.8. *The following assertions hold for a 3-Leibniz algebra T with root spaces decomposition $T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha\right)$.*

(i) *For any $\alpha_0 \in \Lambda^T$, the 3-Leibniz subalgebra $T_{[\alpha_0]}$ of T associated to $[\alpha_0]$ is an ideal of T.*
(ii) *If T is simple, then there exists a connection between any two nonzero roots of T, and $T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^T \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]$.*

Proof. (i) We have to show that $[T_{[\alpha_0]}, T, T] + [T, T_{[\alpha_0]}, T] + [T, T, T_{[\alpha_0]}] \subset T_{[\alpha_0]}$. To do so, we divide the proof in several steps.

Firstly note that $[T_{0,[\alpha_0]}, T_0, T_0] + [T_0, T_{0,[\alpha_0]}, T_0] + [T_0, T_0, T_{0,[\alpha_0]}] \subset [T_0, T_0, T_0] = 0$, by (5). Secondly, given $\alpha, \beta \in \Lambda^T \cup \{0\}$ Lemma 5.2 gives

$$[T_{0,[\alpha_0]}, T_\alpha, T_\beta] + [T_\alpha, T_{0,[\alpha_0]}, T_\beta] + [T_\alpha, T_\beta, T_{0,[\alpha_0]}] \subset T_{[\alpha_0]}.$$

Therefore,

$$[T_{0,[\alpha_0]}, T, T] + [T, T_{0,[\alpha_0]}, T] + [T, T, T_{0,[\alpha_0]}] \subset T_{[\alpha_0]}. \tag{8}$$

For $\alpha, \beta \in \Lambda^T \cup \{0\}$, using Lemma 2.5(v) and Lemma 4.4 we get

$$[\oplus_{\gamma \in [\alpha_0]} T_\gamma, T_\alpha, T_\beta] + [T_\alpha, \oplus_{\gamma \in [\alpha_0]} T_\gamma, T_\beta] + [T_\alpha, T_\beta, \oplus_{\gamma \in [\alpha_0]} T_\gamma] \subset T_{[\alpha_0]},$$

that is, $[V_{[\alpha_0]}, T_\alpha, T_\beta] + [T_\alpha, V_{[\alpha_0]}, T_\beta] + [T_\alpha, T_\beta, V_{[\alpha_0]}] \subset T_{[\alpha_0]}$, which implies

$$[V_{[\alpha_0]}, T, T] + [T, V_{[\alpha_0]}, T] + [T, T, V_{[\alpha_0]}] \subset T_{[\alpha_0]}. \tag{9}$$

Our claim now follows from (8) and (9).

(ii) The simplicity of T applies to get that $T_{[\alpha]} \in \{J, T\}$ for any $\alpha \in \Lambda^T$. If $T_{[\alpha_0]} = T$ for some $\alpha_0 \in \Lambda^T$ then $[\alpha_0] = \Lambda^T$; which implies that any nonzero root of T is connected to α_0, and therefore, any nonzero roots of T are connected. Assume now that $T_{[\alpha]} = T$ for every $\alpha \in \Lambda^T$. Then $[\alpha] = [\beta]$ for any $\alpha, \beta \in T$. Since $\Lambda^T = \bigcup_{[\gamma] \in \Lambda/\sim} [\gamma]$ we get $[\alpha] = \Lambda^T$. That is, all nonzero roots of T are connected. To finish, observe that $T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^T \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]$ follows in any case. $\square$

Notation 4.9. Let us denote

$$T_{0,\Lambda^T} := \mathrm{span}_{\mathbb{K}}\{[T_\alpha, T_\beta, T_\gamma] : \alpha + \beta + \gamma = 0, \quad \text{where} \quad \alpha, \beta, \gamma \in \Lambda^T \cup \{0\}\} \subset T_0.$$

In what follows, we will use the terminology $I_{[\alpha]} := T_{[\alpha]}$ where $T_{[\alpha]}$ is one of the ideals of T described in Theorem 4.8(i).

Theorem 4.10. *If $\mathcal{U}$ is a vector space complement of T_{0,Λ^T}, then $T = \mathcal{U} + \sum_{[\alpha] \in \Lambda^T/\sim} I_{[\alpha]}$. Moreover, $[T, I_{[\alpha]}, I_{[\beta]}] + [I_{[\alpha]}, T, I_{[\beta]}] + [I_{[\alpha]}, I_{[\beta]}, T] = 0$, whenever $[\alpha] \neq [\beta]$.*

Proof. From $T = T_0 \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha\right) = (\mathcal{U} \oplus T_{0,\Lambda^T}) \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha\right)$, it follows that $\bigoplus_{\alpha \in \Lambda^T} T_\alpha = \bigoplus_{[\alpha] \in \Lambda^T/\sim} V_{[\alpha]}$ and $T_{0,\Lambda^T} = \sum_{[\alpha] \in \Lambda^T/\sim} T_{0,[\alpha]}$, which imply

$$T = (\mathcal{U} \oplus T_{0,\Lambda^T}) \oplus \left(\bigoplus_{\alpha \in \Lambda^T} T_\alpha\right) = \mathcal{U} + \sum_{[\alpha] \in \Lambda^T/\sim} I_{[\alpha]},$$

where each $I_{[\alpha]}$ is an ideal of T by Theorem 4.8. Now, given $[\alpha] \neq [\beta]$, the assertion

$$[T, I_{[\alpha]}, I_{[\beta]}] + [I_{[\alpha]}, T, I_{[\beta]}] + [I_{[\alpha]}, I_{[\beta]}, T] = 0,$$

is a consequence of writing

$$[T, I_{[\alpha]}, I_{[\beta]}] = \left[T_0 + \sum_{\gamma \in \Lambda^T} T_\gamma, T_{0,[\alpha]} + V_{[\alpha]}, T_{0,[\beta]} + V_{[\beta]}\right],$$

$$[I_{[\alpha]}, T, I_{[\beta]}] = \left[T_{0,[\alpha]} + V_{[\alpha]}, T_0 + \sum_{\gamma \in \Lambda^T} T_\gamma, T_{0,[\beta]} + V_{[\beta]}\right],$$

$$[I_{[\alpha]}, I_{[\beta]}, T] = \left[T_{0,[\alpha]} + V_{[\alpha]}, T_{0,[\beta]} + V_{[\beta]}, T_0 + \sum_{\gamma \in \Lambda^T} T_\gamma\right],$$

and applying (5), Lemmas 4.4 and 5.2 taking into account that $\alpha \nsim \beta$. $\square$

Corollary 4.11. *If $\mathrm{Ann}(T) = 0$ and $[T, T, T] = T$, then $T = \bigoplus_{[\alpha] \in \Lambda^T/\sim} I_{[\alpha]}$.*

Proof. Since $[T, T, T] = T$, Theorem 4.10 applies to get

$$\left[\mathcal{U} + \sum_{[\alpha]\in\Lambda^T/\sim} I_{[\alpha]},\ \mathcal{U} + \sum_{[\alpha]\in\Lambda^T/\sim} I_{[\alpha]}, \mathcal{U} + \sum_{[\alpha]\in\Lambda^T/\sim} I_{[\alpha]}\right] = \mathcal{U} + \sum_{[\alpha]\in\Lambda^T/\sim} I_{[\alpha]}.$$

Keeping in mind that $\mathcal{U} \subset T_0$ and $[T, I_{[\alpha]}, I_{[\beta]}] + [I_{[\alpha]}, T, I_{[\beta]}] + [I_{[\alpha]}, I_{[\beta]}, T] = 0$ if $[\alpha] \neq [\beta]$, Lemma 4.4 and (5) yield $\mathcal{U} = 0$, that is, $T = \sum_{[\alpha]\in\Lambda^T/\sim} I_{[\alpha]}$. To finish, given $x \in I_{[\alpha]} \cap \sum_{\substack{[\beta]\in\Lambda^T/\sim \\ \beta\nsim\alpha}} I_{[\beta]}$ using again the equation $[T, I_{[\alpha]}, I_{[\beta]}] + [I_{[\alpha]}, T, I_{[\beta]}] + [I_{[\alpha]}, I_{[\beta]}, T] = 0$, for $[\alpha] \neq [\beta]$, we obtain

$$\left[T, x, I_{[\alpha]}\right] + \left[T, x, \sum_{\substack{[\beta]\in\Lambda^T/\sim \\ \beta\nsim\alpha}} I_{[\beta]}\right] = 0,$$

$$\left[x, T, I_{[\alpha]}\right] + \left[x, T, \sum_{\substack{[\beta]\in\Lambda^T/\sim \\ \beta\nsim\alpha}} I_{[\beta]}\right] = 0,$$

$$\left[x, I_{[\alpha]}, T\right] + \left[x, \sum_{\substack{[\beta]\in\Lambda^T/\sim \\ \beta\nsim\alpha}} I_{[\beta]}, T\right] = 0.$$

It imply that $[x, T, T] + [T, x, T] + [T, T, x] = 0$, that is, $x \in \mathrm{Ann}(T) = 0$. Thus $x = 0$, proving that the sum is direct. □

5. Split 3-Leibniz algebras of maximal length. The simple components

In this section we study the simplicity of split 3-Leibniz algebras by focusing our attention in those of maximal length. This terminology has its origin in the theory of gradation of Lie and Leibniz algebras. See, for example, [9,12,15,16,19]. Our goal is to characterize the simplicity of a 3-Leibniz algebra T in terms of connectivity properties of its associated root system Λ^T.

Definition 5.1. We say that a split 3-Leibniz algebra $T = T_0 \oplus \left(\bigoplus_{\alpha\in\Lambda^T} T_\alpha\right)$ is of **maximal length** if $\dim T_\alpha = 1$ for any $\alpha \in \Lambda^T$.

The following lemma is a consequence of the fact that the set of multiplications by elements in H is a commuting set of diagonalizable endomorphisms, and every ideal of T remains invariant under this set.

Lemma 5.2. *Let $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^T} T_\alpha)$ be a split 3-Leibniz algebra. Then every ideal I of T decomposes as $I = (I \cap T_0) \oplus \left(\bigoplus_{\alpha\in\Lambda^T} (I \cap T_\alpha)\right)$.*

In what follows $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^T} T_\alpha)$ denotes a split 3-Leibniz algebra of maximal length. If I is a nonzero ideal of T, then

$$I = (I \cap T_0) \oplus \left(\bigoplus_{\alpha\in\Lambda_I^T} T_\alpha\right),$$

where $\Lambda_I^T := \{\gamma \in \Lambda^T : I \cap T_\gamma \neq 0\}$ by Lemma 5.2. In particular, if we consider the ideal J, (see Definition 2.2), we get

$$J = (J \cap T_0) \oplus \left(\bigoplus_{\alpha\in\Lambda_{\neg J}^T} T_\alpha\right) \oplus \left(\bigoplus_{\beta\in\Lambda_J^T} T_\beta\right)$$

with $\Lambda_{\neg J}^T := \{\alpha \in \Lambda^T : J \cap T_\alpha = 0\}$. Moreover, we can write

$$\Lambda^T = \Lambda_J^T \,\dot\cup\, \Lambda_{\neg J}^T. \tag{10}$$

The notion of connectivity of nonzero roots, introduced in Definition 3.4, is not strong enough to detect whether a nonzero root α of T belongs either to Λ_J^T or $\Lambda_{\neg J}^T$. Therefore, given a root space T_α we have lost all the information about the intersection of T_α with J. More precisely, we cannot assert when $T_\alpha \cap J \neq 0$. Due to the fundamental role played by such of these intersections in the study of the simplicity of T, our next objective is to refine the previous concept of connections of nonzero roots.

Definition 5.3. Let $\alpha, \beta \in \Lambda_\Psi^T$ with $\Psi \in \{\neg J, J\}$. We say that α is J-**connected** to β, denoted by $\alpha \sim_J \beta$, if there exists a family $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ contained in $\pm\Lambda_{\neg J}^T \cup \{0\}$ if $\Psi = \neg J$ or contained in $\pm\Lambda^T \cup \{0\}$ if $\Psi = J$, satisfying the following conditions:

1. $\alpha_1 = \alpha$.
2. An odd number of factors operated under $\dotplus$ belongs to $\pm\Lambda_\Psi^T$, that is,

$$\left\{\alpha_1, (\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3, (((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \alpha_4) \dotplus \alpha_5, \dots, ((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_{2n}) \dotplus \alpha_{2n+1}\right\} \subset \pm\Lambda_\Psi^T.$$

3. The result of the operation of an even number of factors under $\dotplus$ either belongs to $\pm\Lambda^{\mathfrak{L}}$ or is an element of Θ_Ω, that is,

$$\left\{(\alpha_1 \dotplus \alpha_2), ((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \alpha_4, \dots, ((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_{2n-1}) \dotplus \alpha_{2n}\right\} \subset \pm\Lambda^{\mathfrak{L}} \cup \Theta_\Omega.$$

4. $((\cdots((\alpha_1 \dotplus \alpha_2) \dotplus \alpha_3) \dotplus \cdots) \dotplus \alpha_{2n}) \dotplus \alpha_{2n+1} \in \pm\beta$.

The set $\{\alpha_1, \dots, \alpha_n\}$ is called a J-connection from α to β.

The next result can be proved in a similar way to Proposition 3.5.

Proposition 5.4. *The following assertions hold.*

(1) *The relation $\sim_J$ is an equivalence relation in $\Lambda_{\neg J}^T$.*
(2) *The relation $\sim_J$ is an equivalence relation in Λ_J^T.*

Motivated by the theory of split Lie algebras and triple systems [9,12,16], we introduce the following definition.

Definition 5.5. A split 3-Leibniz algebra T is **root-multiplicative** if:

(1) Given $\alpha \in \Lambda_{\neg J}^T$ and $\beta, \gamma \in \Lambda_{\neg J}^T \cup \{0\}$ such that $\alpha \dotplus \beta \in \Lambda^{\mathfrak{L}} \cup \Theta_\Omega$ and $\alpha + \beta + \gamma \in \Lambda^T$ then $[T_\alpha, T_\beta, T_\gamma] \neq 0$.
(2) Given $\alpha \in \Lambda_J^T$ and $\beta, \gamma \in \Lambda^T \cup \{0\}$ such that $\alpha \dotplus \beta \in \Lambda^{\mathfrak{L}} \cup \Theta_\Omega$ and $\alpha + \beta + \gamma \in \Lambda_J^T$ then $[T_\alpha, T_\beta, T_\gamma] \neq 0$.

Next, we would like to distinguish the elements of a 3-Leibniz algebra of maximal length T giving rise to elements in the Lie algebra $\mathfrak{L}$ which annihilate the "Lie type roots" of T, in the following sense:

Definition 5.6. The *Lie-annihilator* of a split 3-Leibniz algebra of maximal length T is the set $\mathcal{Z}_{Lie}(T) := \{v \in T : \mathrm{ad}(v, w)\left(\bigoplus_{\alpha \in \Lambda_{\neg J}^T} T_\alpha\right) = 0 \text{ for some } w \in T\}$.

Observe that $\mathcal{Z}(T) \subset \mathcal{Z}_{Lie}(T)$. Moreover, this is the natural ternary extension of the Lie annihilator of split and graded Leibniz algebras of maximal length. (See [19].)

The symmetry of $\Lambda_{\neg J}^T$ is defined, as usual. To be more precise, $\Lambda_{\neg J}^T$ is **symmetric** if $\alpha \in \Lambda_{\neg J}^T$ implies $-\alpha \in \Lambda^{\neg J}$. That is, $\Lambda_{\neg J}^T = -\Lambda_{\neg J}^T$. The same concept applies to the set $\Lambda^{\mathfrak{L}}$. From now on we will assume that $\Lambda_{\neg J}^T$ and $\Lambda^{\mathfrak{L}}$ are both symmetric.

Proposition 5.7. *Let T be a split 3-Leibniz algebra of maximal length. Assume that T is root-multiplicative, $\mathcal{Z}_{Lie}(T) = 0$ and $T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^T \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]$. If $\Lambda_{\neg J}^T$ has all of its roots J-connected, then any ideal I of T such that $I \not\subseteq T_0 + J$ satisfies that $I = T$.*

Proof. Let I be an ideal of T such that $I \not\subseteq T_0 + J$. Applying Lemma 5.2 and Eq. (10) we can write

$$I = (I \cap T_0) \oplus \left(\bigoplus_{\alpha_i \in \Lambda_{\neg J,I}^T} T_{\alpha_i}\right) \oplus \left(\bigoplus_{\beta_j \in \Lambda_{J,I}^T} T_{\beta_j}\right), \tag{11}$$

where

$$\Lambda_{\neg J,I}^T := \{\alpha \in \Lambda_{\neg J}^T : T_\alpha \cap I \neq 0\} = \{\alpha \in \Lambda_{\neg J}^T : T_\alpha \subseteq I\}, \quad \text{and}$$
$$\Lambda_{J,I}^T := \{\alpha \in \Lambda_J^T : T_\alpha \cap I \neq 0\} = \{\alpha \in \Lambda_J^T : T_\alpha \subseteq I\}.$$

From $I \not\subseteq T_0 + J$ we have $\Lambda_{\neg J,I}^T \neq \emptyset$; thus, we can fix some $\alpha_0 \in \Lambda_{\neg J,I}^T$ such that

$$0 \neq T_{\alpha_0} \subset I. \tag{12}$$

Since α_0 is J-connected to any $\beta \in \Lambda^T_{\neg J}$, we can find a J-connection

$$\{\gamma_1, \gamma_2, \ldots, \gamma_{2n+1}\} \subset \Lambda^T_{\neg J} \cup \{0\} \tag{13}$$

from α_0 to β. Now, by Definition 5.3 and taking into account that $\Lambda^T_{\neg J} = -\Lambda^T_{\neg J}$ and $\Lambda^{\mathfrak{L}} = -\Lambda^{\mathfrak{L}}$, we have $\gamma_1 \dotplus \gamma_2 \in \Lambda^{\mathfrak{L}} \cup \Theta_\Omega$ and $\gamma_1 + \gamma_2 + \gamma_3 = (\gamma_1 \dotplus \gamma_2) \dotplus \gamma_3 \in \Lambda^T_{\neg J}$. Then, the root-multiplicativity and maximality length of T apply to get

$$0 \neq [T_{\gamma_1}, T_{\gamma_2}, T_{\gamma_3}] = T_{\gamma_1+\gamma_2+\gamma_3} = T_{(\gamma_1 \dotplus \gamma_2) \dotplus \gamma_3} \subset I,$$

since $T_{\gamma_1} = T_{\alpha_0} \subset I$. Repeating this process we get

$$0 \neq T_{((\cdots((\gamma_1 \dotplus \gamma_2) \dotplus \gamma_3) \dotplus \cdots) \dotplus \gamma_{2n}) \dotplus \gamma_{2n+1}} = T_{\epsilon\beta} \subset I, \text{ for some } \epsilon \in \pm 1$$

which shows that

$$0 \neq T_{\epsilon_\beta \beta} \subset I \text{ for any } \beta \in \Lambda^T_{\neg J} \text{ and some } \epsilon_\beta \in \pm 1. \tag{14}$$

We claim that $J \cap T_0 \subset \mathcal{Z}_{Lie}(T)$. In fact, for any $v \in J \cap T_0$ and $\alpha \in \Lambda^T_{\neg J}$ we have $[v, w, T_\alpha] \subset T_\alpha \subset J$, for any $w \in T_0$. Thus, if $[v, w, T_\alpha] \neq 0$, then $\alpha \in \Lambda^T_J$, a contradiction. From here $[v, w, T_\alpha] = 0$ and so

$$J \cap T_0 \subset \mathcal{Z}_{Lie}(T) = 0. \tag{15}$$

Let us also prove that

$$H = \mathrm{ad}(T_0, T_0) + \sum_{\alpha \in \Lambda^T_{\neg J}} \mathrm{ad}(T_\alpha, T_{-\alpha}). \tag{16}$$

From Remark 2.7 we can write $H = \mathrm{ad}(T_0, T_0) + \sum_{\gamma \in \Lambda^T} \mathrm{ad}(T_\gamma, T_{-\gamma})$.

If $0 \neq \mathrm{ad}(t_\gamma, t_{-\gamma})$ with either $t_\gamma \in J$ or $t_{-\gamma} \in J$, then proceeding as before we obtain that $0 \neq t_\gamma \in \mathcal{Z}_{Lie}(T)$, a contradiction. Hence, since $\Lambda^T = \Lambda^T_{\neg J} \dot\cup \Lambda^T_J$, Eq. (16) holds.

Our next aim is to prove that

$$T_0 \subset I. \tag{17}$$

To do so we study the triple products $[T_\alpha, T_\beta, T_\gamma]$ which constitute T_0. Notice that Eq. (15) produces

$$T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^T_{\neg J} \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]. \tag{18}$$

Given $\alpha, \beta, \gamma \in \Lambda^T_{\neg J}$ with $\alpha + \beta + \gamma = 0$, from Eqs. (4) and (5) we obtain that $\gamma \neq 0$, and either $\alpha \neq 0$ or $\beta \neq 0$. If $\alpha \neq 0$ and $\beta = 0$, (respectively, $\alpha = 0$ and $\beta \neq 0$), then $\alpha = -\gamma$, (respectively, $\beta = -\gamma$). Eq. (14) implies now $[T_\alpha, T_\beta, T_\gamma] = [T_{-\gamma}, T_0, T_\gamma] \subset I$, (respectively, $[T_\alpha, T_\beta, T_\gamma] = [T_0, T_{-\gamma}, T_\gamma] \subset I$). To finish, if α, β, and γ are nonzero, then $\alpha + \beta \in \Lambda^{\mathfrak{L}}$ by $[T_\alpha, T_\beta, T_\gamma] \neq 0$ and $\alpha + \beta \neq 0$.

Applying the root-multiplicativity of T we get $[T_\alpha, T_\beta, T_{-\beta}] = T_\alpha \subset I$ since some $T_{\epsilon\beta}$ ($\epsilon \in \pm 1$) is contained in I. Thus $[T_\alpha, T_\beta, T_\gamma] \subset I$, which jointly (18) imply $T_0 \subset I$, as desired.

To finish, we claim that $T_\gamma \subset I$ for any $\gamma \in \Lambda^T$. It follows from Eqs. (14), (16), and (17), by noticing that $0 \neq HT_\gamma = T_\gamma$ for any $\gamma \neq 0$. □

Proposition 5.8. *Let T be a split 3-Leibniz algebra of maximal length. Assume that T is root-multiplicative, $\mathcal{Z}_{Lie}(T) = 0$ and $T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^T \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]$. Let I be a nonzero ideal of T contained in J. If Λ^T_J has all of its elements J-connected, then $I = J$.*

Proof. Applying Eq. (10) we can write

$$I = (I \cap T_0) \oplus \left(\bigoplus_{\beta_j \in \Lambda^T_{J,I}} T_{\beta_j} \right), \tag{19}$$

where $\Lambda^T_{J,I} := \{\alpha \in \Lambda^T_J : T_\alpha \cap I \neq 0\} = \{\alpha \in \Lambda^T_J : T_\alpha \subseteq I\}$. Eq. (15) gives

$$J = \bigoplus_{\gamma \in \Lambda^T_J} T_\gamma. \tag{20}$$

On the other hand $I \cap T_0 \subset J \cap T_0 = 0$, we can also write $I = \bigoplus_{\beta_j \in \Lambda_{J,I}^T} T_{\beta_j}$. Thus, we can find some $\beta_0 \in \Lambda_{J,I}^T$ such that $0 \neq T_{\beta_0} \subset I$.

Observe that if $\beta \in \Lambda_J^T$ then $-\beta \notin \Lambda$. Indeed, in the opposite case, the fact $\mathcal{Z}_{Lie}(T) = 0$ would imply $\mathrm{ad}(t_\beta, t_{-\beta})(T_\alpha) \neq 0$ for some $\alpha \in \Lambda_{\neg J}^T$. But then $\alpha \in \Lambda_J^T$, a contradiction, so $-\beta \notin \Lambda$. Next, if $\gamma \in \Lambda_J^T$, a similar argument to the one used in Proposition 5.7 guarantees the existence of a J-connection $\{\delta_1, \delta_2, \dots, \delta_{2r}, \delta_{2r+1}\}$ from β_0 to γ such that $0 \neq [[\dots[T_{\beta_0}, T_{\delta_2}, T_{\delta_3}], \dots], T_{\delta_{2r}}, T_{\delta_{2r+1}}] = T_\gamma \subset I$. The result now follows from Eq. (20). □

Theorem 5.9. *Let T be a split 3-Leibniz algebra of maximal length. Assume that T is root-multiplicative, $\mathcal{Z}_{Lie}(T) = 0$ and $T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in \Lambda^T \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]$. Then T is simple if and only if $\Lambda_{\neg J}^T$ and Λ_J^T have all of its elements J-connected.*

Proof. Suppose first that T is simple. Then H and T_0 are as in Eqs. (16) and (18), respectively. Thus from (5) we obtain that $\Lambda_{\neg J}^T \neq \emptyset$. Given $\alpha \in \Lambda_{\neg J}^T$, let $[\alpha]_J = \{\beta \in \Lambda_{\neg J}^T : \alpha \sim_J \beta\}$ and $I = (\sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma \in [\alpha]_J \cup \{0\}}} [T_\alpha, T_\beta, T_\gamma]) \oplus (\bigoplus_{\beta \in [\alpha]_J} T_\beta) \oplus J$. Proceeding like in Section 4 we can show that I is a non-zero ideal of T distinct from J. The simplicity of T yields that $I = T$, which implies that $[\alpha]_J = \Lambda_{\neg J}^T$, that is, $\Lambda_{\neg J}^T$ has all of its roots J-connected.

Suppose now $J \neq 0$ and take some $\beta \in \Lambda_J^T$. If we define the linear subspace $K = \bigoplus_{\gamma \in \Lambda_J^T :\, \beta \sim_J \gamma} T_\gamma$. We have as above, taking also into account Eq. (15), that K is a non-zero ideal of T different to T. By simplicity $K = J$ and then Λ_J^T has all of its elements J-connected.

Conversely, given I a nonzero ideal of T, we distinguish two cases:

- If $I \not\subseteq T_0 + J$, then Proposition 5.7 gives us $I = T$.
- If $I \subset T_0 + J$, then $I \cap T_0 \subset \mathcal{Z}_{Lie}(T)$. In fact, for any $v \in I \cap T_0$ and $\alpha \in \Lambda_{\neg J}^T$ we have that $[v, T_0, T_\alpha] \subset J$, which implies that $[v, T_0, T_\alpha] = 0$. Thus $I \cap T_0 \subset \mathcal{Z}_{Lie}(T) = 0$. Applying Lemma 5.2 we obtain that $I \subset J$, and so $I = J$ by Proposition 5.8.

We have proved either $I = T$ or $I = J$, which shows that T is simple, as desired. □

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