Algorithm for Testing the Leibniz Algebra Structure

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Abstract. Given a basis of a vector space V over a field $\mathbb{K}$ and a multiplication table which defines a bilinear map on V, we develop a computer program on Mathematica which checks if the bilinear map satisfies the Leibniz identity, that is, if the multiplication table endows V with a Leibniz algebra structure. In case of a positive answer, the program informs whether the structure corresponds to a Lie algebra or not, that is, if the bilinear map is skew-symmetric or not.

The algorithm is based on the computation of a Gröbner basis of an ideal, which is employed in the construction of the universal enveloping algebra of a Leibniz algebra. Finally, we describe a program in the NCAlgebra package which permits the construction of Gröbner bases in non commutative algebras.

1 Introduction

A classical problem in Lie algebras theory is to know how many different (up to isomorphisms) finite-dimensional Lie algebras are for each dimension [11,15].

The classical methods to obtain the classifications essentially solve the system of equations given by the bracket laws, that is, for a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ with basis $\{a_1, \ldots, a_n\}$, the bracket is completely determined by the scalars $c_{ij}^k \in \mathbb{K}$ such that

$$[a_i, a_j] = \sum_{k=1}^{n} c_{ij}^k a_k \tag{1}$$

so that the Lie algebra structure is determined by means of the computation of the structure constants c_{ij}^k. In order to reduce the system given by (1) for $i, j \in \{1, 2, \ldots, n\}$, different invariants as center, derived algebra, nilindex, nilradical, Levi subalgebra, Cartan subalgebra, etc., are used. Nevertheless, a new approach by using Gröbner bases techniques is available [9,10].

On the other hand, in 1993, J.-L. Loday [17] introduced a non-skew symmetric generalization of Lie algebras, the so called Leibniz algebras. They are $\mathbb{K}$-vector

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spaces $\mathfrak{g}$ endowed with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the Leibniz relation holds

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \text{for all} \;\; x,y,z \in \mathfrak{g}\,. \tag{2}$$

When the bracket satisfies $[x,x] = 0$ for all $x \in \mathfrak{g}$, then the Leibniz identity (2) becomes the Jacobi identity, and so a Leibniz algebra is a Lie algebra. From the beginning, the classification problem of finite-dimensional Leibniz algebras is present in a lot of papers [1,2,3,4,5,6,8]. Nevertheless, the space of solutions of the system given by the structure constants becomes very hard to compute, especially for dimensions greater than 3 because the system has new equations coming from the non-skew symmetry of the bracket. In these situations, the literature only collects the classification of specific classes of algebras (solvable, nilpotent, filiform, etc.). In order to simplify the problem, new techniques applying Gröbner bases methods are developed [14]. However, the space of solutions is usually huge and two kind of problems can occur in applications:

1. Given a multiplication table of an algebra for a fixed dimension n, how can we know if it corresponds to a Leibniz algebra structure among the ones obtained by the classification?

2. Since the classification provides a lot of isomorphism classes, how can we be sure that the classification is well done?

The aim of the present paper is to give an answer to the following question: Given a multiplication table of an algebra for a fixed dimension n, how can we know if it corresponds to a Leibniz algebra structure? We present a computer program in NCAlgebra [12] (a package running under Mathematica which permits the construction of Gröbner bases in non commutative algebras) that implements the algorithm to test if a multiplication table for a fixed dimension n corresponds to a Leibniz algebra structure. In case of a positive answer, the program distinguishes between Lie and non-Lie algebras. The algorithm is based on the computation of a Gröbner basis of an ideal, which is employed in the construction of the universal enveloping algebra UL($\mathfrak{g}$) [19] of a Leibniz algebra $\mathfrak{g}$ over a field $\mathbb{K}$ with basis $\{a_1, a_2, \ldots, a_n\}$.

In order to do this, we consider the ideal $J =< \{-\Phi(r_{[a_i,a_j]}) + x_i x_j - x_j x_i, -\Phi(l_{[a_i,a_j]}) + y_i x_j - x_j y_i, i,j = 1,\ldots,n\} >$ of the free associative non-commutative unitary algebra $\mathbb{K} < y_1,\ldots,y_n,x_1,\ldots,x_n >$, where $\Phi : T(\mathfrak{g}^l \oplus \mathfrak{g}^r) \to \mathbb{K} < y_1,\ldots,y_n,x_1,\ldots,x_n >$ is the isomorphism given by $\Phi(l_{a_i}) = y_i, \Phi(r_{a_i}) = x_i$, being $\mathfrak{g}^l$ and $\mathfrak{g}^r$ two copies of $\mathfrak{g}$, and an element $x \in \mathfrak{g}$ corresponds to the elements l_x and r_x in the left and right copies, respectively.

Then $(\mathfrak{g}, [-,-])$ is a Leibniz algebra if and only if the Gröbner basis corresponding to the ideal J with respect to any monomial order does not contain linear polynomials in the variables $y_1,\ldots,y_n$.

We also obtain that $(\mathfrak{g}, [-,-])$ is a Lie algebra if and only if the Gröbner basis with respect to any monomial order of the ideal $J' =< \{-\Phi(r_{[a_i,a_j]}) + x_i x_j - x_j x_i, i,j = 1,\ldots,n\} >$ does not contain linear polynomials in the variables $x_1,\ldots,x_n$.

2 On Leibniz Algebras

Definition 1. *A Leibniz algebra* $\mathfrak{g}$ *is a* $\mathbb{K}$*-vector space equipped with a bilinear map* $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying the Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \textit{for all} \;\; x,y,z \in \mathfrak{g}\,. \tag{2}$$

When the bracket satisfies $[x,x] = 0$ for all $x \in \mathfrak{g}$, then the Leibniz identity (2) becomes the Jacobi identity; so a Leibniz algebra is a Lie algebra. Hence, there is a canonical inclusion functor from the category **Lie** of Lie algebras to the category **Leib** of Leibniz algebras. This functor has as left adjoint the Liezation functor which assigns to a Leibniz algebra $\mathfrak{g}$ the Lie algebra $\mathfrak{g}_{\mathrm{Lie}} = \mathfrak{g}/\mathfrak{g}^{\mathrm{ann}}$, where $\mathfrak{g}^{\mathrm{ann}} = \langle \{[x,x], x \in \mathfrak{g}\} \rangle$.

Example 1.

1. Lie algebras.
2. Let A be a K-associative algebra equipped with a K-linear map $D : A \to A$ satisfying

   $$D(a(Db)) = DaDb = D((Da)b), \quad \text{for all} \;\; a,b \in A\,. \tag{3}$$

   Then A with the bracket $[a,b] = aDb - Dba$ is a Leibniz algebra.
   If $D = \mathrm{Id}$, we obtain the Lie algebra structure associated to an associative algebra. If D is an idempotent algebra endomorphism ($D^2 = D$) or D is a derivation of square zero ($D^2 = 0$), then D satisfies equation (3) and the bracket gives rise to a structure of non-Lie Leibniz algebra.
3. Let D be a dialgebra [18]. Then $(D,[-,-])$ is a Leibniz algebra with respect to the bracket defined by $[x,y] = x \dashv y - y \vdash x, x,y \in D$.
4. Let $\mathfrak{g}$ be a differential Lie algebra, then $(\mathfrak{g},[-,-]_d)$ with $[x,y]_d := [x,dy]$ is a non-Lie Leibniz algebra.

For a Leibniz algebra $\mathfrak{g}$, we consider two copies of $\mathfrak{g}$, left and right, denoted by $\mathfrak{g}^l$ and $\mathfrak{g}^r$, respectively. For an element $x \in \mathfrak{g}$, we denote by l_x and r_x the corresponding elements in the left and right copies, respectively. The universal enveloping algebra of $\mathfrak{g}$ was defined in [19] as

$$\mathrm{UL}(\mathfrak{g}) := T(\mathfrak{g}^l \oplus \mathfrak{g}^r)/I$$

where $T(V)$ is the tensor algebra on V and I is the two-sided ideal spanned by the following relations:

i) $r_{[x,y]} - (r_x r_y - r_y r_x)$
ii) $l_{[x,y]} - (l_x r_y - r_y l_x)$
iii) $(r_y + l_y)l_x$.

Moreover, [19, Theorem (2.3)] establishes that the category of representations (resp. co-representations) of the Leibniz algebra $\mathfrak{g}$ is equivalent to the category of right (resp. left) modules over $\mathrm{UL}(\mathfrak{g})$. After this, [16, Corollary 1.4] establishes the equivalence between the categories of representations and co-representations of a Leibniz algebra.

Let $\mathfrak{g}$ be a finite-dimensional Leibniz algebra with basis $\{a_1, \ldots, a_n\}$. There is an isomorphism of algebras

$$\Phi : T(\mathfrak{g}^l \oplus \mathfrak{g}^r) \to \mathbb{K} < y_1, \ldots, y_n, x_1, \ldots, x_n >$$

given by $\Phi(l_{a_i}) = y_i, \Phi(r_{a_i}) = x_i$, where $\mathbb{K} < y_1, \ldots, y_n, x_1, \ldots, x_n >$ denotes the free associative non-commutative unitary $\mathbb{K}$-algebra of polynomials.

Having in mind the inclusion $\mathfrak{g} \hookrightarrow T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$ and the isomorphism Φ, we obtain that

$$\mathrm{UL}(\mathfrak{g}) \cong \frac{\mathbb{K} < y_1, \ldots, y_n, x_1, \ldots, x_n >}{< -\Phi(r_{[a_i,a_j]}) + x_i x_j - x_j x_i, -\Phi(l_{[a_i,a_j]}) + y_i x_j - x_j y_i, (x_i + y_i) y_j >},$$

and hence we can use the theory of Gröbner bases on $\mathbb{K} < y_1, \ldots, y_n, x_1, \ldots, x_n >$ [20] to obtain results about the structure of $\mathfrak{g}$.

Let $\prec$ be a given monomial order on the noncommutative polynomial ring $\mathbb{K}{<}X{>}$. For an arbitrary polynomial $p \in \mathbb{K}{<}X{>}$, we will use $lm(p)$ to denote the leading monomial of p.

Definition 2. *Let I be a two-sided ideal of $\mathbb{K}{<}X{>}$. A subset $\{0\} \subsetneq G \subset I$ is called a* Gröbner basis *for I if for every $0 \neq f \in I$, there exists $g \in G$, such that $lm(g)$ is a factor of $lm(f)$.*

Lemma 1 (Diamond Lemma [7]). *Let $\prec$ be a monomial order on $\mathbb{K}{<}X{>}$ and let $G = \{g_1, \ldots, g_m\}$ be a set of generators of an ideal I in $\mathbb{K}{<}X{>}$. If all the overlap relations involving members of G reduce to zero modulo G, then G is a Gröbner basis for I.*

We note that the overlap relations are the noncommutative version of the S-polynomials.

Now, we consider the ideal $J =< \{g_{ij} = -\Phi(r_{[a_i,a_j]}) + x_i x_j - x_j x_i, h_{ij} = -\Phi(l_{[a_i,a_j]}) + y_i x_j - x_j y_i, i,j = 1, \ldots, n\} >$. For $i,j,k \in \{1, \ldots, n\}$, let be $P_{ijk} = h_{ij} x_k + y_i g_{kj}$.

Lemma 2. *$P_{ijk} \to_J -\Phi(l_{[[a_i,a_j],a_k]}) - \Phi(l_{[a_i,[a_k,a_j]]}) + \Phi(l_{[[a_i,a_k],a_j]})$ for all $i,j,k \in \{1, \ldots, n\}$.*

Proof. $P_{ijk} = h_{ij} x_k + y_i g_{kj} = -\Phi(l_{[a_i,a_j]}) x_k - x_j y_i x_k - y_i \Phi(r_{[a_k,a_j]}) + y_i x_k x_j \to_J -\Phi(l_{[a_i,a_j]}) x_k - x_j y_i x_k - y_i \Phi(r_{[a_k,a_j]}) + \Phi(l_{[a_i,a_k]}) x_j + x_k y_i x_j \to_J -\Phi(l_{[a_i,a_j]}) x_k - y_i \Phi(r_{[a_k,a_j]}) + \Phi(l_{[a_i,a_k]}) x_j + x_k y_i x_j - x_j \Phi(l_{[a_i,a_k]}) - x_j x_k y_i \to_J -\Phi(l_{[a_i,a_j]}) x_k - y_i \Phi(r_{[a_k,a_j]}) + \Phi(l_{[a_i,a_k]}) x_j + x_k y_i x_j - x_j \Phi(l_{[a_i,a_k]}) + \Phi(r_{[a_k,a_j]}) y_i - x_k x_j y_i \to_J -\Phi(l_{[a_i,a_j]}) x_k - y_i \Phi(r_{[a_k,a_j]}) + \Phi(l_{[a_i,a_k]}) x_j + x_k \Phi(l_{[a_i,a_j]}) - x_j \Phi(l_{[a_i,a_k]}) + \Phi(r_{[a_k,a_j]}) y_i$.

Let $[a_k, a_j] = \alpha_1^{kj} a_1 + \cdots + \alpha_n^{kj} a_n$.

$\Phi(r_{[a_k,a_j]}) y_i - y_i \Phi(r_{[a_k,a_j]}) = (\alpha_1^{kj} x_1 + \cdots + \alpha_n^{kj} x_n) y_i - y_i (\alpha_1^{kj} x_1 + \cdots + \alpha_n^{kj} x_n) = \alpha_1^{kj}(x_1 y_i - y_i x_1) + \cdots + \alpha_n^{kj}(x_n y_i - y_i x_n) \to_J -\alpha_1^{kj} \Phi(l_{[a_i,a_1]}) - \cdots - \alpha_n^{kj} \Phi(l_{[a_i,a_n]}) = -\Phi(l_{[a_i,[a_k,a_j]]})$.

$\Phi(l_{[a_i,a_j]})x_k - x_k\Phi(l_{[a_i,a_j]}) = (\alpha_1^{ij}y_1 + \cdots + \alpha_n^{ij}y_n)x_k - x_k(\alpha_1^{ij}y_1 + \cdots + \alpha_n^{ij}y_n) = \alpha_1^{ij}(y_1x_k - x_ky_1) + \cdots + \alpha_n^{ij}(y_nx_k - x_ky_n) \rightarrow_J \alpha_1^{ij}\Phi(l_{[a_1,a_k]}) + \cdots + \alpha_n^{ij}\Phi(l_{[a_n,a_k]}) = \Phi(l_{[[a_i,a_j],a_k]})$. □

Theorem 1. *Let* $\mathfrak{g}$ *be a finite-dimensional* $\mathbb{K}$*-vector space with basis* $\{a_1, \dots, a_n\}$ *together with a bilinear map* $[-,-] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$*.*
$(\mathfrak{g}, [-,-])$ *is a Leibniz algebra if and only if the Gröbner basis* $G = \{g_1, \dots, g_t\}$ *corresponding to the ideal* J *with respect to any monomial order does not contain linear polynomials* $g_j(y_1, \dots, y_n)$*.*

Proof. If $(\mathfrak{g}, [-,-])$ is a Leibniz algebra, then $P_{ijk} = 0$ for all $i, j, k \in \{1, \dots, n\}$ because of the Leibniz identity (2).

On the other hand, if $(\mathfrak{g}, [-,-])$ is not a Leibniz algebra, then there exist $i, j, k \in \{1, \dots, n\}$ such that $[[a_i, a_j], a_k] + [a_i, [a_k, a_j]] - [[a_i, a_k], a_j] \neq 0$, so J contains a degree 1 polynomial on the variables $y_1, \dots, y_n$. □

Example 2. Let $(\mathfrak{g} =< a_1, a_2, a_3 >_{\mathbb{K}}, [-,-])$ be the vector space such that

$$[a_1, a_3] = a_2,\ [a_2, a_3] = a_1 + a_2,\ [a_3, a_3] = a_3, \text{ and 0 in other case.}$$

In this case, $J =< -x_1x_2 + x_2x_1, -x_1x_3 + x_3x_1 + x_2, x_1x_2 - x_2x_1, -x_2x_3 + x_3x_2 + x_1 + x_2, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2, x_3, x_1y_1 - y_1x_1, x_2y_1 - y_1x_2, x_3y_1 - y_1x_3 + y_2, x_1y_2 - y_2x_1, x_2y_2 - y_2x_2, x_3y_2 - y_2x_3 + y_1 + y_2, x_1y_3 - y_3x_1, x_2y_3 - y_3x_2, x_3y_3 - y_3x_3 + y_3 >\subset \mathbb{K}<y_1, y_2, y_3, x_1, x_2, x_3>$.

The Gröbner basis of J with respect to degree lexicographical ordering with $y_3 > y_2 > y_1 > x_3 > x_2 > x_1$ is $\{y_3, y_2, y_1, x_3, x_2, x_1\}$.

Therefore, $\mathfrak{g}$ is not a Leibniz algebra.

According to Theorem 1, we can know if a finite-dimensional algebra is Leibniz or not. In case of positive answer, it is natural to ask if the Leibniz algebra is a Lie algebra or a non-Lie Leibniz algebra.

As it is well-known [13]

$$\mathfrak{g}^{\text{ann}} =< \{[a_i, a_i], i = 1, \dots, n\} \cup \{[a_i, a_j] + [a_j, a_i], i, j = 1, \dots, n\} >,$$

and having in mind the identification

$$\begin{array}{ccccc} \mathfrak{g} & \hookrightarrow & T(\mathfrak{g}^l \oplus \mathfrak{g}^r) & \overset{\Phi}{\underset{\simeq}{\longrightarrow}} & \mathbb{K} < x_1, \dots, x_n, y_1, \dots, y_n > \\ a_i & \mapsto & r_{a_i} & \mapsto & x_i \end{array}$$

then the generators of $\mathfrak{g}^{\text{ann}}$ can be written as

$$\{\Phi(r_{[a_i,a_i]}), i = 1, \dots, n\} \cup \{\Phi(r_{[a_i,a_j]}) + \Phi(r_{[a_j,a_i]}),\ i, j = 1, \dots, n\}$$

and these generators belongs to J, since

$$g_{ii} = \Phi(r_{[a_i,a_i]}),\ i = 1, \dots, n\,,$$

$$-g_{ij} - g_{ji} = \Phi(r_{[a_i,a_j]}) + \Phi(r_{[a_j,a_i]}),\ i, j = 1, \dots, n\,.$$

Consequently, if we compute the Gröbner basis G with respect to any monomial order for the ideal J, we can obtain two possible results:

1. There are some linear polynomials $g_j(y_1,\dots,y_n)$ in G, so $\mathfrak{g}$ is not a Leibniz algebra.
2. There are not any linear polynomials $g_j(y_1,\dots,y_n)$ in G, so $\mathfrak{g}$ is a Leibniz algebra, and:
   (a) If there are some linear polynomials $g_i(x_1,\dots,x_n)$ in G, then $\mathfrak{g}^{\text{ann}} \neq 0$ and so $\mathfrak{g}$ is a non Lie-Leibniz algebra.
   (b) In other case $\mathfrak{g}$ is a Lie algebra.

Example 3. Let $(\mathfrak{g} =< a_1, a_2, a_3 >_{\mathbb{K}}, [-,-])$ be the vector space such that

$$[a_1, a_3] = a_2,\ [a_2, a_3] = a_1 + a_2,\ [a_3, a_3] = a_1, \text{ and 0 in other case.}$$

In this case, $J =< -x_1x_2 + x_2x_1, -x_1x_3 + x_3x_1 + x_2, x_1x_2 - x_2x_1, -x_2x_3 + x_3x_2 + x_1 + x_2, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2, x_1, x_1y_1 - y_1x_1, x_2y_1 - y_1x_2, x_3y_1 - y_1x_3 + y_2, x_1y_2 - y_2x_1, x_2y_2 - y_2x_2, x_3y_2 - y_2x_3 + y_1 + y_2, x_1y_3 - y_3x_1, x_2y_3 - y_3x_2, x_3y_3 - y_3x_3 + y_1 >\subset \mathbb{K}<y_1, y_2, y_3, x_1, x_2, x_3>$.

The Gröbner basis of J with respect to degree lexicographical ordering with $y_3 > y_2 > y_1 > x_3 > x_2 > x_1$ is $\{x_2, x_1, -x_3y_1 + y_1x_3 - y_2, -x_3y_2 + y_2x_3 - y_1 - y_2, -x_3y_3 + y_3x_3 - y_1\}$.

Therefore, $\mathfrak{g}$ is a Leibniz algebra and not a Lie algebra.

Example 4. Let $(\mathfrak{g} =< a_1, a_2, a_3, a_4 >_{\mathbb{K}}, [-,-])$ be the vector space such that

$$[a_4, a_1]=a_1, [a_1, a_4]=-a_1, [a_4, a_2]=a_2, [a_2, a_4]=-a_2, [a_4, a_3]=a_3, [a_3, a_4]=-a_3,$$

and 0 in other case.

In this case, $J =< -x_1x_2 + x_2x_1, -x_1x_3 + x_3x_1, -x_1x_4 + x_4x_1 - x_1, x_1x_2 - x_2x_1, -x_2x_3 + x_3x_2, -x_2x_4 + x_4x_2 - x_2, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2, -x_3x_4 + x_4x_3 - x_3, x_1x_4 - x_4x_1 + x_1, x_2x_4 - x_4x_2 + x_2, x_3x_4 - x_4x_3 + x_3, x_1y_1 - y_1x_1, x_2y_1 - y_1x_2, x_3y_1 - y_1x_3, x_4y_1 - y_1x_4 - y_1, x_1y_2 - y_2x_1, x_2y_2 - y_2x_2, x_3y_2 - y_2x_3, x_4y_2 - y_2x_4 - y_2, x_1y_3 - y_3x_1, x_2y_3 - y_3x_2, x_3y_3 - y_3x_3, x_4y_3 - y_3x_4 - y_3, x_1y_4 - y_4x_1 + y_1, x_2y_4 - y_4x_2 + y_2, x_3y_4 - y_4x_3 + y_3, x_4y_4 - y_4x_4 >\subset \mathbb{K}<y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4>$.

The Gröbner basis of J with respect to degree lexicographical ordering with $y_4 > y_3 > y_2 > y_1 > x_4 > x_3 > x_2 > x_1$ is $\{-x_1x_2 + x_2x_1, -x_1x_3 + x_3x_1, -x_1x_4 + x_4x_1 - x_1, -x_2x_3 + x_3x_2, -x_2x_4 + x_4x_2 - x_2, -x_3x_4 + x_4x_3 - x_3, -x_1y_1 + y_1x_1, -x_2y_1 + y_1x_2, -x_3y_1 + y_1x_3, -x_4y_1 + y_1x_4 + y_1, -x_1y_2 + y_2x_1, -x_2y_2 + y_2x_2, -x_3y_2 + y_2x_3, -x_4y_2 + y_2x_4 + y_2, -x_1y_3 + y_3x_1, -x_2y_3 + y_3x_2, -x_3y_3 + y_3x_3, -x_4y_3 + y_3x_4 + y_3, -x_1y_4 + y_4x_1 - y_1, -x_2y_4 + y_4x_2 - y_2, -x_3y_4 + y_4x_3 - y_3, -x_4y_4 + y_4x_4\}$.

Therefore, $\mathfrak{g}$ is a Lie algebra.

We can also solve the dichotomy of knowing if a $\mathbb{K}$-vector space with a given multiplication table is a Lie algebra by means of the following ideal

$$J' =< \{g_{ij} = -\Phi(r_{[a_i,a_j]}) + x_ix_j - x_jx_i, i, j = 1,\dots,n\} > .$$

For $i, j, k \in \{1,\dots,n\}$, let be $Q_{ijk} = x_jg_{ki} - g_{ij}x_k$.

Lemma 3. *$Q_{ijk} \rightarrow_{J'} -\Phi(r_{[a_k,[a_i,a_j]]}) + \Phi(r_{[[a_k,a_i],a_j]}) - \Phi(r_{[[a_k,a_j],a_i]})$ for all $i, j, k \in \{1,\dots,n\}$.*

Proof. $x_j g_{ki} - g_{ij} x_k = \Phi(r_{[a_i,a_j]})x_k - x_i x_j x_k - x_j \Phi(r_{[a_k,a_i]}) + x_j x_k x_i \to_{g_{kj} x_i} \Phi(r_{[a_i,a_j]})x_k - x_i x_j x_k - x_j \Phi(r_{[a_k,a_i]}) - \Phi(r_{[a_k,a_j]})x_i + x_k x_j x_i \to_{-x_i g_{kj}} \Phi(r_{[a_i,a_j]})x_k - x_j \Phi(r_{[a_k,a_i]}) - \Phi(r_{[a_k,a_j]})x_i + x_k x_j x_i + x_i \Phi(r_{[a_k,a_j]}) - x_i x_k x_j \to_{-g_{ki} x_j} \Phi(r_{[a_i,a_j]})x_k - x_j \Phi(r_{[a_k,a_i]}) - \Phi(r_{[a_k,a_j]})x_i + x_k x_j x_i + x_i \Phi(r_{[a_k,a_j]}) + \Phi(r_{[a_k,a_i]})x_j - x_k x_i x_j = \Phi(r_{[a_i,a_j]})x_k + x_k(x_j x_i - x_i x_j) + \Phi(r_{[a_k,a_i]})x_j - x_j \Phi(r_{[a_k,a_i]}) - \Phi(r_{[a_k,a_j]})x_i + x_i \Phi(r_{[a_k,a_j]}) \to_{J'} \Phi(r_{[a_i,a_j]})x_k - x_k \Phi(r_{[a_i,a_j]}) + \Phi(r_{[a_k,a_i]})x_j - x_j \Phi(r_{[a_k,a_i]}) - \Phi(r_{[a_k,a_j]})x_i + x_i \Phi(r_{[a_k,a_j]})$.

Let $[a_i, a_j] = \alpha_1^{ij} a_1 + \cdots + \alpha_n^{ij} a_n$.

$\Phi(r_{[a_i,a_j]})x_k - x_k \Phi(r_{[a_i,a_j]}) = (\alpha_1^{ij} x_1 + \cdots + \alpha_n^{ij} x_n)x_k - x_k(\alpha_1^{ij} x_1 + \cdots + \alpha_n^{ij} x_n) = \alpha_1^{ij}(x_1 x_k - x_k x_1) + \cdots + \alpha_{k-1}^{ij}(x_{k-1} x_k - x_k x_{k-1}) + \alpha_k^{ij} \cdot 0 + \alpha_{k+1}^{ij}(x_{k+1} x_k - x_k x_{k+1}) + \cdots + \alpha_n^{ij}(x_n x_k - x_k x_n) \to_{J'} \alpha_1^{ij} \Phi(r_{[a_1,a_k]}) + \cdots + \alpha_{k-1}^{ij} \Phi(r_{[a_{k-1},a_k]}) - \alpha_{k+1}^{ij} \Phi(r_{[a_k,a_{k+1}]}) - \cdots - \alpha_n^{ij} \Phi(r_{[a_k,a_n]}) \to_{J'} \alpha_1^{ij} \Phi(r_{[a_1,a_k]}) + \ldots + \alpha_{k-1}^{ij} \Phi(r_{[a_{k-1},a_k]}) + \alpha_{k+1}^{ij} \Phi(r_{[a_{k+1},a_k]}) - \cdots - \alpha_n^{ij} \Phi(r_{[a_n,a_k]}) = \Phi(r_{[[a_i,a_j],a_k]}) - \alpha_k^{ij} \Phi(r_{[a_k,a_k]}) \to_{J'} \Phi(r_{[[a_i,a_j],a_k]})$, or $\Phi(r_{[a_i,a_j]})x_k - x_k \Phi(r_{[a_i,a_j]}) \to_{J'} -\Phi(r_{[a_k,[a_i,a_j]]})$. □

Theorem 2. *Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{K}$-vector space with basis $\{a_1, \ldots, a_n\}$ together with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.*
$(\mathfrak{g}, [-,-])$ is a Lie algebra if and only if the Gröbner basis $G = \{g_1, \ldots, g_t\}$ corresponding to the ideal J' with respect to any monomial order does not contain linear polynomials $g_j(x_1, \ldots, x_n)$.

Example 5. Let $(\mathfrak{g} =< a_1, a_2, a_3, a_4 >_{\mathbb{K}}, [-,-])$ be the vector space such that

$$[a_4, a_1] = a_2, \quad [a_1, a_4] = -a_2, \quad [a_4, a_2] = a_3, \quad [a_2, a_4] = -a_3,$$
$$[a_4, a_3] = a_1 + a_2, \quad [a_3, a_4] = -a_1 - a_2, \quad \text{and} \quad 0 \quad \text{in other case} .$$

In this case, $J' =< -x_1x_2 + x_2x_1, -x_1x_3 + x_3x_1, -x_1x_4 + x_4x_1 - x_2, x_1x_2 - x_2x_1, -x_2x_3 + x_3x_2, -x_2x_4 + x_4x_2 - x_3, x_1x_3 - x_3x_1, x_2x_3 - x_3x_2, -x_3x_4 + x_4x_3 - x_1 - x_2, x_1x_4 - x_4x_1 + x_2, x_2x_4 - x_4x_2 + x_3, x_3x_4 - x_4x_3 + x_1 + x_2 > \subset \mathbb{K}<x_1, x_2, x_3, x_4>$.

The Gröbner basis of J' with respect to degree lexicographical ordering with $x_4 > x_3 > x_2 > x_1$ is $\{-x_1x_2 + x_2x_1, -x_1x_3 + x_3x_1, -x_1x_4 + x_4x_1 - x_2, -x_2x_3 + x_3x_2, -x_2x_4 + x_4x_2 - x_3, -x_3x_4 + x_4x_3 - x_1 - x_2\}$.

Therefore, $\mathfrak{g}$ is a Lie algebra.

3 Computer Program

In this section we describe a program in NCAlgebra [12] (a package running under Mathematica) that implements the algorithms discussed in the previous section. The program computes the reduced Gröbner basis of the ideal J which determines if the introduced multiplication table of $\mathfrak{g}$ corresponds to a Leibniz algebra or not. In case of positive answer, the program decides whether the algebra is a Lie or non-Lie Leibniz algebra. The Mathematica code together with some examples are available in `http://web.usc.es/~mladra/research.html`.

```
#######################################################################
   (* This program tests if an introduced multiplication table
  corresponds to a Lie algebra, a non-Lie Leibniz algebra or an
 algebra which has not Leibniz algebra structure. To run this code
       properly it is necessary to load the NCGB package*)
#######################################################################
```

(* Let $\mathfrak{g} =< a_1, \ldots, a_n >$ be an algebra of dimension n *)
(* Insert the Bracket represented by Bracket$[\{i1, i2\}] := \{\lambda_1, \ldots, \lambda_n\}$ where $[a_{i1}, a_{i2}] = \lambda_1 a_1 + \cdots + \lambda_n a_n$. In Example 2, e.g., Bracket[{2, 3}] := {1,1,0} *)

```
LeibnizQ[n_]:=Module[{G,A,lengA,Variabs,BaseG,varaux,rvarsx,lvarsy},

    (* First of all we construct the generators of the ideal *)

    G={};
    A=Tuples[Table[i,{i,1,n}],2];
    lengA=Length[A];

    Do[
        G=Join[
          G,{Bracket[A[[i]]].Table[x[i],{i,1,n}]-(x[A[[i,1]]]** x[A[[i,2]]]-
                  x[A[[i,2]]]**x[A[[i,1]]])}]
      ,{i,1,lengA}];

    Do[
        G=Join[
          G,{Bracket[A[[i]]].Table[y[i],{i,1,n}]-(y[A[[i,1]]]** x[A[[i,2]]]-
                  x[A[[i,2]]]**y[A[[i,1]]])}]
      ,{i,1,lengA}];

    rvarsx=Table[x[i],{i,1,n}];
    lvarsy=Table[y[i],{i,1,n}];
    Variabs=Join[rvarsx,lvarsy];

    (* Now we compute a Gröbner Basis of G *)

    SetNonCommutative@@Variabs;

    SetMonomialOrder[Variabs] ;

    BaseG=NCMakeGB[G,15];

    (* Finally, we check the Gröbner Basis *)

    varaux=Select[BaseG,Union[Variables[#],lvarsy]===lvarsy&];
```

```
If[varaux==={},

   (* 𝔤 is a Leibniz algebra but, is 𝔤 a Lie algebra? *)

  varaux=Select[BaseG,Union[Variables[#],rvarsx]===rvarsx&];

  If[varaux==={},

     Print["𝔤 is a Lie algebra"];
    ,
    Print["𝔤 is not a Lie algebra but 𝔤 is a Leibniz algebra"];

    ];
  ,
  Print["𝔤 is not a Leibniz algebra"];
  ];
]
```

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