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On nilpotent index and dibaricity of evolution algebras

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ABSTRACT

An evolution algebra corresponds to a quadratic matrix A of structural constants. It is known the equivalence between nil, right nilpotent evolution algebras and evolution algebras which are defined by upper triangular matrices A . We establish a criterion for an n -dimensional nilpotent evolution algebra to be with maximal nilpotent index $2^{n-1} + 1$. We give the classification of finite-dimensional complex evolution algebras with maximal nilpotent index. Moreover, for any $s = 1, \dots, n - 1$ we construct a wide class of n -dimensional evolution algebras with nilpotent index $2^{n-s} + 1$. We show that nilpotent evolution algebras are not dibaric and establish a criterion for two-dimensional real evolution algebras to be dibaric.

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1. Introduction

In the book [9] the foundations of evolution algebras are developed. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, Markov chains, dynamic systems, knot theory, 3-manifolds and the study of the Riemann-zeta functions [1,3,4,8,9].

Nilpotent algebra is an algebra for which there is a natural number k such that any product of k elements of the algebra is zero. If there is a non-zero product of $k - 1$ elements, then k is called the index of nilpotency of the algebra. Examples of nilpotent algebras are: an algebra with zero multiplication; direct sums of nilpotent algebras, the nilpotent indices of which are uniformly bounded; and the

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tensor product of two algebras, one of which is nilpotent. Nilpotent subalgebras that coincide with their normalizer (Cartan subalgebras) play an essential role in the classification of simple Lie algebras of finite dimension.

The algebraic notions like nilpotency, right nilpotency and solvability might be interpreted in a biological way as a various types of vanishing (“deaths”) populations.

The structural constants of an evolution algebra are given by a quadratic matrix A (see Section 2). In [3] the equivalence between nil, right nilpotent evolution algebras and evolution algebras which are defined by upper triangular matrices A is proved, and the classification of 2-dimensional complex evolution algebras is obtained.

In [2] the derivations of n -dimensional complex evolution algebras, depending on the rank of the matrix A , are studied. For an evolution algebra with non-singular matrix it is proved that the space of derivations is zero. The spaces of derivations for evolution algebras with matrices of rank $n - 1$ are described.

The paper [1] is devoted to the study of finite-dimensional complex evolution algebras. The class of evolution algebras isomorphic to evolution algebras with Jordan form matrices of structural constants is described. For finite-dimensional complex evolution algebras the criteria of nilpotency is established in terms of the properties of the corresponding matrices. Moreover, it is proved that for nilpotent n -dimensional complex evolution algebras the possible maximal nilpotency index is $2^{n-1} + 1$. The criteria of planarity for finite graphs is formulated by means of evolution algebras defined by graphs.

In [5] an evolution algebra E associated to the free population is introduced and using this non-associative baric algebra, many results are obtained in an explicit form, e.g., the explicit description of stationary quadratic operators and the explicit solutions of a non-linear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of a selection.

Dibaric algebras have not non-zero homomorphisms to the set of the real numbers. In [4] a concept of bq-homomorphism (which is given by two linear maps f, g of the algebra to the set of the real numbers) is introduced and it is shown that an algebra is dibaric if and only if it admits a non-zero bq-homomorphism.

In the study of any class of algebras, it is important to describe, up to isomorphism, at least algebras of lower dimensions because such description gives examples to establish or reject certain conjectures. In this way in [6] and [10], the classifications of associative and nilpotent Lie algebras of low dimensions were given.

In this paper we continue the study of algebraic properties of evolution algebras. The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we establish a criterion for an n -dimensional nilpotent evolution algebra to be of maximal nilpotent index $2^{n-1} + 1$. Since these algebras have maximal index of right nilpotency and maximal index of solvability too, then we might say that among vanishing populations, these are the latest vanishing populations.

We give the classification of finite-dimensional complex evolution algebras with maximal nilpotent index. Moreover, for any $s = 1, \dots, n-1$ we construct a wide class of n -dimensional evolution algebras with nilpotent index $2^{n-s} + 1$. Section 4 is devoted to the dibaricity of evolution algebras. We show that nilpotent real evolution algebras are not dibaric and establish a criterion for two-dimensional real evolution algebras to be dibaric.

2. Preliminaries

Evolution algebras. Let (E, \cdot) be an algebra over a field K . If it admits a basis $\{e_1, e_2, \dots\}$, such that

$$e_i \cdot e_j = \begin{cases} 0, & \text{if } i \neq j; \\ \sum_k a_{ik} e_k, & \text{if } i = j, \end{cases}$$

then this algebra is called an *evolution algebra* [9]. The basis is called a natural basis. We denote by $A = (a_{ij})$ the matrix of the structural constants of the evolution algebra E .

The following properties are known [9]:

- (1) Evolution algebras are not associative, in general.
- (2) Evolution algebras are commutative, flexible.
- (3) Evolution algebras are not power-associative, in general.
- (4) The direct sum of evolution algebras is also an evolution algebra.
- (5) The Kronecker product of evolution algebras is an evolution algebra.

Definition 2.1. Let E be an evolution algebra, and E_1 be a subspace of E . If E_1 has a natural basis $\{e_i \mid i \in \Lambda_1\}$, which can be extended to a natural basis $\{e_j \mid j \in \Lambda\}$ of E , E_1 is called an evolution subalgebra, where Λ_1 and Λ are index sets and Λ_1 is a subset of Λ .

Definition 2.2. An element a of an evolution algebra E is called nil if there exists $n(a) \in \mathbb{N}$ such that $(\cdots \underbrace{(a \cdot a) \cdot a}_{n(a)} \cdots a) = 0$. An evolution algebra E is called nil if every element of the algebra is nil.

For an evolution algebra E we introduce the following sequences, $k \geq 1$,

$$E^{[1]} = E^{<1>} = E, \quad E^{[k+1]} = E^{[k]}E^{[k]}, \quad E^{<k+1>} = E^{<k>}E, \\ E^k = \sum_{i=1}^{k-1} E^i E^{k-i}. \quad (2.1)$$

The following inclusions hold

$$E^{<k>} \subseteq E^k, \quad E^{[k+1]} \subseteq E^{2^k}.$$

Since E is a commutative algebra we obtain

$$E^k = \sum_{i=1}^{\lfloor k/2 \rfloor} E^i E^{k-i},$$

where $\lfloor x \rfloor$ denotes the integer part of x .

Definition 2.3. An evolution algebra E is called right nilpotent if there exists some $s \in \mathbb{N}$ such that $E^{<s>} = 0$. The smallest s such that $E^{<s>} = 0$ is called the index of right nilpotency.

Definition 2.4. An evolution algebra E is called nilpotent if there exists some $n \in \mathbb{N}$ such that $E^n = 0$. The smallest n such that $E^n = 0$ is called the index of nilpotency.

In [1], it is proved that the notions of nilpotent and right nilpotent are equivalent.

Definition 2.5. An algebra \mathbf{A} is called solvable if there exists some $t \in \mathbb{N}$ such that $\mathbf{A}^{[t]} = 0$. The smallest t such that $\mathbf{A}^{[t]} = 0$ is called the index of solvability.

Dibaric algebras. A character for an algebra \mathbf{A} is a non-zero multiplicative linear form on \mathbf{A} , that is, a non-zero algebra homomorphism from \mathbf{A} to \mathbb{R} [5]. A pair (\mathbf{A}, σ) consisting of an algebra \mathbf{A} and a character σ on \mathbf{A} is called a *baric algebra*.

As usual, the algebras considered in mathematical biology are not baric.

Definition 2.6. ([7,11]) Let $\mathfrak{A} = \langle w, m \rangle_{\mathbb{R}}$ denote a two-dimensional commutative algebra over \mathbb{R} with multiplicative table

$$w^2 = m^2 = 0, \quad wm = \frac{1}{2}(w + m).$$

Then \mathfrak{A} is called the sex differentiation algebra.

It is clear that $\mathfrak{A}^2 = \langle w + m \rangle_{\mathbb{R}}$ is an ideal of \mathfrak{A} which is isomorphic to the field \mathbb{R} . Hence the algebra \mathfrak{A}^2 is a baric algebra.

Definition 2.7. ([7]) An algebra \mathbf{A} is called dibaric if it admits a homomorphism onto the sex differentiation algebra \mathfrak{A} .

3. Nilpotent evolution algebras

In [3] it is proved that the notions of nil and right nilpotency are equivalent for evolution algebras. Moreover, the matrix A of structural constants for such algebras has upper (or lower, up to permutation of basis elements of the algebra) triangular form.

Let evolution algebra E be a right nilpotent algebra, then it is evident that E is a nil algebra. Therefore for the related matrix $A = (a_{ij})_{i,j=1}^n$, we have

$$a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} = 0,$$

for any $k \in \{1, 2, \dots, n\}$ and arbitrary $i_1, i_2, \dots, i_k \in \{1, 2, \dots, n\}$ with $i_p \neq i_q$ for $p \neq q$ [3].
The following results are known:

Theorem 3.1. ([3]) The following statements are equivalent for an n -dimensional evolution algebra E :

(a) The matrix corresponding to E can be written as

$$\widehat{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix};$$

(b) E is a right nilpotent algebra;

(c) E is a nil algebra.

Lemma 3.2. Let E be a finite-dimensional evolution algebra and $E^j, j \geq 1$, the evolution subalgebras of E defined in (2.1). Then

$$E^{2^k+i} = E^{2^{k+1}}, \quad i = 1, \dots, 2^k, \quad k = 0, 1, \dots \quad (3.1)$$

Proof. We shall use mathematical induction. We have $E^1 = E$, $E^2 = EE$, and for $k = 1$,

$$E^3 = EE^2 = E^2E^2, \quad E^4 = EE^3 + E^2E^2 = E^2E^2 = E^3.$$

Assume for k the equalities (3.1) are true. We shall prove for $k + 1$. Using $E^{i+j} \subset E^i, E^{i+j}E^i = E^{i+j}E^{i+j}$ and assumptions of the induction we get

$$\begin{aligned}
 E^{2^{k+1}+i} &= \sum_{j=1}^{2^k+\lfloor i/2 \rfloor} E^j E^{2^{k+1}+i-j} = \sum_{j=1}^{2^k+\lfloor i/2 \rfloor} E^j E^{2^k+i} \\
 &= \sum_{j=1}^{2^k+\lfloor i/2 \rfloor} E^j E^{2^{k+1}} = E E^{2^{k+1}} = E^{2^{k+1}} E^{2^{k+1}}. \quad \square
 \end{aligned}$$

Lemma 3.3. *If for an evolution algebra E there exists $s \in \mathbb{N}$ such that $E^{2^s+1} = E^{2^{s+1}+1}$, then $E^k = E^{2^s+1}$ for any $k = 2^s + 1, 2^s + 2, \dots, 2^{s+2} + 1$.*

Proof. We have

$$E^{2^s+1} \supseteq E^{2^s+2} \supseteq \dots \supseteq E^{2^{s+1}+1}.$$

Hence by condition of the lemma we get $E^k = E^{2^s+1}$ for any $k = 2^s + 1, 2^s + 2, \dots, 2^{s+1} + 1$. It remains to prove the equality for $k = 2^{s+1} + i + 1, i = 1, \dots, 2^{s+1}$. We have

$$E^{2^{s+1}+1} = \sum_{j=1}^{2^s} E^j E^{2^{s+1}-j+1} = E E^{2^s+1} = E^{2^s+1}.$$

For $i = 1$ using the above obtained equalities we get

$$E^{2^{s+1}+2} = \sum_{j=1}^{2^s+1} E^j E^{2^{s+1}-j+2} = E E^{2^s+1} = E^{2^s+1}.$$

Now assume the assertion is true for i and we shall show it for $i + 1$.

$$E^{2^{s+1}+i+2} = \sum_{j=1}^{2^s+1+\lfloor i/2 \rfloor} E^j E^{2^{s+1}+i-j+2}.$$

Since $2^s + 1 < 2^{s+1} + i - j + 2 \leq 2^{s+1} + i + 1$ for any $j = 1, 2, \dots, 2^s + 1 + \lfloor i/2 \rfloor$, using the assumption of the induction, we get

$$E^{2^{s+1}+i+2} = \sum_{j=1}^{2^s+1+\lfloor i/2 \rfloor} E^j E^{2^s+1} = E E^{2^s+1} = E^{2^s+1}. \quad \square$$

From this lemma we get the following

Corollary 3.4. *If for an evolution algebra E there exists $s \in \mathbb{N}$ such that $E^{2^s+1} = E^{2^{s+1}+1}$, then $E^k = E^{2^s+1}$ for any $k \geq 2^s + 1$.*

Proof. If the condition of Lemma 3.3 is satisfied for s , then it is satisfied for $s + 1$. So, iterating the lemma we get $E^k = E^{2^s+1}$ for any $k \geq 2^s + 1$. \square

From this corollary it follows that an evolution algebra E satisfying the condition of Lemma 3.2 is not nilpotent.

The following is an example satisfying the condition of Lemma 3.3.

Example 3.5. Fix some $r \in \{2, 3, \dots, n-1\}$ and consider the evolution algebra with the multiplication table

$$e_i^2 = e_{i+1}, \quad i = 1, \dots, r-1; \quad e_i^2 = e_r, \quad i = r, \dots, n.$$

It is easy to see that this algebra satisfies the condition of Lemma 3.3 for some $s \geq r$. In this case, $E^k = \{e_r\}$ for all $k \geq 2^s + 1$.

Theorem 3.6. An n -dimensional nilpotent evolution algebra E has maximal nilpotent index, $2^{n-1} + 1$, if and only if

$$a_{12}a_{23} \dots a_{n-1,n} \neq 0.$$

Proof. Necessity. Assume $a_{12}a_{23} \dots a_{n-1,n} = 0$ then $\dim E^2 \leq n-2$. Since E is nilpotent, by Lemma 3.2, for any k we have $E^{2^k+1} \supsetneq E^{2^{k+1}+1}$. Consequently, $\dim E^{2^k+1} \leq n-2-k$. Hence $E^{2^{n-2}+1} = 0$, i.e., E has not maximal nilpotent index.

Sufficiency was proved in [1]. \square

Let E be an n -dimensional nilpotent evolution algebra with maximal nilpotent index. Then by the following scaling of basis

$$\left\{ \begin{array}{l} e'_1 = a_{12}^{-1/2} a_{23}^{-1/4} \dots a_{n-1,n}^{-(1/2)^{n-1}} e_1 \\ e'_2 = a_{23}^{-1/2} a_{34}^{-1/4} \dots a_{n-1,n}^{-(1/2)^{n-2}} e_2 \\ \dots \\ e'_{n-1} = a_{n-1,n}^{-1/2} e_{n-1} \\ e'_n = e_n \end{array} \right. \quad (3.2)$$

the evolution algebra is isomorphic to an evolution algebra E' with matrix of structural constants

$$A' = \begin{pmatrix} 0 & 1 & a'_{13} & \dots & a'_{1n} \\ 0 & 0 & 1 & \dots & a'_{2n} \\ 0 & 0 & 0 & \dots & a'_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Let E be an n -dimensional nilpotent evolution algebra such that the matrix of structural constants satisfies $a_{i_1 i_1+1} = \dots = a_{i_s i_s+1} = 0$, for some $s = 1, \dots, n-1$. Then omitting all multipliers $a_{i_k i_k+1}$, $k = 1, \dots, s$ in (3.2) one can show that the evolution algebra E is isomorphic to an evolution algebra E' with matrix of structural constants $A' = (a'_{ij})$, with $a'_{i_1 i_1+1} = \dots = a'_{i_s i_s+1} = 0$ and $a'_{i_i+1} = 1$, $i \neq i_1, \dots, i_s$.

The following theorem gives the classification of evolution algebras with matrix of structural constants as A' .

Theorem 3.7. Any finite-dimensional complex evolution algebra with maximal nilpotent index is isomorphic to one of pairwise non-isomorphic algebras with matrix of structural constants

$$\begin{pmatrix} 0 & 1 & a_{13} & \dots & a_{1,n-1} & 0 \\ 0 & 0 & 1 & \dots & a_{2,n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{3,n-1} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where one of non-zero a_{ij} can be chosen equal to 1.

Proof. Assume $\varphi = (\alpha_{ij})_{i,j=1,\dots,n}$ is an isomorphism between evolution algebras E and E' with multiplication tables

$$E : \begin{cases} e_i^2 = e_{i+1} + \sum_{j=i+2}^n a_{ij}e_j, & i = 1, \dots, n-2, \\ e_{n-1}^2 = e_n, \\ e_n^2 = 0. \end{cases}$$

$$E' : \begin{cases} e'_i{}^2 = e'_{i+1} + \sum_{j=i+2}^n b_{ij}e'_j, & i = 1, \dots, n-2, \\ e'_{n-1}{}^2 = e'_n, \\ e'_n{}^2 = 0. \end{cases}$$

We shall use the following lemma.

Lemma 3.8. $\varphi = (\alpha_{ij})$ is defined by

$$\alpha_{ii} = \alpha_{11}^{2^{i-1}}, \quad \alpha_{11} \neq 0; \quad \alpha_{ij} = 0, \quad i \neq j, \quad j \neq n; \quad \alpha_{in} \in \mathbb{C}. \quad (3.3)$$

Proof. For $i = n$ consider

$$0 = \varphi(e_n^2) = e_n'^2 = \sum_{j=1}^{n-2} \alpha_{nj}^2 \left(e_{j+1} + \sum_{s=j+2}^n a_{js}e_s \right) + \alpha_{n,n-1}^2 e_n.$$

From this equality it follows that $\alpha_{nj} = 0, j = 1, \dots, n-1$.

For $i = n-1$ we have

$$e'_n = \alpha_{nn}e_n = \varphi(e_{n-1}^2) = \sum_{i=1}^{n-1} \alpha_{n-1,i}^2 e_i^2 = \sum_{i=1}^{n-2} \alpha_{n-1,i}^2 \left(e_{i+1} + \sum_{j=i+2}^n a_{ij}e_j \right) + \alpha_{n-1,n-1}^2 e_n.$$

From this equalities we get

$$\alpha_{n-1,i} = 0, \quad i = 1, \dots, n-2, \quad \alpha_{nn} = \alpha_{n-1,n-1}^2.$$

Similarly from $e'_{n-1}e_j = 0$, we get $\alpha_{j,n-1} = 0$. Hence $\alpha_{n-1,j} = \alpha_{j,n-1} = 0, j \neq n-1$. Assume (3.3) is true for $i \geq k$ and $j \neq n$. We shall check it for $i = k-1$. By the assumptions we have

$$\varphi(e_{k-1}^2) = \sum_{s=1}^{k-1} \alpha_{k-1,s}^2 e_s^2 = \sum_{s=1}^{k-1} \alpha_{k-1,s}^2 \left(e_{s+1} + \sum_{j=s+2}^n a_{sj} e_j \right).$$

On the other hand

$$\begin{aligned} \varphi(e_{k-1}^2) &= e_{k-1}'^2 = e_k' + \sum_{j=k+1}^n b_{k-1,j} e_j' \\ &= \alpha_{kk} e_k + \alpha_{kn} e_n + \sum_{j=k+1}^n b_{k-1,j} (\alpha_{jj} e_j + \alpha_{jn} e_n). \end{aligned}$$

Hence $\alpha_{k-1,j} = 0, j = 1, \dots, k-2$. Using the above equalities we get

$$\alpha_{k-1,k-1}^2 e_k + \left(\sum_{j=k+1}^n a_{k-1,j} e_j \right) \alpha_{k-1,k-1}^2 = \alpha_{kk} e_k + \alpha_{kn} e_n + \sum_{j=k+1}^n b_{k-1,j} (\alpha_{jj} e_j + \alpha_{jn} e_n) \quad (3.4)$$

which gives $\alpha_{kk} = \alpha_{k-1,k-1}^2$.

For $s < k$ using assumptions, we get

$$\begin{aligned} 0 &= \varphi(e_{k-1} e_s) = e_{k-1}' e_s' = (\alpha_{k-1,k-1} e_{k-1} + \alpha_{k-1,n} e_n) \left(\sum_{t=1}^k \alpha_{st} e_t + \alpha_{sn} e_n \right) \\ &= \alpha_{k-1,k-1} \alpha_{s,k-1} e_{k-1}^2 = \alpha_{k-1,k-1} \alpha_{s,k-1} \left(e_k + \sum_{j=k+1}^n a_{kj} e_j \right). \end{aligned}$$

Then $\alpha_{s,k-1} = 0$. Hence

$$\alpha_{j,k-1} = 0, \quad j < k-1, \quad \alpha_{k-1,j} = 0, \quad j \neq k-1. \quad \square$$

Now we shall continue the proof of theorem. From the equality (3.4) we get

$$\begin{aligned} \alpha_{k-1,k-1}^2 a_{k-1,n} e_n + \left(\sum_{j=k+1}^{n-1} a_{k-1,j} e_j \right) \alpha_{k-1,k-1}^2 \\ = \sum_{j=k+1}^{n-1} b_{k-1,j} \alpha_{jj} e_j + \left(\alpha_{kn} + 2b_{k-1,n} \alpha_{nn} + \sum_{j=k+1}^{n-1} b_{k-1,j} \alpha_{jn} \right) e_n. \end{aligned}$$

Consequently,

$$\begin{aligned} \alpha_{k-1,k-1}^2 a_{k-1,n} &= \alpha_{kn} + 2b_{k-1,n} \alpha_{nn} + \sum_{j=k+1}^{n-1} b_{k-1,j} \alpha_{jn}, & k = 2, \dots, n-1. \\ \alpha_{k-1,k-1}^2 a_{k-1,j} &= \alpha_{jj} b_{k-1,j}, & j = k+1, \dots, n-1. \end{aligned}$$

From these formulas using $\alpha_{kk} = \alpha_{11}^{2^{k-1}}$, we obtain

$$\begin{aligned} \alpha_{11}^{2^{k-1}} a_{k-1,n} &= \alpha_{kn} + 2b_{k-1,n} \alpha_{11}^{2^{n-1}} + \sum_{j=k+1}^{n-1} b_{k-1,j} \alpha_{jn}, \quad k = 2, \dots, n-1. \\ \alpha_{11}^{2^{k-1}} a_{k-1,j} &= \alpha_{11}^{2^{j-1}} b_{k-1,j}, \quad j = k+1, \dots, n-1. \end{aligned} \quad (3.5)$$

From the second equation of the system (3.5) we obtain

$$b_{k-1,j} = \alpha_{11}^{2^{k-1}-2^{j-1}} a_{k-1,j}, \quad k = 2, \dots, n-1, \quad j = k+1, \dots, n-1. \quad (3.6)$$

Using (3.6), in order to have $b_{k-1,n} = 0$, in the first equation of (3.5) we put

$$\begin{aligned} \alpha_{kn} &= \alpha_{11}^{2^{k-1}} a_{k-1,n} - \sum_{j=k+1}^{n-1} b_{k-1,j} \alpha_{jn} \\ &= \alpha_{11}^{2^{k-1}} a_{k-1,n} - \sum_{j=k+1}^{n-1} \alpha_{11}^{2^{k-1}-2^{j-1}} a_{k-1,j} \alpha_{jn}, \quad k = 2, \dots, n-1. \end{aligned}$$

If there exist k_0, j_0 such that $a_{k_0,j_0} \neq 0$ then taking $\alpha_{11} = a_{k_0,j_0}^{\frac{-1}{2^{k_0}-2^{j_0-1}}}$, we have

$$b_{k_0,j_0} = 1, \quad k_0 = 1, \dots, n-2. \quad \square$$

Remark 3.9. Note that the evolution algebras of Theorem 3.7 are also algebras of maximal index of solvability and maximal index of right nilpotency.

Example 3.10. Any four-dimensional complex evolution algebra with maximal nilpotent index is isomorphic to one of the algebras with the following matrix of structural constants

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Any five-dimensional complex evolution algebra with maximal nilpotent index is isomorphic to one of the algebras with the following matrix of structural constants

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & b & 0 \\ 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $b, d \in \mathbb{C}$.

Proposition 3.11. Let E be an evolution algebra with matrix of structural constants

$$A = \begin{pmatrix} 0 & 1 & a_{13} & \dots & a_{1,m+1} & a_{1,m+2} & \dots & a_{1,n-1} & a_{1n} \\ 0 & 0 & 1 & \dots & a_{2,m+1} & a_{2,m+2} & \dots & a_{2,n-1} & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3,m+1} & a_{3,m+2} & \dots & a_{3,n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{m-1,m+1} & a_{m-1,m+2} & \dots & a_{m-1,n-1} & a_{m-1,n} \\ 0 & 0 & 0 & \dots & a_{m,m+1} & a_{m,m+2} & \dots & a_{m,n-1} & a_{mn} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

with $a_{i,i+1} = 1$, for $i \neq m$, $i = 1, \dots, n-1$, $a_{m,m+1} = 0$ and $a_{m,m+2} \neq 0$ or $a_{m-1,m+1} \neq 0$. Then

$$E^{2^{k+1}} = E^{2^{k+2}} = \dots = E^{2^{k+1}} = \begin{cases} \langle e_{k+1}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle, & \text{if } k \leq m-2, \\ \langle e_{k+3}, \dots, e_n \rangle, & \text{if } m-1 \leq k \leq n-3, \end{cases} \quad (3.7)$$

and $E^{2^{n-2}+1} = 0$, i.e., its nilpotent index is $2^{n-2} + 1$.

Proof. We have

$$e_i^2 = e_{i+1} + \sum_{j=i+2}^n a_{ij} e_j, \quad i = 1, \dots, m-1, m+2, \dots, n-1; \quad e_m^2 = \sum_{j=m+2}^n a_{mj} e_j.$$

$$E^2 = \langle e_1^2, e_2^2, \dots, e_n^2 \rangle.$$

It is easy to see that $\langle e_i^2, i = m+1, \dots, n \rangle = \langle e_i, i = m+2, \dots, n \rangle$ and $e_1^2, e_2^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n$ are linearly independent. Thus

$$E^2 = \langle e_1^2, e_2^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle.$$

$$E^3 = EE^2 = \langle e_i^2 e_j, e_k^2 \mid i = 1, \dots, m-1, j = i+1, \dots, n, k = m+2, \dots, n \rangle.$$

We have

$$e_i^2 e_j = \begin{cases} e_j^2, & \text{if } j = i+1, \\ a_{ij} e_j^2, & \text{if } j > i+1. \end{cases}$$

In case $a_{m,m+2} \neq 0$, from $e_m^2 = a_{m,m+2} e_{m+2} + \sum_{j=m+3}^n a_{mj} e_j$, we obtain $e_{m+2} \in E^3$.

If $a_{m-1,m+1} \neq 0$, then since $e_{m-1}^2 e_{m+1} = a_{m-1,m+1} e_{m+1}^2$, we conclude that $e_{m+2} \in E^3$. Thus we obtain

$$E^3 = \langle e_2^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle.$$

Now we shall compute $E^4 = EE^3 + E^2E^2$. Similarly as in the case EE^2 , we get

$$\begin{aligned} EE^3 &= \langle e_i^2 e_j, e_k^2 \mid i = 2, \dots, m-1, j = i+1, \dots, n, k = m+2, \dots, n \rangle \\ &= \langle e_3^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle. \\ E^2E^2 &= \langle e_i^2, e_j^2 e_k, e_p^2 e_q^2 \mid i, k = m+2, \dots, n, j, p, q = 1, \dots, m-1 \rangle. \end{aligned}$$

It is easy to see that

$$\begin{aligned} e_i^2 e_k &= a_{ik} e_k^2, \quad i = 1, \dots, m-1, k = m+2, \dots, n. \\ e_p^2 e_q^2 &= \begin{cases} e_{p+1}^2, & \text{if } p = q = 1, \dots, m-1, \\ a_{p,q+1} e_{q+1}^2 + \sum_{j=q+2}^n a_{pj} a_{qj} e_j^2, & \text{if } p < q. \end{cases} \end{aligned}$$

Using these equalities, we obtain

$$E^2E^2 = \langle e_2^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle = E^3.$$

Consequently, $E^4 = E^3$.

Lets assume that the equalities (3.7) are true for k , we shall prove it for $k+1$.

$$\begin{aligned} E^{2^{k+1}+1} &= EE^{2^{k+1}} + E^2E^{2^{k+1}-1} + \dots + E^{2^k}E^{2^k+1} = (E + E^2 + \dots + E^{2^k})E^{2^k+1} \\ &= EE^{2^k+1} = \langle e_1, \dots, e_n \rangle \langle e_{k+1}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle \\ &= \langle e_{k+2}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle. \end{aligned}$$

We also have

$$\begin{aligned} E^{2^{k+2}} &= EE^{2^{k+2}-1} + E^2E^{2^{k+2}-2} + \dots + E^{2^{k+1}}E^{2^{k+1}} \supseteq E^{2^{k+1}}E^{2^{k+1}} \\ &= \langle e_{k+2}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \langle e_{k+2}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle &= E^{2^{k+1}+1} \supseteq E^{2^{k+1}+2} \supseteq \dots \supseteq E^{2^{k+2}} \\ &\supseteq \langle e_{k+2}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle. \end{aligned}$$

Consequently

$$E^{2^{k+1}+1} = E^{2^{k+1}+2} = \dots = E^{2^{k+2}} = \langle e_{k+2}^2, \dots, e_{m-1}^2, e_{m+2}, \dots, e_n \rangle.$$

This gives the formula (3.7). Using the formula, we get $E^{2^{n-2}+1} = 0$. \square

The following example shows that the nilpotent index may be $2^{n-s} + 1$ for any $s = 1, \dots, n-1$.

Example 3.12. Consider an n -dimensional nilpotent evolution algebra E with matrix of structural constants $A = (a_{ij})$ satisfying

$$\begin{aligned} a_{ikj} &= 0, & j &= 1, 2, \dots, n-1, & k &= 1, \dots, s-1; \\ a_{jik} &= 0, & j &= 1, 2, \dots, n, & k &= 1, \dots, s-1, \end{aligned}$$

where $1 \leq i_1 < i_2 < \dots < i_{s-1} \leq n-2$, $s < n$. Then the nilpotent index of E is $2^{n-s} + 1$.

The following proposition generalizes Example 3.12.

Proposition 3.13. Let E be an n -dimensional evolution algebra with matrix of structural constants $A = (a_{ij})$ such that for some $s < n-1$ and $1 \leq i_1 < i_2 < \dots < i_{s-1} \leq n-2$ satisfies

$$\begin{aligned} a_{jik} &= a_{ikj} = 0, & \text{for all } j \notin \{i_1, \dots, i_{s-1}, n\}, & k = 1, \dots, s-1, \\ a_{ik i_{k+1}} &= 1, & \text{for all } k = 1, \dots, s-1. \end{aligned}$$

i.e., A has the following form

$$\begin{pmatrix} 0 & 1 & a_{13} & \dots & a_{1i_1-1} & 0 & a_{1i_1+1} & \dots & a_{1i_{s-1}-1} & 0 & a_{1i_{s-1}+1} & \dots & a_{1n} \\ 0 & 0 & 1 & \dots & a_{2i_1-1} & 0 & a_{2i_1+1} & \dots & a_{2i_{s-1}-1} & 0 & a_{2i_{s-1}+1} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & a_{i_1-1i_1+1} & \dots & a_{i_1-1i_{s-1}-1} & 0 & a_{i_1-1i_{s-1}+1} & \dots & a_{i_1-1n} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & a_{i_1i_{s-1}} & 0 & \dots & a_{i_1n} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & a_{i_1+1i_{s-1}-1} & 0 & a_{i_1+1i_{s-1}+1} & \dots & a_{i_1+1n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then the nilpotent index of E is equal to $2^{\max\{s-1, n-s\}} + 1$.

Proof. The evolution algebra E can be written as $E = A + B$, where $A = \langle e_i \mid i \neq i_1, \dots, i_{s-1} \rangle$, $B = \langle e_{i_1}, e_{i_2}, \dots, e_{i_{s-1}} \rangle$. It is easy to see that $AB = 0$, this implies $A^i B^j = 0$, $i, j = 1, 2, \dots$. Consequently, $E^k = A^k + B^k$. Using similar arguments as above (for computation of the maximal nilpotent index) one can see that the nilpotent index of A is $2^{n-s} + 1$ and the nilpotent index of B is $2^{s-1} + 1$. This completes the proof. \square

Remark 3.14. By [1, Proposition 4.7] if E is an n -dimensional nilpotent evolution algebra with index of nilpotency not equal to $2^{n-1} + 1$, then it is not greater than $2^{n-2} + 1$. Moreover, in the paper [1], there is an example of evolution algebra with nilpotent index $3 \cdot 2^{k-4} + 1$, where $4 \leq k \leq n$. Therefore it is interesting to know all possible values of the nilpotent index for nilpotent evolution algebras. This problem is difficult, but for small values of n one can do exact calculations. For example, if $n = 3$ then

the nilpotent index can be 2,3,5. For $n = 4$ all possible values of the nilpotent index are 2,3,4,5,9. The 4-dimensional evolution algebra E , with the following matrix

$$A = \begin{pmatrix} 0 & 1 & b & c \\ 0 & 0 & 0 & -b^2f \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

has nilpotent index 4, where $bf \neq 0$. This case is interesting since it has not the form $2^k + 1$.

4. Dibaric algebras

In this section we will study some dibaricity properties of arbitrary algebras and evolution algebras.

Theorem 4.1. Any finite-dimensional nilpotent evolution algebra E is not dibaric.

Proof. Assume $\varphi = (b_{ij})_{i=1,\dots,n; j=1,2}$ is a homomorphism $\varphi: E \rightarrow \mathfrak{A}$. We shall use the following \square

Lemma 4.2. For any $i, j = 1, \dots, n$, we have

$$b_{i1}b_{j2} = b_{i2}b_{j1} = 0. \quad (4.1)$$

Proof. Without loss of generality we assume $i \leq j$ and use mathematical induction. Let C_k denote all cases of (4.1) with $2n - (i + j) + 1 = k$. For $k = 1$, i.e., $i = j = n$, from $\varphi(e_n^2) = 0$ we get

$$b_{n1}b_{n2} = 0. \quad (4.2)$$

For $k = 2$ we have $i = n - 1$ and $j = n$. We get

$$0 = \varphi(e_{n-1}e_n) = \frac{1}{2}(b_{n-1,1}b_{n2} + b_{n-1,2}b_{n1})(m + w).$$

This by (4.2) gives

$$b_{n-1,1}b_{n2} = b_{n-1,2}b_{n1} = 0.$$

Assuming that C_k holds, we have to prove C_{k+1} , that is, Eq. (4.1), for any $i, j = 1, \dots, n$, $i \leq j$, which satisfy $2n - (i + j) + 1 = k + 1$.

Case $i < j$: From $\varphi(e_i e_j) = \frac{1}{2}(b_{i1}b_{j2} + b_{i2}b_{j1})(m + w) = 0$ we get

$$b_{i1}b_{j2} + b_{i2}b_{j1} = 0. \quad (4.3)$$

By assumptions we have $i < j$, $2n - 2j + 1 \leq k$ and $b_{j1}b_{j2} = 0$. This by (4.3) gives (4.1).

Case $i = j$: Consider

$$\begin{aligned} \varphi(e_i^2) &= \varphi\left(\sum_{s=i+1}^n a_{is}e_s\right) = \sum_{s=i+1}^n a_{is}\varphi(e_s) = \left(\sum_{s=i+1}^n a_{is}b_{s1}\right)m + \left(\sum_{s=i+1}^n a_{is}b_{s2}\right)w \\ &= \varphi(e_i)^2 = \frac{1}{2}b_{i1}b_{i2}(m + w). \end{aligned}$$

Consequently,

$$\begin{cases} 2 \sum_{s=i+1}^n a_{is} b_{s1} = b_{i1} b_{i2}, \\ 2 \sum_{s=i+1}^n a_{is} b_{s2} = b_{i1} b_{i2}. \end{cases} \quad (4.4)$$

Since $2n - (i + s) + 1 \leq k$ for any $s = i + 1, i + 2, \dots, n$, by the assumption of the induction we get

$$b_{s1} b_{i2} = b_{s2} b_{i1} = 0. \quad (4.5)$$

Now, multiplying both sides of the first equation of (4.4) by b_{i2} , then by (4.5) we get $b_{i1} b_{i2} = 0$. \square

Now we shall continue the proof of the theorem. By Lemma 4.2, if there exists i_0 such that $b_{i_0 1} \neq 0$ then $b_{j2} = 0$ for all j , i.e., $\varphi(e_i) = b_{i1} m$. Such φ is not onto.

The following result gives a sufficient condition for an arbitrary algebra to be non dibaric.

Theorem 4.3. Let \mathbf{A} be a finite-dimensional real algebra with table of multiplication $e_i e_j = \sum_k a_{ij}^k e_k$,

where $(a_{ij}^k)_{i,j,k=1,\dots,n}$ is the matrix of structural constants, and such that the matrix $A = (a_{ii}^k)_{i,k=1,\dots,n}$ has $\det(A) \neq 0$. Then \mathbf{A} is not dibaric.

Proof. Assume $\varphi = (\alpha_{ij})_{i=1,\dots,n; j=1,2}$ is a homomorphism $\varphi: \mathbf{A} \rightarrow \mathfrak{A}$. We have

$$\begin{aligned} \varphi(e_i^2) &= \sum_{s=1}^n a_{ii}^s (\alpha_{s1} m + \alpha_{s2} w). \\ \varphi(e_i^2) &= \frac{1}{2} (\alpha_{i1} \alpha_{i2}) (m + w). \end{aligned}$$

Consequently,

$$\begin{cases} 2 \sum_{s=1}^n a_{ii}^s \alpha_{s1} = \alpha_{i1} \alpha_{i2}, \\ 2 \sum_{s=1}^n a_{ii}^s \alpha_{s2} = \alpha_{i1} \alpha_{i2}. \end{cases} \quad (4.6)$$

Subtracting from first equation of the system (4.6) the second one, we obtain

$$\sum_{s=1}^n a_{ii}^s (\alpha_{s1} - \alpha_{s2}) = 0, \quad i = 1, \dots, n. \quad (4.7)$$

If $\det(A) \neq 0$ we get from the system (4.7) that $\alpha_{i1} = \alpha_{i2}$ for all i . Hence $\varphi(e_i) = \alpha_{i1} (m + w)$, but such φ is not onto. \square

Remark 4.4. Non dibaric algebras given by Theorem 4.1 show that the condition $\det(A) \neq 0$ is not necessary to be non dibaric.

Corollary 4.5. Let E be an evolution algebra with matrix A of structural constants. If $\det(A) \neq 0$ then E is not dibaric.

Definition 4.6. ([4]). For a given algebra \mathbf{A} , a pair (f, g) of linear forms $f: \mathbf{A} \rightarrow \mathbb{R}, g: \mathbf{A} \rightarrow \mathbb{R}$ is called bq-homomorphism if

$$f(xy) = g(xy) = \frac{f(x)g(y) + f(y)g(x)}{2} \text{ for any } x, y \in \mathbf{A}. \quad (4.8)$$

Note that if $f = g$ then condition (4.8) implies that f is a homomorphism.

A bq-homomorphism (f, g) is called non-zero if both f and g are non-zero.

Theorem 4.7. ([4]). An algebra \mathbf{A} is dibaric if and only if there is a non-zero bq-homomorphism (f, g) .

In case of Theorem 4.7 the homomorphism $\varphi: \mathbf{A} \rightarrow \mathfrak{A}$ has the form $\varphi(x) = f(x)m + g(x)w$. Lets denote

$$V_n = \{x \in \mathbf{A} : \varphi(x^n) = 0\}.$$

Proposition 4.8. For any $n \geq 3$, we have $V_n = V_3$.

Proof. Using $(m + w)^n = m + w$ and mathematical induction, one can prove the following formula

$$\varphi(x^n) = \frac{f(x)g(x)}{2^{n-1}}(f(x) + g(x))^{n-2}(m + w).$$

From this formula, for $n \geq 3$, we get $\varphi(x^n) = 0$ if and only if $f(x)g(x)(f(x) + g(x)) = 0$. This completes the proof. \square

Remark 4.9. Any solvable algebra \mathbf{A} is not dibaric. Indeed, there is an homomorphism φ onto \mathfrak{A} . It is easy to see that

$$\varphi(\mathbf{A}^{[k]}) = \varphi(\mathbf{A})^{[k]} = \mathfrak{A}^{[k]} = \mathfrak{A}^2 \cong \mathbb{R}, \quad \text{for all } k \geq 2. \quad (4.9)$$

By the solvability of \mathbf{A} there exists k such that $\mathbf{A}^{[k]} = 0$. Then from (4.9) we get $0 = \mathfrak{A}^2 \cong \mathbb{R}$, this is a contradiction.

Two-dimensional dibaric evolution algebras. In this subsection we find a criterion for two-dimensional real evolution algebra to be dibaric.

Let the two-dimensional real evolution algebra E be given by the matrix of structural constants

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proposition 4.10. The two-dimensional real evolution algebra E is dibaric if and only if one of the following conditions hold

- (1) $b = d = 0$ and $ac < 0$;
- (2) $b \neq 0, ad = bc, D \geq 0$ and $B^2 + C^2 \neq 0$,

where $D = (8a - 1)^2 - 32(bd + a^2)$, $B = 4a^2 + 4bd - a + a\sqrt{D}$ and $C = 4a^2 + 4bd - a - a\sqrt{D}$.

Proof. Let $\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a homomorphism. It is onto if $\alpha\delta \neq \gamma\beta$. Moreover, φ must satisfy

$$\begin{cases} 2(a\alpha + b\gamma) = \alpha\beta \\ 2(a\beta + b\delta) = \alpha\beta \\ 2(c\alpha + d\gamma) = \gamma\delta \\ 2(c\beta + d\delta) = \gamma\delta \\ \alpha\delta + \beta\gamma = 0 \end{cases} \quad (4.10)$$

The proof follows from the elementary analysis of the system (4.10). \square

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