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# ON Lie-CENTRAL EXTENSIONS OF LEIBNIZ ALGEBRAS

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**Abstract:** Basing ourselves on the categorical notions of central extensions and commutators in the framework of semi-abelian categories relative to a Birkhoff subcategory, we study central extensions of Leibniz algebras with respect to the Birkhoff subcategory of Lie algebras, called Lie-central extensions. We construct a Lie-homology of Leibniz algebras and obtain a six-term exact homology sequence associated to a Lie-central extension. This sequence, together with the relative commutators, allows us to characterize several classes of Lie-central extensions, such as Lie-trivial, Lie-stem and Lie-stem cover, to introduce and characterize Lie-unicentral, Lie-capable, Lie-solvable and Lie-nilpotent Leibniz algebras.

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**Key words:** Lie-central extension, relative commutator, Lie-unicentral, Lie-capable, Lie-solvable, Lie-nilpotent Leibniz algebra

## 1 Introduction

A general theory of central extensions relative to a chosen subcategory of a base category was introduced in [15]. Recently, in [7] were analyzed in details the categorical concepts of central extensions, perfect objects and commutators in a semi-abelian category [2], relative to a Birkhoff subcategory. Some examples (e.g. groups vs. abelian groups, Lie (Leibniz) algebras vs. vector spaces) are absolute, meaning that they are fitted into the relative case with respect to the subcategory of all abelian objects. In the absolute case, some results were already investigated in [14].

In this paper we deal with the 'non absolute' case: Leibniz algebras vs. Lie algebras. In particular, the goal of the present paper is to consider the relative notions of central extension and commutator when the semi-abelian category is *Leib*, the category of Leibniz algebras, and its Birkhoff subcategory is *Lie*, the

category of Lie algebras, together with the Liezation functor  $(-)\text{Lie} : \text{Leib} \rightarrow \text{Lie}$ , which is left adjoint to the inclusion functor  $\text{Lie} \hookrightarrow \text{Leib}$ .

Under these circumstances, the notion of central extension relative to  $\text{Lie}$ , so called  $\text{Lie}$ -central extension, and the notion of commutator relative to  $\text{Lie}$ , provide the necessary ingredients to introduce the notions of unicentrality, capability, solvability and nilpotency of Leibniz algebras relative to  $\text{Lie}$ , all of them named with the prefix  $\text{Lie}$ -.

Homological machinery relative to  $\text{Lie}$ , coming from the categorical semi-abelian framework [11, 12, 13], allows us to characterize these new notions by means of the six-term exact sequence

$$\mathfrak{n} \otimes \mathfrak{g}_{\text{Lie}} \longrightarrow HL_2^{\text{Lie}}(\mathfrak{g}) \longrightarrow HL_2^{\text{Lie}}(\mathfrak{q}) \longrightarrow \mathfrak{n} \longrightarrow \mathfrak{g}_{\text{Lie}} \longrightarrow \mathfrak{q}_{\text{Lie}} \longrightarrow 0$$

associated to the  $\text{Lie}$ -central extension  $0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g} \rightarrow \mathfrak{q} \rightarrow 0$ , where  $HL_*^{\text{Lie}}(-)$  denotes the  $\text{Lie}$ -homology of a Leibniz algebra (see Subsection 3.2).

In particular, we organize the paper as follows: in Section 2 we recall the necessary categorical background from [7, 9]. Then we particularize in Subsection 3.1 these notions for the category  $\text{Leib}$  and its Birkhoff subcategory  $\text{Lie}$ . So we recall the concepts of  $\text{Lie}$ -perfect Leibniz algebras,  $\text{Lie}$ -trivial and  $\text{Lie}$ -central extensions of Leibniz algebras and give the five-term exact sequence in homology relative to the Liezation functor, associated to an extension of Leibniz algebras. In Subsection 3.2, given a Leibniz algebra  $\mathfrak{q}$ , we construct an explicit complex whose first homology  $HL_1^{\text{Lie}}(\mathfrak{q})$  is nothing else but the Liezation  $\mathfrak{q}_{\text{Lie}}$  of  $\mathfrak{q}$  and the second homology  $HL_2^{\text{Lie}}(\mathfrak{g})$  is isomorphic to the second homology  $HL_2(\mathfrak{q}, (-)_{\text{Lie}})$  of  $\mathfrak{q}$  with respect to the Liezation functor. Then we obtain the above mentioned six-term exact sequence associated to a  $\text{Lie}$ -central extension. In Subsection 3.3 we introduce and study the notions of  $\text{Lie}$ -stem extension and  $\text{Lie}$ -stem cover of Leibniz algebras. In Section 4 we introduce the notions of  $\text{Lie}$ -unicentral,  $\text{Lie}$ -capable and precise  $\text{Lie}$ -center of Leibniz algebras and investigate interrelationships between them. In Section 5, by using the relative commutators, we introduce  $\text{Lie}$ -central and  $\text{Lie}$ -abelian series of Leibniz algebras. Then we introduce and study the concepts of  $\text{Lie}$ -solvability and  $\text{Lie}$ -nilpotency of Leibniz algebras. In Section 6 we obtain a homological characterization of  $\text{Lie}$ -nilpotency of a Leibniz algebra, which is similar to the Stallings' theorem for Leibniz algebras in the absolute case [6].

## 2 Categorical background

In this section we give an overview of needed definitions and results from [7, 8, 9] on central extensions, perfect objects and commutators in semi-abelian categories.

In their article [15], Janelidze and Kelly introduced a general theory of central extensions relative to a chosen subcategory  $\mathcal{B}$  of the base category  $\mathcal{A}$ . This theory holds when  $\mathcal{B}$  is a Birkhoff subcategory of a semi-abelian category  $\mathcal{A}$  [9].

Recall from [16] that a category  $\mathcal{A}$  is *semi-abelian* if it is pointed, Barr exact and Bourn protomodular with binary coproducts. A subcategory  $\mathcal{B}$  of a semi-abelian category  $\mathcal{A}$  is a *Birkhoff subcategory* if it is full, reflective and closed under subobjects and regular quotients.

From now on, we consider  $\mathcal{B}$  a fixed Birkhoff subcategory of a semi-abelian category  $\mathcal{A}$ . We denote by  $I : \mathcal{A} \rightarrow \mathcal{B}$  the left adjoint to the inclusion functor  $\mathcal{B} \hookrightarrow \mathcal{A}$ , and write the components of its unit by  $\eta_A : A \rightarrow I(A)$ .

Then an *extension* in  $\mathcal{A}$  is a regular epimorphism in  $\mathcal{A}$ .

An extension  $f : B \twoheadrightarrow A$  in  $\mathcal{A}$  is said to be *trivial (with respect to  $\mathcal{B}$ )* or  *$\mathcal{B}$ -trivial* [15] when the induced square

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \eta_B \downarrow & & \downarrow \eta_A \\ I(B) & \xrightarrow{I(f)} & I(A) \end{array} \quad (\mathbf{A})$$

is a pullback.

An extension  $f : B \twoheadrightarrow A$  in  $\mathcal{A}$  is called *central (with respect to  $\mathcal{B}$ )* or  *$\mathcal{B}$ -central* [15] when either one of the projections  $f_0, f_1$  in the kernel pair  $(R[f], f_0, f_1)$  of  $f$  is  $\mathcal{B}$ -trivial. That is to say,  $f$  is  $\mathcal{B}$ -central if and only if in the diagram

$$\begin{array}{ccccc} R[f] & \xrightarrow{f_0} & B & \xrightarrow{f} & A \\ & \searrow f_1 & \downarrow \eta_B & & \\ \eta_{R[f]} \downarrow & & & & \\ IR[f] & \xrightarrow{If_0} & IB & & \\ & \searrow If_1 & & & \end{array}$$

either one of the left hand side squares is a pullback.

An object  $P$  of  $\mathcal{A}$  is called *perfect with respect to  $\mathcal{B}$*  or  *$\mathcal{B}$ -perfect* when  $I(P)$  is the zero object  $0$  of  $\mathcal{B}$ .

Since, in a semi-abelian category, a regular epimorphism is always the cokernel of its kernel, an appropriate notion of short exact sequence exists. Such will be any sequence  $K \xrightarrow{k} B \xrightarrow{f} A$  that satisfies  $k = \text{Ker}(f)$  and  $f = \text{Coker}(k)$ . We denote this situation by  $0 \longrightarrow K \xrightarrow{k} B \xrightarrow{f} A \longrightarrow 0$ .

For any object  $A$  of  $\mathcal{A}$ , the adjunction  $\mathcal{A} \xrightleftharpoons[\perp]{I} \mathcal{B}$  induces a short exact sequence

$$0 \longrightarrow [A, A]_{\mathcal{B}} \xrightarrow{\mu_A} A \xrightarrow{\eta_A} IA \longrightarrow 0.$$

Here the object  $[A, A]_{\mathcal{B}}$ , defined as the kernel of  $\eta_A$ , acts as a *zero-dimensional commutator relative to  $\mathcal{B}$* . Of course,  $IA = A/[A, A]_{\mathcal{B}}$ , so that  $A$  is an object of  $\mathcal{B}$  if and only if  $[A, A]_{\mathcal{B}}$  is zero. On the other hand, an object  $A$  of  $\mathcal{A}$  is called  *$\mathcal{B}$ -perfect* when  $[A, A]_{\mathcal{B}} = A$ .

Let us remark that an extension  $f: B \twoheadrightarrow A$  in  $\mathcal{A}$  is  $\mathcal{B}$ -central if and only if the restrictions  $[f_0, f_0]_{\mathcal{B}}, [f_1, f_1]_{\mathcal{B}}: [R[f], R[f]]_{\mathcal{B}} \rightarrow [B, B]_{\mathcal{B}}$  of the kernel pair projections  $f_0, f_1: R[f] \rightarrow B$  coincide. This is the case precisely when  $[f_0, f_0]_{\mathcal{B}}$  and  $[f_1, f_1]_{\mathcal{B}}$  are isomorphisms, or, equivalently, when  $\text{Ker}[f_0, f_0]_{\mathcal{B}}: L_1[f] \rightarrow [R[f], R[f]]_{\mathcal{B}}$  is zero. In the following diagram

$$\begin{array}{ccccccc}
& & L_1[f] & & & & \\
& & \downarrow \text{Ker}[f_0, f_0]_{\mathcal{B}} & \nearrow & & & \\
0 & \longrightarrow & [R[f], R[f]]_{\mathcal{B}} & \xrightarrow{\mu_{R[f]}} & R[f] & \xrightarrow{\eta_{R[f]}} & IR[f] \longrightarrow 0 \\
& & \downarrow [f_0, f_0]_{\mathcal{B}} \quad \downarrow [f_1, f_1]_{\mathcal{B}} & & \downarrow f_0 \quad \downarrow f_1 & & \downarrow If_0 \quad \downarrow If_1 \\
0 & \longrightarrow & [B, B]_{\mathcal{B}} & \xrightarrow{\mu_B} & B & \xrightarrow{\eta_B} & IB \longrightarrow 0,
\end{array}$$

through the composite  $f_1 \circ \mu_{R[f]} \circ \text{Ker}[f_0]_{\mathcal{B}}$  the object  $L_1[f]$  may be considered as a normal subobject of  $B$ . It acts as a *one-dimensional commutator relative to  $\mathcal{B}$*  and, if  $K$  denotes the kernel of  $f$ , it is usually written  $[K, B]_{\mathcal{B}}$ .

Let  $A$  be an object of a semi-abelian category  $\mathcal{A}$  with enough projectives and  $f: B \twoheadrightarrow A$  be a projective presentation with kernel  $K$ . The induced objects

$$\frac{[B, B]_{\mathcal{B}}}{[K, B]_{\mathcal{B}}} \quad \text{and} \quad \frac{K \cap [B, B]_{\mathcal{B}}}{[K, B]_{\mathcal{B}}}$$

are independent of the chosen projective presentation of  $A$  as explained for instance in [12]. The object  $(K \cap [B, B]_{\mathcal{B}})/[K, B]_{\mathcal{B}}$  is called (the Hopf formula for) the second homology object of  $A$  (with coefficients in  $\mathcal{B}$ ) and it is denoted by  $H_2(A, I)$ . We also write  $H_1(A, I)$  for  $I(A)$ .

When  $\mathcal{A}$  is a semi-abelian monadic category, the objects  $H_1(A, I)$  and  $H_2(A, I)$  may be computed using comonadic homology as in [13] and they are fitted into the semi-abelian homology theory (see [11]). Moreover, [12, Theorem 5.9] states that, given a short exact sequence  $0 \rightarrow K \rightarrow B \rightarrow A \rightarrow 0$ , there is the following five-term exact sequence

$$H_2(B, I) \longrightarrow H_2(A, I) \xrightarrow{\theta^{*(B)}} \frac{K}{[K, B]_{\mathcal{B}}} \longrightarrow H_1(B, I) \longrightarrow H_1(A, I) \longrightarrow 0. \quad (\mathbf{B})$$

### 3 Central extensions of Leibniz algebras with respect to Lie algebras

In this section we consider the particular case where  $\mathcal{A}$  is the semi-abelian category **Leib** of Leibniz algebras, and the Birkhoff subcategory  $\mathcal{B}$  is **Lie**, the category of Lie algebras.

### 3.1 Preliminary results on Leibniz algebras

We fix  $\mathbb{K}$  as a ground field such that  $\frac{1}{2} \in \mathbb{K}$ . All vector spaces and tensor products are considered over  $\mathbb{K}$ .

A *Leibniz algebra* [18, 19] is a vector space  $\mathfrak{q}$  equipped with a bilinear map  $[-, -] : \mathfrak{q} \otimes \mathfrak{q} \rightarrow \mathfrak{q}$ , usually called the *Leibniz bracket* of  $\mathfrak{q}$ , satisfying the *Leibniz identity*:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad x, y, z \in \mathfrak{q}.$$

Leibniz algebras form a semi-abelian category [7, 16], denoted by  $\mathbf{Leib}$ , whose morphisms are linear maps that preserve the Leibniz bracket.

A subalgebra  $\mathfrak{h}$  of a Leibniz algebra  $\mathfrak{q}$  is said to be *left (resp. right) ideal* of  $\mathfrak{q}$  if  $[h, q] \in \mathfrak{h}$  (resp.  $[q, h] \in \mathfrak{h}$ ), for all  $h \in \mathfrak{h}$ ,  $q \in \mathfrak{q}$ . If  $\mathfrak{h}$  is both left and right ideal, then  $\mathfrak{h}$  is called *two-sided ideal* of  $\mathfrak{q}$ . In this case  $\mathfrak{q}/\mathfrak{h}$  naturally inherits a Leibniz algebra structure.

For a Leibniz algebra  $\mathfrak{q}$ , we denote by  $\mathfrak{q}^{\text{ann}}$  the subspace of  $\mathfrak{q}$  spanned by all elements of the form  $[x, x]$ ,  $x \in \mathfrak{q}$ . Further, we consider

$$Z^r(\mathfrak{q}) = \{a \in \mathfrak{q} \mid [x, a] = 0, x \in \mathfrak{q}\}, \quad Z^l(\mathfrak{q}) = \{a \in \mathfrak{q} \mid [a, x] = 0, x \in \mathfrak{q}\},$$

and call the *right center* and the *left center* of  $\mathfrak{q}$  [17], respectively. It is proved in [17, Lemma 1.1] that both  $\mathfrak{q}^{\text{ann}}$  and  $Z^r(\mathfrak{q})$  are two-sided ideals of  $\mathfrak{q}$ . Just by using the Leibniz identity, it is easy to see that  $Z^l(\mathfrak{q})$  is also two-sided ideal of  $\mathfrak{q}$ . Of course, the intersection

$$Z(\mathfrak{q}) = Z^l(\mathfrak{q}) \cap Z^r(\mathfrak{q})$$

is again two-sided ideal of  $\mathfrak{q}$  and it is called the *center* of  $\mathfrak{q}$ .

Given a Leibniz algebra  $\mathfrak{q}$ , it is clear that the quotient  $\mathfrak{q}_{\text{Lie}} = \mathfrak{q}/\mathfrak{q}^{\text{ann}}$  is a Lie algebra. This defines the so-called *Lieization functor*  $(-)\text{Lie} : \mathbf{Leib} \rightarrow \mathbf{Lie}$ , which assigns to a Leibniz algebra  $\mathfrak{q}$  the Lie algebra  $\mathfrak{q}_{\text{Lie}}$ . Moreover, the canonical epimorphism  $\mathfrak{q} \twoheadrightarrow \mathfrak{q}_{\text{Lie}}$  is universal among all homomorphisms from  $\mathfrak{q}$  to a Lie algebra, implying that the Lieization functor is left adjoint to the inclusion functor  $\mathbf{Lie} \hookrightarrow \mathbf{Leib}$ .

It is an easy task to check that the category  $\mathbf{Lie}$  is a Birkhoff subcategory of  $\mathbf{Leib}$ . Focusing our attention in the adjoint pair

$$\mathbf{Leib} \begin{array}{c} \xrightarrow{(-)\text{Lie}} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathbf{Lie}, \quad (\mathbf{C})$$

we particularize the general theory described in Section 2 to the case when  $\mathcal{A}$  is the category  $\mathbf{Leib}$ ,  $\mathcal{B}$  is its Birkhoff subcategory  $\mathbf{Lie}$  and the functor  $I$  is precisely the Lieization functor  $(-)\text{Lie}$ . First let us remark the following

**Remark 3.1** A Leibniz algebra  $\mathfrak{q}$  is Lie-perfect if  $\mathfrak{q}_{\text{Lie}} = 0$ , that is  $\mathfrak{q} \cong \mathfrak{q}^{\text{ann}}$ . It follows by [5, Lemma 3.1] that any Lie-perfect Leibniz algebra  $\mathfrak{q}$  is the trivial one. Then, by [9, Theorem 3.5], we get that a Leibniz algebra  $\mathfrak{q}$  admits a universal Lie-central extension if and only if  $\mathfrak{q} = 0$ .

It is clear that an extension in Leib is just an epimorphism  $f: \mathfrak{g} \rightarrow \mathfrak{q}$  of Leibniz algebras, which is a Lie-trivial extension if and only if it induces an isomorphism  $\mathfrak{g}^{\text{ann}} \cong \mathfrak{q}^{\text{ann}}$ .

**Example 3.2** Let  $\mathfrak{g}$  and  $\mathfrak{q}$  be three and two-dimensional (as vector spaces) Leibniz algebras with  $\mathbb{K}$ -linear bases  $\{a_1, a_2, a_3\}$  and  $\{e_1, e_2\}$ , respectively, with the Leibniz brackets given respectively by  $[a_1, a_3] = a_1$  and  $[e_1, e_2] = e_1$  and zero elsewhere (see [5, 10]). Consider the homomorphism of Leibniz algebras  $f: \mathfrak{g} \rightarrow \mathfrak{q}$  defined by  $f(a_1) = e_1$ ,  $f(a_2) = 0$ ,  $f(a_3) = e_2$ . Obviously  $f$  is surjective,  $\mathfrak{g}^{\text{ann}} = \langle \{a_1\} \rangle$  and  $\mathfrak{q}^{\text{ann}} = \langle \{e_1\} \rangle$ , so they are isomorphic through  $f$ . Consequently  $f: \mathfrak{g} \rightarrow \mathfrak{q}$  is a Lie-trivial extension.

For a Leibniz algebra  $\mathfrak{q}$  and two-sided ideals  $\mathfrak{m}$  and  $\mathfrak{n}$  of  $\mathfrak{q}$ , we put

$$C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n}) = \{q \in \mathfrak{q} \mid [q, m] + [m, q] \in \mathfrak{n}, \text{ for all } m \in \mathfrak{m}\}.$$

Further, we denote by  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$  the subspace of  $\mathfrak{q}$  spanned by all elements of the form  $[m, n] + [n, m]$ ,  $m \in \mathfrak{m}$ ,  $n \in \mathfrak{n}$ .

**Lemma 3.3** Let  $\mathfrak{q}$  be a Leibniz algebra and  $\mathfrak{m}, \mathfrak{n}$  be two-sided ideals of  $\mathfrak{q}$ . Then both  $C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n})$  and  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$  are two-sided ideals of  $\mathfrak{q}$ . Moreover,  $Z(\mathfrak{q}) \subseteq C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n})$  and  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}} \subseteq Z^r(\mathfrak{q})$ .

*Proof.* It is clear that both  $C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n})$  and  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$  are subspaces of  $\mathfrak{q}$ .

Take any elements  $q \in C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n})$  and  $x \in \mathfrak{q}$ . Then, for any  $m \in \mathfrak{m}$ , we have

$$\begin{aligned} [[q, x], m] + [m, [q, x]] &= -[q, [m, x]] + [[q, m], x] + [[m, q], x] - [[m, x], q] \\ &= -([q, [m, x]] + [[m, x], q]) + [q, m] + [m, q], x \in \mathfrak{n} \end{aligned}$$

and, if we denote  $-([q, [m, x]] + [[m, x], q]) + [q, m] + [m, q], x$  by  $n$ , we get

$$\begin{aligned} [[x, q], m] + [m, [x, q]] &= [[x, q], m] - [m, [q, x]] = [[x, q], m] - n + [[q, x], m] \\ &= [[x, q] + [q, x], m] - n \in \mathfrak{n}. \end{aligned}$$

Consequently,  $C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n})$  is a two-sided ideal of  $\mathfrak{q}$ . The inclusion  $Z(\mathfrak{q}) \subseteq C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{m}, \mathfrak{n})$  is obvious.

Now, for any  $m \in \mathfrak{m}$ ,  $n \in \mathfrak{n}$  and  $q \in \mathfrak{q}$ , we have

$$[m, n] + [n, m], q = [m, [n, q]] + [[n, q], m] + [n, [m, q]] + [[m, q], n] \in [\mathfrak{m}, \mathfrak{n}]_{\text{Lie}},$$

and

$$[q, [m, n] + [n, m]] = 0.$$

Consequently,  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$  is a two-sided ideal of  $\mathfrak{q}$  contained in  $Z^r(\mathfrak{q})$ .  $\square$

In particular, the two-sided ideal  $C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{q}, 0)$  is the *Lie-center* of the Leibniz algebra  $\mathfrak{q}$  and it will be denoted by  $Z_{\text{Lie}}(\mathfrak{q})$ , that is,

$$Z_{\text{Lie}}(\mathfrak{q}) = \{z \in \mathfrak{q} \mid [q, z] + [z, q] = 0 \text{ for all } q \in \mathfrak{q}\}.$$

The following proposition is an immediate consequence of the discussion in [7, Example 1.9].

**Proposition 3.4** *Given an extension of Leibniz algebras  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  with  $\mathfrak{n} = \text{Ker}(f)$ , the following conditions are equivalent:*

- (a)  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is *Lie-central*;
- (b)  $\mathfrak{n} \subseteq Z_{\text{Lie}}(\mathfrak{q})$ ;
- (c)  $[\mathfrak{n}, \mathfrak{g}]_{\text{Lie}} = 0$ .

**Remark 3.5** *Obviously every central extension  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  (i.e.  $\text{Ker}(f) \subseteq Z(\mathfrak{g})$ ) is a Lie-central extension, but the converse is not true in general as the following example shows: consider the three and two-dimensional Leibniz algebras  $\mathfrak{g}$  and  $\mathfrak{q}$  with  $\mathbb{K}$ -linear bases  $\{a_1, a_2, a_3\}$  and  $\{e_1, e_2\}$ , respectively, and with the Leibniz brackets given respectively by  $[a_1, a_3] = a_1$  and  $[e_1, e_2] = e_1$  and zero elsewhere. Then the homomorphism of Leibniz algebras  $f : \mathfrak{g} \rightarrow \mathfrak{q}$  given by  $f(a_1) = 0, f(a_2) = e_1, f(a_3) = e_2$  is surjective,  $\text{Ker}(f) = \langle \{a_1\} \rangle$ ,  $Z(\mathfrak{g}) = \langle \{a_2\} \rangle$  and  $Z_{\text{Lie}}(\mathfrak{g}) = \langle \{a_1\} \rangle$ . Consequently the extension  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is not central, but it is Lie-central.*

The five-term exact sequence **(B)** associated to an extension of Leibniz algebras  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ , with  $\mathfrak{n} = \text{Ker}(f)$ , turns to

$$H_2(\mathfrak{g}, (-)_{\text{Lie}}) \longrightarrow H_2(\mathfrak{q}, (-)_{\text{Lie}}) \xrightarrow{\theta^*(\mathfrak{g})} \frac{\mathfrak{n}}{[\mathfrak{n}, \mathfrak{g}]_{\text{Lie}}} \longrightarrow \mathfrak{g}_{\text{Lie}} \longrightarrow \mathfrak{q}_{\text{Lie}} \longrightarrow 0. \quad (\mathbf{D})$$

If  $0 \rightarrow \mathfrak{r} \rightarrow \mathfrak{f} \xrightarrow{\rho} \mathfrak{g} \rightarrow 0$  is a free presentation of  $\mathfrak{g}$ , then  $0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{f} \xrightarrow{\tau=f \circ \rho} \mathfrak{q} \rightarrow 0$  is a free presentation of  $\mathfrak{q}$  and we have the following commutative diagram of Leibniz



algebras

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & \swarrow & & \\
 & & & \mathfrak{r} & & & \\
 & & \swarrow & \downarrow & & & \\
 0 & \longrightarrow & \mathfrak{s} & \longrightarrow & \mathfrak{f} & & \\
 & & \downarrow & & \downarrow & \searrow & \\
 & & \mathfrak{n} & \longrightarrow & \mathfrak{g} & \xrightarrow{f} & \mathfrak{q} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \searrow \\
 & & 0 & & 0 & & 0
 \end{array}
 \quad \text{(E)}$$

$\tau = f \circ \rho$

Now suppose that  $f$  is a Lie-central extension, hence  $[\mathfrak{n}, \mathfrak{g}]_{\text{Lie}} = 0$  and the sequence (D) actually is

$$\frac{\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}} \longrightarrow \frac{\mathfrak{s} \cap [\mathfrak{f}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \longrightarrow \mathfrak{g}_{\text{Lie}} \longrightarrow \mathfrak{q}_{\text{Lie}} \longrightarrow 0 \quad \text{(F)}$$

### 3.2 Lie-homology of Leibniz algebras

Recall from [19, 20] that the Leibniz homology  $HL_*(\mathfrak{q})$  of a Leibniz algebra  $\mathfrak{q}$  with trivial coefficients is defined to be the homology of the Leibniz complex  $(CL_*(\mathfrak{q}), d_*)$ , where  $CL_n(\mathfrak{q}) = \mathfrak{q}^{\otimes n}$  and  $d_n : CL_n(\mathfrak{q}) \rightarrow CL_{n-1}(\mathfrak{q})$ ,  $n \geq 1$ , is given by

$$d_n(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i < j \leq n} (-1)^{j+1} x_1 \otimes \cdots \otimes [x_i, x_j] \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n.$$

As usual  $\widehat{x_j}$  means that the variable  $x_j$  is omitted.

It is well known that the first Leibniz homology of a Leibniz algebra  $\mathfrak{q}$  is the abelianization of  $\mathfrak{q}$ , i.e.  $HL_1(\mathfrak{q}) = \mathfrak{q}/[\mathfrak{q}, \mathfrak{q}]$ . Below we change the boundary operator of  $(CL_*(\mathfrak{q}), d_*)$  in such a way that we get a chain complex whose first homology is the Liezation of  $\mathfrak{q}$ .

In the sequel we denote the elements  $[x, y] + [y, x]$  by  $\langle x, y \rangle$ , for all  $x, y \in \mathfrak{q}$ . The following lemma is an easy exercise.

**Lemma 3.6** *Given a Leibniz algebra  $\mathfrak{q}$  the following identities hold:*

- (a)  $\langle \langle a, b \rangle, c \rangle = \langle b, [a, c] \rangle + \langle a, [b, c] \rangle = [\langle a, b \rangle, c] = \langle [a, c], b \rangle + \langle a, [b, c] \rangle;$
- (b)  $\langle a, b \rangle = \langle b, a \rangle;$
- (c)  $[a, \langle b, c \rangle] = 0.$

for all  $a, b, c \in \mathfrak{q}$ .

For a Leibniz algebra  $\mathfrak{q}$  and  $n \geq 1$ , we define  $\delta_n : CL_n(\mathfrak{q}) \rightarrow CL_{n-1}(\mathfrak{q})$  by

$$\begin{aligned} \delta_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) &= \langle x_1, x_2 \rangle \otimes x_3 \otimes \cdots \otimes x_n \\ &+ \sum_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,2)}} (-1)^j x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n. \end{aligned}$$

We claim that  $(CL_*(\mathfrak{q}), \delta_*)$  is again a chain complex. In effect, for any  $y \in \mathfrak{q}$  and  $n \geq 1$ , define  $\mathbb{K}$ -linear maps:

$$\theta_n(y) : CL_n(\mathfrak{q}) \rightarrow CL_n(\mathfrak{q}), \quad \theta_n(y)(q_1 \otimes \cdots \otimes q_n) = \sum_{i=1}^n q_1 \otimes \cdots \otimes [q_i, y] \otimes \cdots \otimes q_n$$

and

$$i_n(y) : CL_n(\mathfrak{q}) \rightarrow CL_{n+1}(\mathfrak{q}), \quad i_n(y)(q_1 \otimes \cdots \otimes q_n) = (-1)^n q_1 \otimes \cdots \otimes q_n \otimes y.$$

Then we have

**Proposition 3.7** *The following identities hold:*

- (a)  $\delta_{n+1}i_n(y) + i_{n-1}(y)\delta_n = -\theta_n(y)$ ,  $n \geq 2$ .
- (b)  $\theta_n(x)\theta_n(y) - \theta_n(y)\theta_n(x) = -\theta_n([x, y])$ ,  $n \geq 1$ .
- (c)  $\theta_n(x)i_{n-1}(y) - i_{n-1}(y)\theta_{n-1}(x) = i_{n-1}([y, x])$ ,  $n \geq 2$ .
- (d)  $\theta_{n-1}(y)\delta_n = \delta_n\theta_n(y)$ ,  $n \geq 2$ .
- (e)  $\delta_n\delta_{n+1} = 0$ ,  $n \geq 2$ .

*Proof.* Thanks to the equalities in Lemma 3.6, the proof can be done by induction on  $n$ , similarly to the proof of Proposition (3.1) in [20].  $\square$

**Definition 3.8** *We call the homology of the complex  $(CL_*(\mathfrak{q}), \delta_*)$  the Lie-homology of the Leibniz algebra  $\mathfrak{q}$  and denote by  $HL_*^{\text{Lie}}(\mathfrak{q})$ , that is,*

$$HL_*^{\text{Lie}}(\mathfrak{q}) := H_*(CL_*(\mathfrak{q}), \delta_*).$$

It is clear that  $HL_1^{\text{Lie}}(\mathfrak{q}) = \mathfrak{q}/\mathfrak{q}^{\text{ann}} = \mathfrak{q}_{\text{Lie}} = H_1(\mathfrak{q}, (-)_{\text{Lie}})$ .

Furthermore, by considering a free presentation  $0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \rightarrow 0$  of the Leibniz algebra  $\mathfrak{q}$ , and having in mind that

$$H_2(\mathfrak{q}, (-)_{\text{Lie}}) = \frac{\mathfrak{s} \cap [\mathfrak{f}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} = \frac{\mathfrak{s} \cap \mathfrak{f}^{\text{ann}}}{\langle \{[s, f] + [f, s], s \in \mathfrak{s}, f \in \mathfrak{f}\} \rangle},$$

we define the  $\mathbb{K}$ -linear map  $\Phi : HL_2^{\text{Lie}}(\mathfrak{q}) \rightarrow H_2(\mathfrak{q}, (-)_{\text{Lie}})$  by  $\Phi(\overline{q_1 \otimes q_2}) = \overline{x}$ , where  $x = [f_1, f_2] + [f_2, f_1]$ , with  $\tau(f_1) = q_1$  and  $\tau(f_2) = q_2$ . Moreover, it is easy to see that this map is bijective and there is an isomorphism of  $\mathbb{K}$ -vector spaces

$$HL_2^{\text{Lie}}(\mathfrak{q}) \approx H_2(\mathfrak{q}, (-)_{\text{Lie}}).$$

**Proposition 3.9** *Given a Lie-central extension  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  of Leibniz algebras with  $\mathfrak{n} = \text{Ker}(f)$ , there is a six-term exact sequence*

$$\mathfrak{n} \otimes \mathfrak{g}_{\text{Lie}} \longrightarrow HL_2^{\text{Lie}}(\mathfrak{g}) \longrightarrow HL_2^{\text{Lie}}(\mathfrak{q}) \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \longrightarrow HL_1^{\text{Lie}}(\mathfrak{g}) \longrightarrow HL_1^{\text{Lie}}(\mathfrak{q}) \longrightarrow 0. \quad (\mathbf{G})$$

*Proof.* By considering the free presentations as in **(E)**, we can rewrite the sequence **(F)** as follows:

$$HL_2^{\text{Lie}}(\mathfrak{g}) \longrightarrow HL_2^{\text{Lie}}(\mathfrak{q}) \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \longrightarrow HL_1^{\text{Lie}}(\mathfrak{g}) \longrightarrow HL_1^{\text{Lie}}(\mathfrak{q}) \longrightarrow 0. \quad (\mathbf{H})$$

Now we extend this sequence by one more term. For that, we consider again the diagram of free presentations **(E)**. Then we have

$$\text{Ker}(HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})) \approx \text{Ker}\left(\frac{\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}} \rightarrow \frac{\mathfrak{s} \cap [\mathfrak{f}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}}\right) \approx \frac{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}.$$

Now we define  $\sigma : \mathfrak{n} \otimes \mathfrak{g}_{\text{Lie}} \rightarrow \frac{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}$  by  $\sigma(n \otimes \overline{x}) = [s, f] + [f, s] + [\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}$ , where  $s \in \mathfrak{r}$  and  $f \in \mathfrak{f}$  such that  $\rho(s) = n$  and  $\rho(f) = x$ . It is easy to check that  $\sigma$  is a well-defined epimorphism. Then the assertion follows.  $\square$

Note that there is a morphism of chain complexes  $\varphi_* : (CL_*(\mathfrak{q}), \delta_*) \rightarrow (CL_*(\mathfrak{q}), d_*)$  given inductively by

$$\begin{aligned} \varphi_1 &= id_{\mathfrak{q}}, \\ \varphi_2(x_1 \otimes x_2) &= x_1 \otimes x_2 + x_2 \otimes x_1, \\ \varphi_n(x_1 \otimes \cdots \otimes x_n) &= \varphi_{n-1}(x_1 \otimes \cdots \otimes x_{n-1}) \otimes x_n \\ &\quad + (-1)^n x_n \otimes \varphi_{n-1}(x_1 \otimes \cdots \otimes x_{n-1}), \quad n \geq 3. \end{aligned}$$

### 3.3 Some properties of Lie-central extensions

In this subsection we establish some properties of Lie-central extensions of Leibniz algebras by using the six-term exact sequence **(G)** associated to the given extension.

**Proposition 3.10** *Let  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  be an extension of Leibniz algebras with  $\mathfrak{n} = \text{Ker}(f)$ . The following statements are equivalent:*

- (a)  $f: \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is a Lie-trivial extension.
- (b)  $\mathfrak{n} \cap \mathfrak{g}^{\text{ann}} = 0$ .
- (c)  $0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{g}_{\text{Lie}} \rightarrow \mathfrak{q}_{\text{Lie}} \rightarrow 0$  is exact in Lie.

*Proof.* The equivalences follow from the definition of Lie-trivial extension and by the  $3 \times 3$  Lemma [3] applied to the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathfrak{n} \cap \mathfrak{g}^{\text{ann}} & \longrightarrow & \mathfrak{g}^{\text{ann}} & \longrightarrow & \mathfrak{q}^{\text{ann}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{q} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \frac{\mathfrak{n}}{\mathfrak{n} \cap \mathfrak{g}^{\text{ann}}} & \longrightarrow & HL_1^{\text{Lie}}(\mathfrak{g}) & \longrightarrow & HL_1^{\text{Lie}}(\mathfrak{q}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

□

**Proposition 3.11** *Let  $f: \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  be a Lie-trivial extension with  $\mathfrak{n} = \text{Ker}(f)$ , then  $\theta^*(\mathfrak{g}): HL_2^{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{n}$  is the zero map and  $\mathfrak{n} \otimes \mathfrak{g}_{\text{Lie}} \rightarrow HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q}) \rightarrow 0$  is exact.*

*Proof.* Since  $[\mathfrak{n}, \mathfrak{g}]_{\text{Lie}} \subseteq \mathfrak{n} \cap \mathfrak{g}^{\text{ann}}$ , by Proposition 3.10 we have that  $[\mathfrak{n}, \mathfrak{g}]_{\text{Lie}} = 0$ , then  $\text{Im}(\theta^*(\mathfrak{g})) = 0$  in sequence  $(\mathbf{G})$ . □

**Corollary 3.12** *Let  $f: \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  be a Lie-trivial extension with  $\mathfrak{n} = \text{Ker}(f)$ , then  $\mathfrak{n} = Z_{\text{Lie}}(\mathfrak{n})$ .*

*Proof.* Since  $\mathfrak{n} \cap \mathfrak{g}^{\text{ann}} = 0$ , we have  $[n, n'] + [n', n] = [n + n', n + n'] = 0$  for any  $n, n' \in \mathfrak{n}$ , which means that  $\mathfrak{n} \subseteq Z_{\text{Lie}}(\mathfrak{n})$ . □

**Definition 3.13** *A Lie-central extension  $f: \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is said to be:*

- (a) a Lie-stem extension if  $\mathfrak{g}_{\text{Lie}} \cong \mathfrak{q}_{\text{Lie}}$ .
- (b) a Lie-stem cover if  $\mathfrak{g}_{\text{Lie}} \cong \mathfrak{q}_{\text{Lie}}$  and the induced map  $HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  is the zero map.

**Example 3.14**

- (a) Let  $\mathfrak{g}$  and  $\mathfrak{q}$  be three and two-dimensional Leibniz algebra with  $\mathbb{K}$ -linear bases  $\{a_1, a_2, a_3\}$  and  $\{e_1, e_2\}$ , with the Leibniz brackets given respectively by  $[a_2, a_3] = -[a_3, a_2] = a_2$ ,  $[a_3, a_3] = a_1$  and  $[e_1, e_2] = -[e_2, e_1] = -e_1$  and zero elsewhere. Then the surjective homomorphism of Leibniz algebras  $f : \mathfrak{g} \rightarrow \mathfrak{q}$  defined by  $f(a_1) = 0$ ,  $f(a_2) = e_1$  and  $f(a_3) = e_2$  is a Lie-stem extension.
- (b) Let  $0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \rightarrow 0$  be a free presentation of a Leibniz algebra  $\mathfrak{q}$ . Then  $[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}$  is a two-sided ideal of  $\mathfrak{f}$  and the  $3 \times 3$  Lemma [3] provides the sequence  $0 \rightarrow \mathfrak{s}/[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}} \rightarrow \mathfrak{f}/[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}} \rightarrow \mathfrak{q} \rightarrow 0$  which is a Lie-stem cover of  $\mathfrak{q}$ .

**Proposition 3.15** For a Lie-central extension  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ , with  $\mathfrak{n} = \text{Ker}(f)$ , the following statements are equivalent:

- (a)  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is a Lie-stem extension.
- (b) The induced map  $\mathfrak{n} \rightarrow HL_1^{\text{Lie}}(\mathfrak{q})$  is the zero map.
- (c)  $\theta^*(\mathfrak{g}) : HL_2^{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{n}$  is an epimorphism.
- (d) The following sequence  $\mathfrak{n} \otimes \mathfrak{g}_{\text{Lie}} \rightarrow HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q}) \xrightarrow{\theta^*(\mathfrak{g})} \mathfrak{n} \rightarrow 0$  is exact.
- (e)  $\mathfrak{n} \subseteq \mathfrak{g}^{\text{ann}}$ .

*Proof.* The equivalences between (a), (b), (c) and (d) follow from the exact sequence (G). The equivalence between (a) and (e) is a consequence of the following  $3 \times 3$  diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathfrak{n} \cap \mathfrak{g}^{\text{ann}} & \longrightarrow & \mathfrak{g}^{\text{ann}} & \longrightarrow & \mathfrak{q}^{\text{ann}} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{q} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & HL_1^{\text{Lie}}(\mathfrak{g}) = HL_1^{\text{Lie}}(\mathfrak{q}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

□

**Proposition 3.16** For a Lie-central extension  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  the following statements are equivalent:

- (a)  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is a Lie-stem cover.
- (b)  $\theta^*(\mathfrak{g}) : HL_2^{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{n}$  is an isomorphism.

*Proof.* This is a direct consequence of the five-term exact sequence (H). □

## 4 The precise Lie-center of a Leibniz algebra

In this section we introduce the notions of Lie-unicentrality, Lie-capability and precise Lie-center of a Leibniz algebra and analyze the relationships between them.

**Definition 4.1** *A Leibniz algebra  $\mathfrak{q}$  is said to be Lie-unicentral if every Lie-central extension  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  satisfies  $f(Z_{\text{Lie}}(\mathfrak{g})) = Z_{\text{Lie}}(\mathfrak{q})$ , that is, the following diagram with exact rows*

$$\begin{array}{ccccc} \text{Ker}(f) & \hookrightarrow & Z_{\text{Lie}}(\mathfrak{g}) & \twoheadrightarrow & Z_{\text{Lie}}(\mathfrak{q}) \\ \parallel & & \downarrow & & \downarrow \\ \text{Ker}(f) & \hookrightarrow & \mathfrak{g} & \xrightarrow{f} & \mathfrak{q} \end{array}$$

*is commutative.*

**Definition 4.2** *A Leibniz algebra  $\mathfrak{q}$  is said to be Lie-capable if there exists a Lie-central extension*

$$0 \longrightarrow Z_{\text{Lie}}(\mathfrak{g}) \longrightarrow \mathfrak{g} \xrightarrow{f} \mathfrak{q} \longrightarrow 0.$$

**Definition 4.3** *The precise Lie-center  $Z_{\text{Lie}}^*(\mathfrak{q})$  of a Leibniz algebra  $\mathfrak{q}$  is the intersection of all two-sided ideals  $f(Z_{\text{Lie}}(\mathfrak{g}))$ , where  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is a Lie-central extension.*

**Remark 4.4**

- (a)  $Z_{\text{Lie}}^*(\mathfrak{q}) \subseteq Z_{\text{Lie}}(\mathfrak{q})$ .
- (b)  $Z_{\text{Lie}}^*(\mathfrak{q}) = Z_{\text{Lie}}(\mathfrak{q})$  if and only if  $f(Z_{\text{Lie}}(\mathfrak{g})) = Z_{\text{Lie}}(\mathfrak{q})$  for every Lie-central extension of Leibniz algebras  $f : \mathfrak{g} \twoheadrightarrow \mathfrak{q}$ , or equivalently, if and only if  $\mathfrak{q}$  is Lie-unicentral.

Given a free presentation  $0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \rightarrow 0$  of the Leibniz algebra  $\mathfrak{q}$ , consider the Lie-central extension (Lie-stem cover of  $\mathfrak{q}$ )

$$0 \longrightarrow \frac{\mathfrak{s}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \longrightarrow \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \xrightarrow{\bar{\tau}} \mathfrak{q} \longrightarrow 0$$

as in Example 3.14 (b). Then we have

**Lemma 4.5**  $Z_{\text{Lie}}^*(\mathfrak{q}) = \bar{\tau} \left( Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \right).$

*Proof.* We need to show that, for any Lie-central extension  $0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{h} \xrightarrow{\varphi} \mathfrak{q} \rightarrow 0$ , the inclusion  $\bar{\tau} \left( Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \right) \subseteq \varphi (Z_{\text{Lie}}(\mathfrak{h}))$  holds.

Since  $\mathfrak{f}$  is a free Leibniz algebra, there exists (uniquely defined epimorphism)  $\alpha : \mathfrak{f} \rightarrow \mathfrak{h}$  such that  $\varphi \circ \alpha = \tau$ . Then  $\alpha(\mathfrak{s}) \subseteq \mathfrak{a}$  and  $\alpha([\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}) \subseteq [\mathfrak{a}, \mathfrak{h}]_{\text{Lie}} = 0$ . Hence,  $\alpha$  induces  $\bar{\alpha} : \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \rightarrow \mathfrak{h}$  such that  $\bar{\alpha} \circ \pi = \alpha$ , where  $\pi : \mathfrak{f} \twoheadrightarrow \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}}$  is the natural projection. It is straightforward to see that  $\bar{\alpha} \left( Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \right) \subseteq Z_{\text{Lie}}(\mathfrak{h})$ . Now, since  $\bar{\tau} \circ \pi = \tau = \varphi \circ \alpha = \varphi \circ \bar{\alpha} \circ \pi$ , it follows that

$$\bar{\tau} \left( Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \right) = (\varphi \circ \bar{\alpha}) \left( Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \right) \subseteq \varphi (Z_{\text{Lie}}(\mathfrak{h})).$$

□

**Corollary 4.6**  $Z_{\text{Lie}}^*(\mathfrak{q}) = 0$  if and only if  $\mathfrak{q}$  is a Lie-capable Leibniz algebra.

*Proof.* If  $\mathfrak{q}$  is a Lie-capable Leibniz algebra, then there exists a Lie-central extension  $0 \rightarrow Z_{\text{Lie}}(\mathfrak{g}) \rightarrow \mathfrak{g} \xrightarrow{f} \mathfrak{q} \rightarrow 0$ , then  $Z_{\text{Lie}}^*(\mathfrak{q}) \subseteq f(Z_{\text{Lie}}(\mathfrak{g})) = 0$ .

Conversely, if  $Z_{\text{Lie}}^*(\mathfrak{q}) = 0$ , for any free presentation  $0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{f} \xrightarrow{\tau} \mathfrak{q} \rightarrow 0$  of  $\mathfrak{q}$ , we have  $\bar{\tau} \left( Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \right) = 0$  by Lemma 4.5. Then

$$0 \longrightarrow Z_{\text{Lie}} \left( \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \right) \longrightarrow \frac{\mathfrak{f}}{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}} \xrightarrow{\bar{\tau}} \mathfrak{q} \longrightarrow 0$$

is a Lie-central extension.

□

**Lemma 4.7** Let  $\pi : \mathfrak{g} \rightarrow \mathfrak{q}$  be an epimorphism of Leibniz algebras, then  $\pi(Z_{\text{Lie}}^*(\mathfrak{g})) \subseteq Z_{\text{Lie}}^*(\mathfrak{q})$ .

*Proof.* For any Lie-central extension  $0 \rightarrow \mathfrak{n} \rightarrow \mathfrak{h} \xrightarrow{\varphi} \mathfrak{q} \rightarrow 0$  of  $\mathfrak{q}$ , consider the pull-back diagram over  $\pi$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{g} \times_{\mathfrak{q}} \mathfrak{h} & \xrightarrow{\bar{\varphi}} & \mathfrak{g} \longrightarrow 0 \\ & & \parallel & & \downarrow \bar{\pi} & & \downarrow \pi \\ 0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{h} & \xrightarrow{\varphi} & \mathfrak{q} \longrightarrow 0. \end{array}$$

Clearly the upper row is again a Lie-central extension of  $\mathfrak{q}$ . Then we have  $\pi(Z_{\text{Lie}}^*(\mathfrak{g})) \subseteq \pi \circ \bar{\varphi}(Z_{\text{Lie}}(\mathfrak{g} \times_{\mathfrak{q}} \mathfrak{h})) = \varphi \circ \bar{\pi}(Z_{\text{Lie}}(\mathfrak{g} \times_{\mathfrak{q}} \mathfrak{h})) \subseteq \varphi(Z_{\text{Lie}}(\mathfrak{h}))$ . This implies that  $\pi(Z_{\text{Lie}}^*(\mathfrak{g})) \subseteq \bigcap_{\varphi} \varphi(Z_{\text{Lie}}(\mathfrak{h})) = Z_{\text{Lie}}^*(\mathfrak{q})$ . □

**Proposition 4.8**

- (a)  $Z_{\text{Lie}}^*(\mathfrak{q})$  is the smallest two-sided ideal  $\mathfrak{n}$  of the Leibniz algebra  $\mathfrak{q}$  such that  $\mathfrak{q}/\mathfrak{n}$  is Lie-capable. In particular,  $\mathfrak{q}/Z_{\text{Lie}}^*(\mathfrak{q})$  is Lie-capable.

- (b) Let  $\mathfrak{n}$  be a two-sided ideal of a Leibniz algebra  $\mathfrak{q}$  such that  $\mathfrak{n} \cap Z_{\text{Lie}}^*(\mathfrak{q}) = 0$ . If  $\mathfrak{q}/\mathfrak{n}$  is Lie-capable, then  $\mathfrak{q}$  is Lie-capable as well.

*Proof.* (a) Given a two-sided ideal  $\mathfrak{n}$  of  $\mathfrak{q}$  such that  $\mathfrak{q}/\mathfrak{n}$  is Lie-capable, consider the epimorphism  $\pi : \mathfrak{q} \twoheadrightarrow \mathfrak{q}/\mathfrak{n}$ . By Lemma 4.7 and Corollary 4.6 we have  $\pi(Z_{\text{Lie}}^*(\mathfrak{q})) \subseteq Z_{\text{Lie}}^*(\mathfrak{q}/\mathfrak{n}) = 0$ , then  $Z_{\text{Lie}}^*(\mathfrak{q}) \subseteq \text{Ker}(\pi) = \mathfrak{n}$ .

(b) By (a),  $Z_{\text{Lie}}^*(\mathfrak{q}) \subseteq \mathfrak{n}$ . Since  $Z_{\text{Lie}}^*(\mathfrak{q}) \cap \mathfrak{n} = 0$ , then  $Z_{\text{Lie}}^*(\mathfrak{q}) = 0$ . Then Corollary 4.6 completes the proof.  $\square$

**Theorem 4.9** Let  $\mathfrak{a}$  be a two-sided ideal of a Leibniz algebra  $\mathfrak{q}$  such that  $\mathfrak{a} \subseteq Z_{\text{Lie}}(\mathfrak{q})$ . Then  $\mathfrak{a} \subseteq Z_{\text{Lie}}^*(\mathfrak{q})$  if and only if the map  $C : \mathfrak{a} \otimes \mathfrak{q}_{\text{Lie}} \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  in sequence  $(\mathbf{G})$  associated to the Lie-central extension  $0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{q} \xrightarrow{\pi} \mathfrak{q}/\mathfrak{a} \rightarrow 0$  is the zero map.

*Proof.* Consider the free presentations  $0 \rightarrow \mathfrak{r} \rightarrow \mathfrak{f} \xrightarrow{\rho} \mathfrak{q} \rightarrow 0$  and  $0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{f} \xrightarrow{\pi \circ \rho} \mathfrak{q}/\mathfrak{a} \rightarrow 0$ . We know from the proof of Proposition 3.9 that  $\text{Im}(C) \cong \frac{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}$ , then by Proposition 4.5 we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & Z_{\text{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}\right) & \longrightarrow & Z_{\text{Lie}}^*(\mathfrak{q}) & \\
 & \nearrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \frac{\mathfrak{r}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}} & \xrightarrow{\quad \bar{\rho} \quad} & \mathfrak{q} & \longrightarrow & 0 \\
 & & \downarrow \epsilon & & \downarrow \gamma & & \\
 & & \bullet & \xrightarrow{\sim} & \mathfrak{q}/Z_{\text{Lie}}^*(\mathfrak{q}) & & 
 \end{array}$$

Hence  $C = 0 \Leftrightarrow \frac{[\mathfrak{s}, \mathfrak{f}]_{\text{Lie}}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}} = 0 \Leftrightarrow \frac{\mathfrak{s}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}} \subseteq Z_{\text{Lie}}\left(\frac{\mathfrak{f}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}\right) \Leftrightarrow \gamma \circ \bar{\rho}\left(\frac{\mathfrak{s}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}\right) = \epsilon\left(\frac{\mathfrak{s}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}\right) = 0 \Leftrightarrow \mathfrak{a} = \rho(\mathfrak{s}) = \bar{\rho}\left(\frac{\mathfrak{s}}{[\mathfrak{r}, \mathfrak{f}]_{\text{Lie}}}\right) \subseteq \text{Ker}(\gamma) = Z_{\text{Lie}}^*(\mathfrak{q})$ .  $\square$

**Corollary 4.10** For a Leibniz algebra  $\mathfrak{q}$  the following statements are equivalent:

- (a)  $\mathfrak{q}$  is Lie-unicentral.
- (b) The map  $C : Z_{\text{Lie}}(\mathfrak{q}) \otimes \mathfrak{q}_{\text{Lie}} \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  in the sequence  $(\mathbf{G})$  associated to the Lie-central extension  $0 \rightarrow Z_{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{q} \rightarrow \mathfrak{q}/Z_{\text{Lie}}(\mathfrak{q}) \rightarrow 0$  is the zero map.
- (c) The canonical homomorphism  $HL_2^{\text{Lie}}(\mathfrak{q}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q}/Z_{\text{Lie}}(\mathfrak{q}))$  is injective.

*Proof.* This is a consequence of the exactness of the sequence  $(\mathbf{G})$  associated to the Lie-central extension  $0 \rightarrow Z_{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{q} \rightarrow \mathfrak{q}/Z_{\text{Lie}}(\mathfrak{q}) \rightarrow 0$ .  $\square$

**Remark 4.11** If  $\mathfrak{q}$  is a Lie-unicentral Leibniz algebra, by Corollary 4.10 and the sequence  $(\mathbf{G})$  associated to the Lie-central extension  $0 \rightarrow Z_{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{q} \rightarrow \mathfrak{q}/Z_{\text{Lie}}(\mathfrak{q}) \rightarrow 0$ , we have

$$HL_2^{\text{Lie}}(\mathfrak{q}) = \text{Ker}(\theta^*(\mathfrak{q}) : HL_2^{\text{Lie}}(\mathfrak{q}/Z_{\text{Lie}}(\mathfrak{q})) \rightarrow Z_{\text{Lie}}(\mathfrak{q})).$$



## 5 Lie-solvable and Lie-nilpotent Leibniz algebras

In this section, by using the relative commutators, we introduce the notions of Lie-solvability and Lie-nilpotency of Leibniz algebras and investigate their properties.

**Definition 5.1** Let  $\mathfrak{m}$  be a two-sided ideal of a Leibniz algebra  $\mathfrak{q}$ . A series from  $\mathfrak{m}$  to  $\mathfrak{q}$  is a finite sequence of two-sided ideals  $\mathfrak{m}_i$ ,  $0 \leq i \leq k$ , of  $\mathfrak{q}$  such that

$$\mathfrak{m} = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{k-1} \trianglelefteq \mathfrak{m}_k = \mathfrak{q}.$$

$k$  is called the length of this series.

A series from  $\mathfrak{m}$  to  $\mathfrak{q}$  of length  $k$  is said to be Lie-central (resp. Lie-abelian) if  $[\mathfrak{m}_i, \mathfrak{q}]_{\text{Lie}} \subseteq \mathfrak{m}_{i-1}$ , or equivalently  $\mathfrak{m}_i/\mathfrak{m}_{i-1} \subseteq Z_{\text{Lie}}(\mathfrak{q}/\mathfrak{m}_{i-1})$  (resp. if  $[\mathfrak{m}_i, \mathfrak{m}_i]_{\text{Lie}} \subseteq \mathfrak{m}_{i-1}$ , or equivalently  $[\mathfrak{m}_i/\mathfrak{m}_{i-1}, \mathfrak{m}_i/\mathfrak{m}_{i-1}]_{\text{Lie}} = 0$ ) for  $1 \leq i \leq k$ .

A series from 0 to  $\mathfrak{q}$  is called a series of the Leibniz algebra  $\mathfrak{q}$ .

**Definition 5.2** A Leibniz algebra  $\mathfrak{q}$  is said to be Lie-solvable if it has a Lie-abelian series. If  $k$  is the minimal length of such series, then  $k$  is called the class of Lie-solvability of  $\mathfrak{q}$ .

We show below that among all Lie-abelian series of a Lie-solvable Leibniz algebra there is one which descends most rapidly.

**Definition 5.3** The Lie-derived series of a Leibniz algebra  $\mathfrak{q}$  is the sequence

$$\cdots \trianglelefteq \mathfrak{q}^{(i)} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{(1)} \trianglelefteq \mathfrak{q}^{(0)}$$

of two-sided ideals of  $\mathfrak{q}$  defined inductively by

$$\mathfrak{q}^{(0)} = \mathfrak{q} \quad \text{and} \quad \mathfrak{q}^{(i)} = [\mathfrak{q}^{(i-1)}, \mathfrak{q}^{(i-1)}]_{\text{Lie}}, \quad i \geq 1.$$

### Theorem 5.4

- (a) Let  $\mathfrak{q}$  be a Leibniz algebra and  $\mathfrak{m} = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{j-1} \trianglelefteq \mathfrak{m}_j = \mathfrak{q}$  be a Lie-abelian series from  $\mathfrak{m}$  to  $\mathfrak{q}$ , then  $\mathfrak{q}^{(i)} \subseteq \mathfrak{m}_{j-i}$ ,  $0 \leq i \leq j$ .
- (b) A Leibniz algebra  $\mathfrak{q}$  is Lie-solvable with class of Lie-solvability  $k$  if and only if  $\mathfrak{q}^{(k)} = 0$  and  $\mathfrak{q}^{(k-1)} \neq 0$ .

*Proof.* (a) This easily follows by induction on  $i$ .

(b) If  $\mathfrak{q}^{(k)} = 0$  and  $\mathfrak{q}^{(k-1)} \neq 0$ , then  $0 = \mathfrak{q}^{(k)} \trianglelefteq \mathfrak{q}^{(k-1)} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{(1)} \trianglelefteq \mathfrak{q}^{(0)} = \mathfrak{q}$  is a Lie-abelian series and by (a) its length is minimal. Therefore  $\mathfrak{q}$  is a Lie-solvable with class of solvability  $k$ . The converse statement directly follows from (a).  $\square$

### Example 5.5

- (a) Abelian Leibniz algebras are Lie-solvable Leibniz algebras of class 1.
- (b) Lie algebras are Lie-solvable Leibniz algebras of class 1.
- (c) The three-dimensional (non-Lie) Leibniz algebra with  $\mathbb{K}$ -linear basis  $\{a_1, a_2, a_3\}$  and Leibniz bracket given by

$$[a_1, a_3] = [a_2, a_3] = a_2, [a_3, a_3] = a_1$$

and zero elsewhere (see [5]), is a Lie-solvable Leibniz algebra of class 2 and solvable of class 2.

- (d) The five-dimensional perfect (non-Lie) Leibniz algebra with  $\mathbb{K}$ -linear basis  $\{a_1, a_2, a_3, a_4, a_5\}$  and Leibniz bracket given by

$$\begin{aligned} [a_2, a_1] &= -a_3, & [a_1, a_2] &= a_3, & [a_1, a_3] &= -2a_1, \\ [a_3, a_1] &= 2a_1, & [a_3, a_2] &= -2a_2, & [a_2, a_3] &= 2a_2, \\ [a_5, a_1] &= a_4, & [a_4, a_2] &= a_5, & [a_4, a_3] &= -a_4, \\ & & & & [a_5, a_3] &= a_5, \end{aligned}$$

and zero elsewhere (see [22]) is Lie-solvable of class 2, but it is not a solvable Leibniz algebra.

- (e) Subalgebras and images by homomorphisms of Lie-solvable Leibniz algebras are Lie-solvable as well.
- (f) Intersection and sum of Lie-solvable two-sided ideals of a Leibniz algebra are Lie-solvable two-sided ideals as well.

### Proposition 5.6

- (a) Let  $\mathfrak{n}$  be a Lie-solvable ideal of a Leibniz algebra  $\mathfrak{q}$  such that  $\mathfrak{q}/\mathfrak{n}$  is Lie-solvable, then  $\mathfrak{q}$  itself is Lie-solvable.
- (b) Let  $\mathfrak{m}$  and  $\mathfrak{n}$  be Lie-solvable two-sided ideals of a Leibniz algebra  $\mathfrak{q}$ , then  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$  is a Lie-solvable two-sided ideal of  $\mathfrak{q}$ .

*Proof.* (a) Since  $\mathfrak{q}/\mathfrak{n}$  is Lie-solvable, then there exists  $k \in \mathbb{N}$  such that  $0 = (\mathfrak{q}/\mathfrak{n})^{(k)} = \mathfrak{q}^{(k)}/\mathfrak{n}$ , that is  $\mathfrak{q}^{(k)} = \mathfrak{n}$ . On the other hand, since  $\mathfrak{n}$  is Lie-solvable, then there exists  $j \in \mathbb{N}$  such that  $\mathfrak{n}^{(j)} = 0$ . Hence  $\mathfrak{q}^{(k+j)} = (\mathfrak{q}^{(k)})^{(j)} = \mathfrak{n}^{(j)} = 0$ .

(b) By Lemma 3.3  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}}$  is a two-sided ideal of  $\mathfrak{q}$ . Moreover,  $[\mathfrak{m}, \mathfrak{n}]_{\text{Lie}} \subseteq \mathfrak{m} \cap \mathfrak{n}$ . Then the statements (e) and (f) in Example 5.5 complete the proof.  $\square$

**Definition 5.7** A Leibniz algebra  $\mathfrak{q}$  is said to be Lie-nilpotent if it has a Lie-central series. If  $k$  is the minimal length of such series, then  $k$  is called the class of Lie-nilpotency of  $\mathfrak{q}$ .

We show below that among all Lie-central series of a Lie-nilpotent Leibniz algebra there is one which descends most rapidly.

**Definition 5.8** *The lower Lie-central series of a Leibniz algebra  $\mathfrak{q}$  is the sequence*

$$\cdots \trianglelefteq \mathfrak{q}^{[i]} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{[2]} \trianglelefteq \mathfrak{q}^{[1]}$$

*of two-sided ideals of  $\mathfrak{q}$  defined inductively by*

$$\mathfrak{q}^{[1]} = \mathfrak{q} \quad \text{and} \quad \mathfrak{q}^{[i]} = [\mathfrak{q}^{[i-1]}, \mathfrak{q}]_{\text{Lie}}, \quad i \geq 2.$$

**Theorem 5.9**

- (a) *Let  $\mathfrak{q}$  be a Leibniz algebra and  $0 = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{j-1} \trianglelefteq \mathfrak{m}_j = \mathfrak{q}$  be a Lie-central series of  $\mathfrak{q}$ , then  $\mathfrak{q}^{[i]} \subseteq \mathfrak{m}_{j-i+1}$ ,  $1 \leq i \leq j+1$ .*
- (b) *A Leibniz algebra  $\mathfrak{q}$  is Lie-nilpotent with class of nilpotency  $k$  if and only if  $\mathfrak{q}^{[k+1]} = 0$  and  $\mathfrak{q}^{[k]} \neq 0$ .*

*Proof.* (a) This follows by induction on  $i$ .

(b) If  $\mathfrak{q}^{[k+1]} = 0$  and  $\mathfrak{q}^{[k]} \neq 0$ , then  $0 = \mathfrak{q}^{[k+1]} \trianglelefteq \mathfrak{q}^{[k]} \trianglelefteq \cdots \trianglelefteq \mathfrak{q}^{[2]} \trianglelefteq \mathfrak{q}^{[1]} = \mathfrak{q}$  is a Lie-central series, which has minimal length by (a). Therefore  $\mathfrak{q}$  is Lie-nilpotent of class  $k$ . The inverse statement directly follows from (a).  $\square$

**Definition 5.10** *The upper Lie-central series of a Leibniz algebra  $\mathfrak{q}$  is the sequence of two-sided ideals*

$$\mathcal{Z}_0^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_1^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \cdots$$

*defined inductively by*

$$\mathcal{Z}_0^{\text{Lie}}(\mathfrak{q}) = 0 \quad \text{and} \quad \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q}) = C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{q}, \mathcal{Z}_{i-1}^{\text{Lie}}(\mathfrak{q})), \quad i \geq 1.$$

Let us observe that  $\mathcal{Z}_1^{\text{Lie}}(\mathfrak{q}) = Z_{\text{Lie}}(\mathfrak{q})$  and, by Lemma 3.3,  $\mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$  indeed is a two-sided ideal of  $\mathfrak{q}$ .

**Lemma 5.11** *Let  $\mathfrak{m}$  and  $\mathfrak{n}$  be two-sided ideals of a Leibniz algebra  $\mathfrak{q}$ . If  $[\mathfrak{m}, \mathfrak{q}]_{\text{Lie}} \subseteq \mathfrak{n}$ , then  $\mathfrak{m} \subseteq C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{q}, \mathfrak{n})$ .*

*Proof.* This is straightforward.  $\square$

Now we show that among all Lie-central series of a nilpotent Leibniz algebra there is one which ascends most rapidly.

**Proposition 5.12** *Let  $0 = \mathfrak{m}_0 \trianglelefteq \mathfrak{m}_1 \trianglelefteq \cdots \trianglelefteq \mathfrak{m}_{k-1} \trianglelefteq \mathfrak{m}_k = \mathfrak{q}$  be a Lie-central series of a Lie-nilpotent Leibniz algebra  $\mathfrak{q}$ , then  $\mathfrak{m}_i \subseteq \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$ ,  $0 \leq i \leq k$ .*

*Proof.* Obviously the assertion holds for  $i = 0$ . Proceeding by induction on  $i$ , we assume that the assertion is true for  $i - 1$ , then using the Lie-centrality and Lemma 5.11, we have  $\mathfrak{m}_i \subseteq C_q^{\text{Lie}}(\mathfrak{q}, \mathfrak{m}_{i-1}) \subseteq C_q^{\text{Lie}}(\mathfrak{q}, \mathcal{Z}_{i-1}^{\text{Lie}}(\mathfrak{q})) = \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$ .  $\square$

**Theorem 5.13** *A Leibniz algebra  $\mathfrak{q}$  is Lie-nilpotent with class of Lie-nilpotency  $k$  if and only if  $\mathcal{Z}_k^{\text{Lie}}(\mathfrak{q}) = \mathfrak{q}$  and  $\mathcal{Z}_{k-1}^{\text{Lie}}(\mathfrak{q}) \neq \mathfrak{q}$ .*

*Proof.* If  $\mathfrak{q}$  is a Lie-nilpotent Leibniz algebra with class of Lie-nilpotency  $k$ , then Proposition 5.12 implies that  $\mathfrak{q} = \mathfrak{m}_k \subseteq \mathcal{Z}_k^{\text{Lie}}(\mathfrak{q}) \subseteq \mathfrak{q}$ . Moreover, in this case  $0 = \mathcal{Z}_0^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_1^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_{k-1}^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_k^{\text{Lie}}(\mathfrak{q}) = \mathfrak{q}$  is a Lie-central series of length  $k$  of  $\mathfrak{q}$ . Hence  $\mathcal{Z}_{k-1}^{\text{Lie}}(\mathfrak{q}) \neq \mathfrak{q}$ .

Conversely, if  $\mathcal{Z}_k^{\text{Lie}}(\mathfrak{q}) = \mathfrak{q}$  and  $\mathcal{Z}_{k-1}^{\text{Lie}}(\mathfrak{q}) \neq \mathfrak{q}$  then  $0 = \mathcal{Z}_0^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_1^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_{k-1}^{\text{Lie}}(\mathfrak{q}) \trianglelefteq \mathcal{Z}_k^{\text{Lie}}(\mathfrak{q}) = \mathfrak{q}$  is a Lie-central series of  $\mathfrak{q}$  and by Proposition 5.12 its length is minimal.  $\square$

#### Example 5.14

- (a) *Abelian Leibniz algebras are Lie-nilpotent Leibniz algebras of class 1.*
- (b) *Lie algebras are Lie-nilpotent Leibniz algebras of class 1.*
- (c) *The three-dimensional non-Lie Leibniz algebra with  $\mathbb{K}$ -linear basis  $\{a_1, a_2, a_3\}$  and Leibniz bracket given by*

$$[a_3, a_3] = a_1$$

*and zero elsewhere (see [5]), is a Lie-nilpotent Leibniz algebra of class 2.*

- (d) *The non-Lie Leibniz algebra given in Example 5.5 (c) is non Lie-nilpotent.*
- (e) *The four-dimensional (non-Lie) Leibniz algebra with  $\mathbb{K}$ -linear basis  $\{a_1, a_2, a_3, a_4\}$  and Leibniz bracket given by*

$$[e_1, e_1] = e_3, [e_2, e_4] = e_2, [e_4, e_2] = -e_2$$

*and zero elsewhere (see [4]) is Lie-nilpotent of class 2, but it is not a nilpotent Leibniz algebra.*

- (f) *Subalgebras and images by homomorphisms of Lie-nilpotent Leibniz algebras are Lie-nilpotent Leibniz algebras.*
- (g) *Intersection and sum of Lie-nilpotent two-sided ideals of a Leibniz algebra are Lie-nilpotent two-sided ideals as well.*

#### Proposition 5.15

- (a) *If  $\mathfrak{q}/\mathcal{Z}_{\text{Lie}}(\mathfrak{q})$  is a Lie-nilpotent Leibniz algebra, then  $\mathfrak{q}$  is a Lie-nilpotent Leibniz algebra.*

- (b) If  $\mathfrak{q}$  is a Lie-nilpotent and non trivial Leibniz algebra, then  $Z_{\text{Lie}}(\mathfrak{q}) \neq 0$ .
- (c) If  $\mathfrak{g} \twoheadrightarrow \mathfrak{q}$  is a Lie-central extension of a Lie-nilpotent Leibniz algebra  $\mathfrak{q}$ , then  $\mathfrak{g}$  is Lie-nilpotent as well.
- (d) A Lie-nilpotent Leibniz algebra is Lie-solvable as well.

*Proof.* (a) There exists  $k \in \mathbb{N}$  such that  $(\mathfrak{q}/Z_{\text{Lie}}(\mathfrak{q}))^{[k]} = 0$ , then  $\mathfrak{q}^{[k]} \subseteq Z_{\text{Lie}}(\mathfrak{q})$ , hence  $\mathfrak{q}^{[k+1]} \subseteq [Z_{\text{Lie}}(\mathfrak{q}), \mathfrak{q}]_{\text{Lie}} = 0$ .

(b) Assume that  $\mathfrak{q}$  has Lie-nilpotency class equal to  $k$ , that is  $[\mathfrak{q}^{[k]}, \mathfrak{q}]_{\text{Lie}} = \mathfrak{q}^{[k+1]} = 0$ , then Lemma 5.11 implies that  $0 \neq \mathfrak{q}^{[k]} \subseteq C_{\mathfrak{q}}^{\text{Lie}}(\mathfrak{q}, 0) = Z_{\text{Lie}}(\mathfrak{q})$ .

(c) There exists  $k \in \mathbb{N}$  such that  $\mathfrak{q}^{[k]} = 0$ . Then  $(\mathfrak{g}/\mathfrak{n})^{[k]} = \mathfrak{g}^{[k]}/\mathfrak{n} = 0$ , where  $\mathfrak{n} = \text{Ker}(\mathfrak{g} \twoheadrightarrow \mathfrak{q})$ . Hence  $\mathfrak{g}^{[k]} \subseteq \mathfrak{n} \subseteq Z_{\text{Lie}}(\mathfrak{g})$  and  $\mathfrak{g}^{[k+1]} = [\mathfrak{g}^{[k]}, \mathfrak{g}]_{\text{Lie}} = 0$ .

(d) By induction on  $i$ , it is easy to see that  $\mathfrak{q}^{(i)} \subseteq \mathfrak{q}^{[i+1]}$ ,  $i \geq 0$ .  $\square$

## 6 Homological criterion for Lie-nilpotency

**Definition 6.1** Let  $\mathfrak{n}$  be a two-sided ideal of a Leibniz algebra  $\mathfrak{q}$ . The lower Lie-central series of  $\mathfrak{q}$  relative to  $\mathfrak{n}$  is the sequence

$$\dots \trianglelefteq \mathfrak{n}^{[i]} \trianglelefteq \dots \trianglelefteq \mathfrak{n}^{[2]} \trianglelefteq \mathfrak{n}^{[1]}$$

of two-sided ideals of  $\mathfrak{q}$  defined inductively by

$$\mathfrak{n}^{[1]} = \mathfrak{n} \quad \text{and} \quad \mathfrak{n}^{[i]} = [\mathfrak{n}^{[i-1]}, \mathfrak{q}]_{\text{Lie}}, \quad i \geq 2.$$

Note that  $[\mathfrak{n}^{[i]}/\mathfrak{n}^{[i+1]}, \mathfrak{n}^{[i]}/\mathfrak{n}^{[i+1]}]_{\text{Lie}} = 0$ . When  $\mathfrak{n} = \mathfrak{q}$  we obtain Definition 5.8. If  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  is a homomorphism of Leibniz algebras such that  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ , where  $\mathfrak{m}$  is a two-sided ideal of  $\mathfrak{g}$  and  $\mathfrak{n}$  is a two-sided ideal of  $\mathfrak{q}$ , then  $\varphi(\mathfrak{m}^{[i]}) \subseteq \mathfrak{n}^{[i]}$ ,  $i \geq 1$ .

**Theorem 6.2** Let  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  be a homomorphism of Leibniz algebras such that  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ , where  $\mathfrak{m}$  is a two-sided ideal of  $\mathfrak{g}$  and  $\mathfrak{n}$  is a two-sided ideal of  $\mathfrak{q}$ , and the following properties hold:

- (a) the induced homomorphism  $HL_1^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_1^{\text{Lie}}(\mathfrak{q})$  is an isomorphism;
- (b) the induced homomorphism  $HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  is an epimorphism;
- (c) the induced homomorphism  $\varphi_1 : \mathfrak{g}/\mathfrak{m} \rightarrow \mathfrak{q}/\mathfrak{n}$  is an isomorphism.

Then  $\varphi$  induces a natural isomorphism  $\varphi_k : \mathfrak{g}/\mathfrak{m}^{[k]} \rightarrow \mathfrak{q}/\mathfrak{n}^{[k]}$ ,  $k \geq 1$ .

*Proof.* We prove by induction on  $k$ . For  $k = 1$  we have the statement (c). Suppose the theorem is true for  $k - 1$ . Applying sequence **(D)**, which is natural [12], to the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{m}^{[k-1]} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{m}^{[k-1]} \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \wr \varphi_{k-1} \\ 0 & \longrightarrow & \mathfrak{n}^{[k-1]} & \longrightarrow & \mathfrak{q} & \longrightarrow & \mathfrak{q}/\mathfrak{n}^{[k-1]} \longrightarrow 0 \end{array}$$

we get the following commutative diagram

$$\begin{array}{ccccccccc} HL_2^{\text{Lie}}(\mathfrak{g}) & \longrightarrow & HL_2^{\text{Lie}}(\frac{\mathfrak{g}}{\mathfrak{m}^{[k-1]}}) & \longrightarrow & \frac{\mathfrak{m}^{[k-1]}}{[\mathfrak{m}^{[k-1]}, \mathfrak{g}]_{\text{Lie}}} & \longrightarrow & HL_1^{\text{Lie}}(\mathfrak{g}) & \longrightarrow & HL_1^{\text{Lie}}(\frac{\mathfrak{g}}{\mathfrak{m}^{[k-1]}}) \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ HL_2^{\text{Lie}}(\mathfrak{q}) & \longrightarrow & HL_2^{\text{Lie}}(\frac{\mathfrak{q}}{\mathfrak{n}^{[k-1]}}) & \longrightarrow & \frac{\mathfrak{n}^{[k-1]}}{[\mathfrak{n}^{[k-1]}, \mathfrak{q}]_{\text{Lie}}} & \longrightarrow & HL_1^{\text{Lie}}(\mathfrak{q}) & \longrightarrow & HL_1^{\text{Lie}}(\frac{\mathfrak{q}}{\mathfrak{n}^{[k-1]}}) \longrightarrow 0 \end{array}$$

By the Five Lemma, which holds in a semi-abelian category [1, 21], we get  $\frac{\mathfrak{m}^{[k-1]}}{[\mathfrak{m}^{[k-1]}, \mathfrak{g}]_{\text{Lie}}} \cong \frac{\mathfrak{n}^{[k-1]}}{[\mathfrak{n}^{[k-1]}, \mathfrak{q}]_{\text{Lie}}}$ , i. e.  $\frac{\mathfrak{m}^{[k-1]}}{\mathfrak{m}^{[k]}} \cong \frac{\mathfrak{n}^{[k-1]}}{\mathfrak{n}^{[k]}}$ . Then the short Five Lemma applied to the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \frac{\mathfrak{m}^{[k-1]}}{\mathfrak{m}^{[k]}} & \longrightarrow & \frac{\mathfrak{g}}{\mathfrak{m}^{[k]}} & \longrightarrow & \frac{\mathfrak{g}}{\mathfrak{m}^{[k-1]}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \frac{\mathfrak{n}^{[k-1]}}{\mathfrak{n}^{[k]}} & \longrightarrow & \frac{\mathfrak{q}}{\mathfrak{n}^{[k]}} & \longrightarrow & \frac{\mathfrak{q}}{\mathfrak{n}^{[k-1]}} \longrightarrow 0 \end{array}$$

and the induction complete the proof.  $\square$

**Corollary 6.3** *Let  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  be a homomorphism of Leibniz algebras such that  $\varphi_{\text{Lie}} : \mathfrak{g}_{\text{Lie}} \rightarrow \mathfrak{q}_{\text{Lie}}$  is an isomorphism and the induced homomorphism  $HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  is an epimorphism. If  $\mathfrak{g}$  and  $\mathfrak{q}$  are Lie-nilpotent Leibniz algebras, then  $\varphi$  is an isomorphism.*

*Proof.* Take  $\mathfrak{m} = \mathfrak{g}$  and  $\mathfrak{n} = \mathfrak{q}$  in Theorem 6.2. Then the assertion follows by keeping in mind that  $HL_1^{\text{Lie}}(\mathfrak{g}) \cong \mathfrak{g}_{\text{Lie}}$ ,  $HL_1^{\text{Lie}}(\mathfrak{q}) \cong \mathfrak{q}_{\text{Lie}}$  and there exists  $k \geq 1$  such that  $\mathfrak{g}^{[k]} = \mathfrak{q}^{[k]} = 0$ .  $\square$

**Corollary 6.4** *Suppose  $\mathfrak{q}$  and  $\mathfrak{q}_{\text{Lie}}$  both are Lie-nilpotent Leibniz algebras. Then necessarily  $\mathfrak{q} \cong \mathfrak{q}_{\text{Lie}}$ , that is,  $\mathfrak{q}$  is a Lie algebra.*

*Proof.* This follows by applying Corollary 6.3 to the canonical epimorphism  $\mathfrak{q} \twoheadrightarrow \mathfrak{q}_{\text{Lie}}$ .  $\square$

**Lemma 6.5** *Let  $\mathfrak{n}$  be a two-sided ideal of a Leibniz algebra  $\mathfrak{q}$ . The lower Lie-central series determined by  $\mathfrak{n}$  vanishes if and only if there exists  $i \geq 0$  such that  $\mathfrak{n} \subseteq \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$ .*

*Proof.* It is enough to use the following obvious equivalence:

$$\mathfrak{n}^{[i]} \subseteq \mathcal{Z}_k^{\text{Lie}}(\mathfrak{q}) \Leftrightarrow \mathfrak{n}^{[i+1]} \subseteq \mathcal{Z}_{k-1}^{\text{Lie}}(\mathfrak{q}).$$

□

**Theorem 6.6** *Let  $\mathfrak{m}$  be a two-sided ideal of  $\mathfrak{g}$  and  $\mathfrak{n}$  be a two-sided ideal of  $\mathfrak{q}$  such that  $\mathfrak{m} \subseteq \mathcal{Z}_i^{\text{Lie}}(\mathfrak{g})$  and  $\mathfrak{n} \subseteq \mathcal{Z}_j^{\text{Lie}}(\mathfrak{q})$ ,  $i, j \geq 0$ . Let  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  be a homomorphism of Leibniz algebras such that  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$  and satisfying the following conditions:*

- (a) *the induced homomorphism  $HL_1^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_1^{\text{Lie}}(\mathfrak{q})$  is an isomorphism;*
- (b) *the induced homomorphism  $HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  is an epimorphism;*
- (c) *the induced homomorphism  $\mathfrak{g}/\mathfrak{m} \rightarrow \mathfrak{q}/\mathfrak{n}$  is an isomorphism.*

*Then  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  is an isomorphism.*

*Proof.* This is a consequence of Theorem 6.2 and Lemma 6.5. □

**Corollary 6.7** *Let  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  be a homomorphism of Leibniz algebras such that  $\varphi(\mathcal{Z}_i^{\text{Lie}}(\mathfrak{g})) \subseteq \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$  for any  $i \geq 0$  and satisfying the following conditions:*

- (a) *the induced homomorphism  $HL_1^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_1^{\text{Lie}}(\mathfrak{q})$  is an isomorphism;*
- (b) *the induced homomorphism  $HL_2^{\text{Lie}}(\mathfrak{g}) \rightarrow HL_2^{\text{Lie}}(\mathfrak{q})$  is an epimorphism;*
- (c) *the induced homomorphism  $\mathfrak{g}/\mathcal{Z}_i^{\text{Lie}}(\mathfrak{g}) \rightarrow \mathfrak{q}/\mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$  is an isomorphism.*

*Then  $\varphi : \mathfrak{g} \rightarrow \mathfrak{q}$  is an isomorphism.*

*Proof.* This follows by applying Theorem 6.6 to the case  $\mathfrak{m} = \mathcal{Z}_i^{\text{Lie}}(\mathfrak{g})$  and  $\mathfrak{n} = \mathcal{Z}_i^{\text{Lie}}(\mathfrak{q})$ . □

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