The completion of partial Latin squares

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Abstract

In recent times there has been some interest in studying partial latin squares which have no completions or precisely one completion, and which are critical with respect to this property. Such squares are called, respectively, *premature partial latin squares* and *critical sets*. There has also been interest in related *maximal* partial latin squares.

This paper will explore the connection between these three structures and review some of the literature in this area. A number of open problems are presented.

1 Introduction

The subject of this paper is the completion or in some cases the non–completion of partial latin squares. A partial latin square, P, of order n is an $n \times n$ array in which each of the symbols $1,\ 2, \ldots, n$ occurs at most once in each row and at most once in each column of the array. Using this definition it can be seen that a partial latin square may contain a number of empty cells and this fact leads to the question:

> Given a partial latin square P, is it possible to fill the empty cells of P so that each of the symbols $1,\ 2, \ldots, n$ occurs exactly once in each row and exactly once in each column of the array?

If such a process is possible, it is said to complete the partial latin square and the exploration of the completion of partial latin squares is the central issue under discussion in this paper. Section 2 provides the reader with the appropriate definitions and notation, while the remaining sections deal with some old and some not so old results concerning the completion of partial latin squares. Throughout the paper a number of open questions will be raised and it is hoped that the discussion presented will stimulate the reader's interest in seeking a resolution to some of these questions.

2 Definitions

Let $N = \{1, 2, \ldots, n\}$ and N^2 and N^3 denote, respectively, the cartesian products $N \times N$ and $N \times N \times N$. An ordered pair $(i, j) \in N^2$ will be called a *cell*. Let P be a collection of ordered triples selected from N^3 satisfying the property that:

1. for any pair $x, y \in N$, P contains at most one triple of the form (x, y, u), where $u \in N$.

A subset P of N^3 is said to be *row latin* if, in addition to Property 1, for any pair $x, y \in N$,

- there is at most one triple of the form $(x, u, y) \in P$,

and P is said to be *column latin* if, in addition to Property 1, for any pair $x, y \in N$,

- there is at most one triple of the form $(u, x, y) \in P$.

A *partial latin square* is a collection P of ordered triples selected from N^3 that satisfy Property 1 above, as well as being both row latin and column latin. If $N = \{1, 2, 3\}$ then the sets T, Q, R and S are examples of partial latin squares.

$$\begin{aligned}
T &= \{(1,1,1), (2,2,2), (3,3,1)\} \\
Q &= \{(1,1,1), (2,2,2), (3,3,1), (1,3,3), (3,1,3)\} \\
R &= \{(1,1,1), (2,2,2), (3,3,3)\} \\
S &= \{(1,1,1), (2,2,2)\}
\end{aligned}$$

To facilitate understanding of these formal definitions it will be useful to relate them to the more intuitive ideas based on arrays. That is, the collection P may be thought of as an $n \times n$ array in which each cell contains at most one symbol selected from N. The array P is row latin, if each symbol of N occurs at most once in each row. The array P is column latin if each symbol of N occurs at most once in each column. The array is termed a partial latin square if each of the n symbols belonging to N occurs at most once in each row and at most once in each column. The arrays corresponding to the above examples are:

1		
	2	
		1

T

1		3
	2	
3		1

Q

1		
	2	
		3

R

1		
	2	

S

A partial latin square $P \subseteq N^3$ is said to be of *order* n and *size* $|P|$. The order of the above partial latin squares is 3. The partial latin squares T, Q, R and S have, respectively, sizes 3, 5, 3 and 2.

Cell $(i, j) \in N^2$ is said to be *filled* with respect to the partial latin square P, of order n, if for some symbol $k \in N$, $(i, j, k) \in P$. Otherwise the cell is *empty*. By way of example, it can be seen that the partial latin square T has cells $(1, 1), (2, 2), (3, 3)$

filled and cells $(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$ empty. The symbol $k \in N$ *occurs in* cell (i,j) if $(i,j,k) \in P$.

A partial latin square P is an $r \times n$ *($n \times c$) latin rectangle*, $0 \leq r \leq n$ $(0 \leq c \leq n)$, if,

- for each $(x,y) \in N^2$ such that $1 \leq x \leq r$ and $1 \leq y \leq n$ $(1 \leq x \leq n$ and $1 \leq y \leq c)$, there exists $k \in N$ such that $(x,y,k) \in P$, and
- for each $(x,y) \in N^2$ such that $r+1 \leq x \leq n$ and $1 \leq y \leq n$ $(1 \leq x \leq n$ and $c+1 \leq y \leq n)$, $\forall k \in N$, $(x,y,k) \notin P$.

If $N = \{1,2,3\}$ then the following sets U and V provide examples of latin rectangles:

$$U = \{(1,1,1),(1,2,3),(1,3,2),(2,1,3),(2,2,2),(2,3,1)\}$$
$$V = \{(1,1,1),(1,2,2),(2,1,3),(2,2,1),(3,1,2),(3,2,3)\}.$$

Represented as arrays they are:

1	3	2
3	2	1

U

1	2	
3	1	
2	3	

V

A *latin square* is a partial latin square in which Property 1 is replaced by:

1′. for each pair $x,y \in N$ there exists precisely one $u \in N$ such that $(x,y,u) \in P$.

So an $n \times n$ array is termed a latin square, of order n, if each of the n symbols belonging to N occurs precisely once in each row and once in each column. Once again, given $N = \{1,2,3\}$ the sets K, M, G and H are examples of latin squares.

$$K = \{(1,1,1),(1,2,3),(1,3,2),(2,1,3),(2,2,2),(2,3,1),(3,1,2),(3,2,1),(3,3,3)\}$$
$$M = \{(1,1,1),(1,2,2),(1,3,3),(2,1,2),(2,2,3),(2,3,1),(3,1,3),(3,2,1),(3,3,2)\}$$
$$G = \{(1,1,1),(1,2,2),(1,3,3),(2,1,3),(2,2,1),(2,3,2),(3,1,2),(3,2,3),(3,3,1)\}$$
$$H = \{(1,1,1),(1,2,3),(1,3,2),(2,1,2),(2,2,1),(2,3,3),(3,1,3),(3,2,2),(3,3,1)\}$$

Or equivalently:

1	3	2
3	2	1
2	1	3

K

1	2	3
2	3	1
3	1	2

M

1	2	3
3	1	2
2	3	1

G

1	3	2
2	1	3
3	2	1

H

Let P and Q be two partial latin squares of order n. The square Q is an *extension* of P if $P \subset Q$, and it is said that P can be *extended* to Q. If there exist α distinct extensions of P, then P is termed *α–extendable*. A partial latin square P is *maximal* if $\alpha = 0$. If Q is a latin square of order n, then the extension Q is termed a *completion* of P. It is also said that P is *completable* or that P can be *completed to Q*. If there exist β distinct completions of P, then P is termed *β-completable.* If $\beta = 0$ the partial latin square is not completable and if $\beta = 1$ the partial latin square is said to have a *unique completion. Premature partial latin square* are those for which $\beta = 0$, but with the additional property that for any $(i, j, k) \in P$, $P \setminus \{(i, j, k)\}$ is completable. If $\beta = 1$ and for any $(i, j, k) \in P$, $P\setminus\{(i, j, k)\}$ is γ–completable, where $\gamma > 1$, then P is said to be a *critical set.* In the above examples, Q is a maximal partial latin square. The partial latin square T has no completion, however it is not maximal as $T \cup \{(1, 3, 3)\}$ is an extension of T. On the other hand T is premature as the sets $T \setminus \{(1, 1, 1)\}$, $T \setminus \{(2, 2, 2)\}$, $T \setminus \{(3, 3, 1)\}$ all have completions. The sets R and S have unique completion to K. However, R is not a critical set as $R\setminus\{(3, 3, 3)\}$ has a unique completion. But S is critical as both K and H are completions of $S \setminus \{(1, 1, 1)\}$ and K, M and G are completions of $S \setminus \{(2, 2, 2)\}$.

3 Some early results

This paper begins with some early results by Hall, Lindner and Smetaniuk. In order to give an outline of the proof of these results, the definition of a system of distinct representatives is required.

Let m be a natural number and $S_1, \ldots, S_m$ be a collection of finite sets. A *system of distinct representatives*, or *SDR*, for the sets $S_1, \ldots, S_m$, is a sequence $\langle s_1, \ldots, s_m \rangle$ of m distinct elements such that $s_i \in S_i$. For each i, s_i is called a *representative* of S_i. For example, the set $\langle 1, 3, 2 \rangle$ is a SDR for the sets $S_1 = \{1\}$, $S_2 = \{1, 3\}$ and $S_3 = \{2, 3\}$.

In 1935, P. Hall proved the following theorem for SDRs. The proof of this result can be found in **[13]**.

THEOREM 1 *(P. Hall, 1935,* **[13]***) Let $S_1, S_2, \ldots, S_m$ be a collection of m finite sets. Then an SDR for these sets exists if and only if, for all $k \in \{0, 1, \ldots, m\}$, $|S_{i1} \cup S_{i2} \cup \ldots \cup S_{ik}| \geq k$, where the k sets $S_{i1}, \ldots, S_{ik}$ represent any collection of k sets chosen from the m sets $S_1, S_2, \ldots, S_m$.*

The above theorem states that a SDR exists if and only if, for all k, the union of any k sets in the collection contains at least k elements.

In 1945 M. Hall, **[12]**, used the above theorem to prove that every latin rectangle of order n can be completed to a latin square.

THEOREM 2 *(M. Hall, 1945,* **[12]***) Every $r \times n$ latin rectangle, $0 \leq r \leq n$, can be completed to a latin square of order n.*

Proof: The case where $r = 0$ is trivial, as is the case where $r = n$. So it will be assumed that P is an $r \times n$ latin rectangle, where $0 < r < n$. It will be shown that P can be extended to an $(r + 1) \times n$ latin rectangle, and hence ultimately to a latin square of order n.

For $1 \leq j \leq n$, let S_j denote the set of all $x \in \{1, 2, \ldots, n\}$ such that x does not occur in the column j of P. Note that $|S_j| = n - r$, and since P is a latin rectangle each $x \in \{1, 2, \ldots, n\}$ belongs to exactly $n - r$ of the sets $S_1, S_2, \ldots, S_n$. It will be shown that there exists a SDR $\langle s_1, s_2, \ldots s_n \rangle$ for the collection $S_1, S_2, \ldots, S_n$ and consequently that the $(r+1) \times n$ latin rectangle $P \cup \{(r+1, 1, s_1), (r+1, 2, s_2), \ldots, (r+1, n, s_n)\}$ is an extension of P.

Assume that such a SDR does not exist. Then there exist some $k \in \{1, 2, \ldots, n\}$ and some choice of $S_{j1}, S_{j2}, \ldots, S_{jk}$ for which $|S_{j1} \cup S_{j2} \cup \ldots \cup S_{jk}| < k$. But $|S_{j1}| + |S_{j2}| + \ldots + |S_{jk}| = k(n-r)$ implies that there exists a symbol $x \in \{1, 2, \ldots, n\}$ occuring in more than $n - r$ of the sets $S_{j1}, S_{j2}, \ldots, S_{jk}$. However this is a contradiction as each $x \in \{1, 2, \ldots, n\}$ belongs to exactly $n - r$ of the sets $S_1, S_2, \ldots, S_n$. Hence a SDR must exist and so the $r \times n$ latin rectangle P can be extended to $P \cup \{(r + 1, 1, s_1), (r + 1, 2, s_2), \ldots, (r + 1, n, s_n)\}$. If $P \cup \{(r + 1, 1, s_1), (r + 1, 2, s_2), \ldots, (r + 1, n, s_n)\}$ is a latin square of order n the argument is complete, otherwise the above process is repeated and $P \cup \{(r+1, 1, s_1), (r+1, 2, s_2), \ldots, (r+1, n, s_n)\}$ is extended to an $(r+2) \times n$ latin rectangle, and so on until a latin square is obtained. □

The fact that each symbol of N occurs precisely r times in the first r rows of the partial latin square P makes the proof of the above result fairly straight forward. But, if this condition is removed the strength of the statement is weakened. For example, there exist partial latin squares of order n and size n that can not be completed. The partial latin square T given above testifies to this, as it is a special case of the partial latin square $\{(1, 1, 1), (i, i, 2) \mid 2 \leq i \leq n\}$. So one may ask for an arbitrary partial latin square of order n and size s, what are the restrictions which must be placed on s to guarantee the existence of a completion? This question was explored by T. Evans in 1960 **[9]**, and resulted in the following conjecture:

CONJECTURE (Evans, 1960, **[9]**) Every partial latin square of order n containing at most $n - 1$ filled cells is completable.

Analysis of relatively small examples suggested that this conjecture was true, but it was not until 1970 that C. Lindner proved a special case of Evans' conjecture. Lindner **[15]** proved that if the $n-1$ filled cells are restricted to $n/2$ rows or columns, there is always a completion. In 1978, R. Haggkvist **[11]** moved a step closer and proved the Evans' conjecture for all $n \geq 1111$. Then in the early 80's complete solutions were given independently by B. Smetaniuk **[18]**, and by I. Andersen and A. Hilton **[1]**. In this paper a brief discription of both Lindner's and Smetaniuk's results will be given.

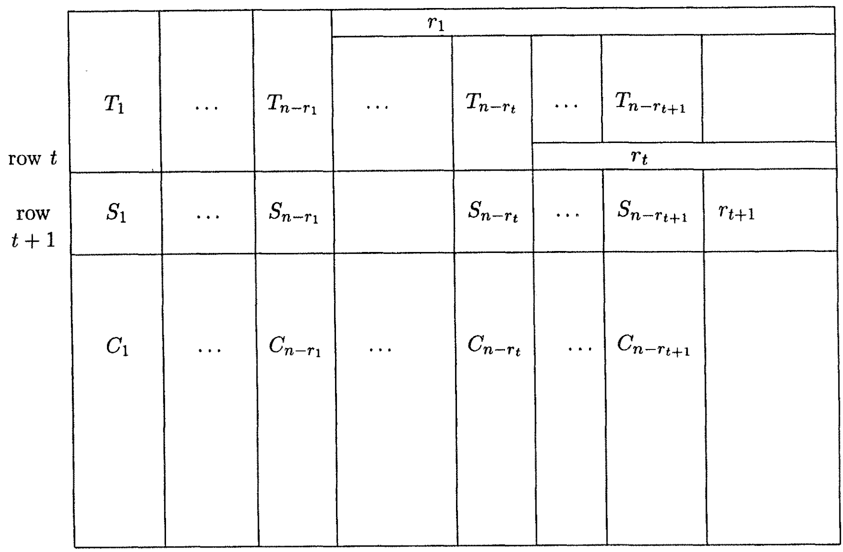
THEOREM 3 *(Lindner, 1970,* **[15]***) Let P be a partial latin square of order n satisfying the conditions*

- $|P| \leq n - 1$, *and*
- $|\{i \mid (i, j, k) \in P\}| = m \leq n/2$,

then P is completable.

Outline of Proof: Lindner begins by assuming that $|P| = n - 1$ and presents a proof that is inductive in nature and proceeds by constructing a series of extension P_{t+1} $(1 \leq t \leq m-1)$ to permutations of P. In the inductive step, it is assumed that there exists a partial latin square P_t (basically an extension of P) for which every cell in rows 1 to t is filled. This partial latin square is then extended to a partial latin square P_{t+1} for which every cell in rows 1 to $t+1$ is filled.

The proof begins by focusing on P_t and by defining sets: T_p containing all symbols occurring in rows 1 to t of column p, C_p containing all symbols occurring in rows $t+2$ to m of column p and S_p the set of symbols not occurring in either row $t+1$ or column p. Equivalently, S_p is the set of symbols which can be used to fill cell $(t+1, p)$ while still maintaining the row latin and column latin conditions. The relationship between these sets and P_t is summarised in the diagram given below.



It is shown that for row $t+1$ of P_t, with empty cells $(t+1, 1), (t+1, 2), \ldots, (t+1, n-r_{t+1})$, the union of any collection of k sets chosen from $S_1, S_2, \ldots, S_{n-r_{t+1}}$ contains at least k elements.

To this end let r_i denote the number of filled cells in row i of P and assume that without loss of generality $r_i \leq r_{i-1}$. Then there exists a positive integer $m \leq n/2$, such that $r_i > 0$ whenever $i \leq m$, and $r_i = 0$, whenever $i > m$. Since $|P| = n - 1$ and $m \leq n/2$, $r_1 \geq 2$. Further, it can be shown that for any t, $r_1 + r_2 + \ldots + r_t \geq 2t$.

So if $k = 1$, then for any $q = 1, \ldots, n - r_{t+1}$, $|T_q \cup C_q| \leq m - 1$, and $r_{t+1} \leq n/2$. So $|S_q| \geq n - (m-1) - (n/2) \geq k$, as required.

If $2 \leq k \leq n-t-r_{t+1}$ and C_q is empty, then it is immediate that $|S_q| \geq n-t-r_{t+1} \geq k$. Otherwise, whenever $2 \leq k \leq n-t-r_{t+1}$ any C_q selected from a collection of k sets chosen from $C_1, C_2, \ldots, C_{n-r_{t+1}}$ satisfies $|C_q| \leq (n-1)-(r_1+\ldots+r_t)-r_{t+1}-(k-1)$ and since $r_1+\ldots+r_t \geq 2t$, $|S_{jq}| \geq n-((n-1)-(r_1+\ldots+r_t)-r_{t+1}-(k-1))-t-r_{r+1} \geq k$, as required.

If $k > n-t-r_{t+1}$ it can be shown that for any collection of k sets chosen from $C_1, C_2, \ldots, C_{n-r_{t+1}}$ at least $1+t$ of them are empty. Denote these sets by $C_{j1}, C_{j2}, \ldots, C_{j(t+1)}$. Using this information it is shown that the elements in the set $S_{j1} \cup S_{j2} \cup \ldots \cup S_{jk}$ constitute the set of elements not already occuring in row $r+1$. Thus $|S_{j1} \cup S_{j2} \cup \ldots \cup S_{jk}| = n - r_{t+1} \geq k$.

And so in all cases the union of any collection of k set chosen from $S_1, S_2, \ldots, S_{n-r_{t+1}}$ contains at least k elements, and so there exists a SDR $\langle s_1, s_2, \ldots, s_{n-r_{t+1}} \rangle$ for the sets $S_1, S_2, \ldots, S_{n-r_{t+1}}$. Consequently, $P_{t+1} = P_t \cup \{(t+1, 1, s_1), (t+1, 2, s_2), \ldots, (t+1, n-r_{t+1}, s_{n-r_{t+1}})\}$ is an extension of P_t.

This process is repeated until $t = m$, yielding an $m \times n$ latin rectangle which, by Theorem 2, can be extended to a latin square of order n that is basically a completion of P.

If $|P| \leq n-1$, it is possible to fill $n-1-|P|$ cells and then proceed as above.

□

Lindner's result can be restated in a number of ways as indicated by the next corollary.

COROLLARY 4 *(Lindner, 1970,* **[15]***) Let P be a partial latin square of order n satisfying the conditions*

- $|P| \leq n-1$, *and*
- $|\{i \mid (i,j,k) \in P\}| \leq n/2$, *or* $|\{j \mid (i,j,k) \in P\}| \leq n/2$, *or* $|\{k \mid (i,j,k) \in P\}| \leq n/2$,

then P is completable.

While not completely settling the Evans' Conjecture the theorem by Lindner made a signficant contribution and led to the complete solution by Smetaniuk in 1981. To understand the essence of Smetaniuk's proof a number of definitions are needed.

The back diagonal of a partial latin square P of order n is formed by the cells $\{(1,n), (2,n-1), \ldots, (n,1)\}$. An entry (x,y,s) of P, is said to lie on the back diagonal of P if $(x,y,s) \in P$ and $x+y = n+1$, (x,y,s) lies above the back diagonal if $(x,y,s) \in P$ and $x+y < n+1$, and (x,y,s) lies below the back diagonal if $(x,y,s) \in P$ and $x+y > n+1$. Let L be a latin square of order n, then L can be used to construct a new partial latin square $P(L)$ as follows:

- If (x,y,s) lies on or above the back diagonal in L, then $(x,y,s) \in P(L)$.
- For all $i = 1, \ldots, n+1$, $(i, n+1-i, n+1) \in P(L)$.

Note that in $P(L)$ all cells below the back diagonal are empty as illustrated in the following example.

$$L \quad = \quad \begin{array}{|c|c|c|c|}\hline 3 & 2 & 4 & 1 \\ \hline 1 & 4 & 3 & 2 \\ \hline 2 & 3 & 1 & 4 \\ \hline 4 & 1 & 2 & 3 \\ \hline \end{array} \qquad\qquad P(L) \quad = \quad \begin{array}{|c|c|c|c|c|}\hline 3 & 2 & 4 & 1 & 5 \\ \hline 1 & 4 & 3 & 5 & \\ \hline 2 & 3 & 5 & & \\ \hline 4 & 5 & & & \\ \hline 5 & & & & \\ \hline \end{array}$$

The proof of Smetaniuk's result relies on two key ideas. These are:

- Let L be any latin square, of order n. The partial latin square $P(L)$, of order $n+1$ (as defined above), can always be completed.
- Let P be a partial latin square, of order n, with at most $n-1$ filled cells. Suppose that for some $k \in N$, k occurs in exactly one cell of P. Then the rows and columns of P can be rearranged and the symbols relabelled to produce an partial latin square P_0 of order n that has the following properties:
  - the symbol n occurs in precisely one cell of P_0 and this occurs on the back diagonal of P_0, and
  - all other filled cells occur above the back diagonal.

(For a clear and concise proof of these ideas the reader is referred to Lindner **[16]**.)

Having established these results Smetaniuk completed the proof of Evans' conjecture as follows.

THEOREM 5 *(Smetaniuk, 1981,* **[18]***) An $n \times n$ partial latin square P, for which $|P| \leq n-1$, can always be completed to a latin square of order n.*

Proof: Assume $|P| = n-1$. If the number of distinct symbols occuring in P is less than or equal to $n/2$, the result is true by Corollary 4.

If the number of distinct symbols occuring in P is greater than $n/2$, then the proof proceeds by induction on n.

The Evans' conjecture is true for partial latin squares of order 1 or 2. So it will be assumed that $m \geq 3$ and that the Evans' conjecture is true for all partial latin square of order less than or equal to m. Let P be a partial latin square of order $m+1$, containing precisely m filled cells. Since $m \geq n/2$, there exists a symbol which occurs in precisely one cell of P, using the above statements the array can be rearrange to form a partial latin square in which the symbol $m+1$ occurs in precisely one cell, which is on the back diagonal, and all other remaining $m-1$ symbols occur in cells above the back diagonal. In this new partial latin square delete the symbol $m+1$ from the back diagonal, and delete the last row and the last column. The resulting partial latin square, of order m, is such that it consists of $m-1$ filled cells and so, by the inductive hypothesis, can be completed. Label the completion by L and construct $P(L)$. It is immediate that the completion of $P(L)$ is a completion of a permutation of P.

If $|P| < n-1$, one fills $n-1-|P|$ cells and proceeds as before, thus verifying that the Evans' conjecture is true in all cases. $\square$

So together the results by Lindner and Smetaniuk prove that the Evans' conjecture is true, and this leads one to ask, "If the size of the partial latin square is increased, what can be said about the possible completion or non-completion of the partial latin square?" It turns out that there are many open problems related to this question. In closing this section one such open problem (which arose out of discussion with D. Street and C. Lindner) is stated. Then in the next two sections the focus of the paper turns to the study of partial latin squares which have no completion.

Open Problem: For which values of h and n does there exist a partial latin square satisfying:

1. each row of P contains exactly h symbols of N;
2. each column of P contains exactly h symbols of N;
3. each symbol of N occurs in exactly h cells of P;
4. P is not completable.

In 1997, B. Burton investigated this problem in his honours thesis **[5]**. Burton showed that for $h = 2$ and $n \in \{3, 4, \ldots, 7\}$ there always exists a partial latin square with properties 1, 2, 3 and 4. However, when $h = 2$ and $n = 8$ all partial latin squares satisfying properties 1, 2 and 3 are completable. This led Burton to conjecture that for $h \geq 2$, $n \geq 4h$ there do not exist partial latin squares of order n for which properties 1 to 4 hold. [1]

4 Maximal Partial Latin Squares

A maximal partial latin square is a partial latin square that can not be extended. In 1993, P. Horak and A. Rosa **[14]** studied the spectrum of maximal partial latin squares. That is, they tried to determine the set

$$S = \{s \mid \text{ there exists a maximal partial latin square of order } n \text{ and size } s\}.$$

It is easy to see that for a maximal partial latin square P of order n, $|P| < n^2 - 1$, and so $n^2 - 1 \notin S$. The definition also implies that if (x, y) is an unoccupied cell, then it can not be filled without violating the row latin or column latin condition. So the number of occupied cells in row x or column y must be greater than or equal to n, and each $k \in N$ must occur in at least one of the occupied cells in row x or column y. This fact implies that if $s \in S$ then $s \geq \frac{n^2}{2}$. However, Horak and Rosa were able to achieve significantly better bounds than this by partitioning the maximal partial latin square, P, into four subarrays and then applying clever counting arguments. They were able to show that:

[1] Recently Ian Wanless has made further progress on this problem. For his results please contact him on ian@maths.ox.ac.uk.

THEOREM 6 *(Horak and Rosa, 1993,* **[14]***) There does not exist a maximal partial latin square of order n and size s where*

(i) $s < \frac{n^2}{2}$,

(ii) $s = \frac{n^2}{2} + h,\ 1 \le h \le \frac{n}{2},\ h$ *odd,* n *even,*

(iii) $s = \frac{n^2+1}{2} + h,\ 1 \le h \le \frac{n-1}{2},\ h$ *odd,* n *odd,*

(iv) $s = n^2 - 1$.

Horak and Rosa also gave a number of constructions for families of maximal partial latin squares. Most of these constructions relied on partitioning the partial latin square into subarrays with appropriate properties. The following result, taken from **[14]**, illustrates the type of construction employed by Horak and Rosa.

LEMMA 7 *(Horak and Rosa, 1993,* **[14]***) If n is even, then there exists a maximal partial latin square of order n and size $\frac{n^2}{2} + 2h$, where $h = 0, 1, \ldots, \frac{n^2}{4}$.*

Proof: Let $n = 2m$. Let $P \subset M^3$, where $M = \{1, 2, \ldots, m\}$, and $Q = \{(i, j, k+m) \mid (i, j, k) \in P\}$ be two latin squares of order m. For $0 \le h \le m^2$, let $C_h \subset M^2$ such that $|C_h| = h$. That is, C_h is a set of h cells selected from P. The set C_h is used to construct two arrays P_h and R_h as follows. For all $(i, j) \in M^2$ there exists a $k \in M$ such that $(i, j, k) \in P$.

- If $(i, j) \in C_h$, then $(i, j, k+m) \in P_h$ and $(i, j, k) \in R_h$.
- If $(i, j) \notin C_h$, then $(i, j, k) \in P_h$.

So P_h and R_h are both $m \times m$ arrays which contain h distinguished cells. In P_h, if (i, j) is a distinguished cell and $(i, j, k) \in P$ then place $k + m$ in this cell, and in the remaining cells place the symbol which occurs in the corresponding cell of P. In R_h, if (i, j) is a distinguished cell place the symbol k, where $(i, j, k) \in P$, in cell (i, j) of R_h, otherwise all remaining cells are left empty. Finally, for each $h \in \{0, 1, \ldots, m^2\}$ take the partial latin square L_h constructed as follows:

$$L_h = \begin{array}{|c|c|}\hline P_h & R_h \\ \hline R_h & Q \\ \hline\end{array}$$

It is easy to see that L_h contains $2(m^2) + 2h = \frac{n^2}{2} + 2h$ filled cells, where $0 \le h \le m^2$ $(= (\frac{n}{2})^2 = \frac{n^2}{4})$. The only empty cells occur in rows 1 to m of columns $m+1$ to $2m$, or in columns 1 to m of rows $m+1$ to $2m$. For any such empty cell (x, y) it is clear that row x contains at least m filled cells and column y contains at least m filled cells. So row x and column y together contain at least n filled cells, and these filled cells, between them, must contain n distinct symbols. Hence L_h is maximal. $\square$

Using arguments similar to that presented above Horak and Rosa proved the following.

THEOREM 8 *(Horak and Rosa, 1993, [14]) If* $S^+ = \{s \mid \frac{n^2}{2} \leq s \leq n^2, s \neq n^2 - 1\}$ *and* $S^- = \{\lceil \frac{n^2}{2} \rceil + h \mid h \text{ odd}, 1 \leq h \leq n-1\}$, *then maximal partial latin squares of order* n *and size* s *exist for all*

$$s \in S^+ \setminus S^-.$$

However, they did not completely settle the question of the spectrum of maximal partial latin squares and the following problem remains open.

Open Problem: Do there exist maximal partial latin squares of order n and size s where

$$s = \tfrac{n^2}{2} + h, \; h \text{ odd}, \; \tfrac{n}{2} < h \leq n-1, \; n \text{ even, and}$$

$$s = \tfrac{n^2+1}{2} + h, \; h \text{ odd}, \; \tfrac{n-1}{2} \leq h \leq n-1, \; n \text{ odd?}$$

5 Premature Partial Latin Squares

A partial latin square, P, is **premature** if it can not be completed, but removing any of its entries destroys this property.

The proof of the Evans' conjecture verifies that if P is a partial latin square of order n where $|P| \leq n-1$, then P is completable. Thus if P is a premature partial latin square of order n, $|P| \geq n$. In fact, it is easy to see that there exist premature partial latin squares of order n and size n. For example, consider the partial latin square, of order $n > 1$, with symbol 1 in cell $(1,1)$ and symbol 2 in cells (i,i), $i = 2, \ldots, n$. If one considers the array displayed below then it is easy to see that symbol 2 must be placed in row 1 and in column 1, but this is not possible, so no completion exists. However, removal of any entry yields a partial latin square of size $n-1$ which has a completion.

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
1 & & & & & & \\
\hline
 & 2 & & & & & \\
\hline
 & & 2 & & & & \\
\hline
 & & & \ddots & & & \\
 & & & \ddots & & & \\
\hline
 & & & & 2 & & \\
\hline
 & & & & & 2 & \\
\hline
 & & & & & & 2 \\
\hline
\end{array}$$

It can be observed that any row of a premature partial latin square of order n must contain at most $n-1$ filled cells. To see this assume that it is not the case. That is, there exists a premature partial latin square P of order n for which there is a row,

r, consisting of n filled cells. By the definition of premature , $P \setminus \{(r, 1, e) \mid e \in N\}$, must have a completion. But any completion of $P \setminus \{(r, 1, e) \mid e \in N\}$ must contain the entry $(r, 1, e)$. This implies that P also has a completion, a contradiction.

Such ideas lead us to ask what is the range of the possible sizes of premature partial latin squares. This is precisely the question asked by L. Brankovic, P. Horak, M. Miller and A. Rosa, in 1998, when they studied the spectrum of premature partial latin squares. That is, they investigated the set,

$$S = \{s \mid \text{ there exists a premature partial latin square of order } n \text{ and size } s\}.$$

Brankovic, Horak, Miller and Rosa, extended the above arguments to show that the size of the largest premature partial latin square, of order n, is less than or equal to $n^2 - 3n + 4$. However, they conjectured that the size of the largest premature partial latin square of order n is less than or equal to $n^2 - n^{3/2}$.

Brankovic, Horak, Miller and Rosa also gave an number of constructions which they later used to establish the existence of premature partial latin squares of various sizes. The validation of the correctness of these constructions relies on the following result by Ryser.

THEOREM 9 *(Ryser, 1951,* [17]*) An $h \times g$ latin rectangle R can be completed to a latin square of order n if and only if each element $1, 2, \ldots, n$ occurs in R at least $h + g - n$ times.*

To give the reader the general flavour of Brankovic, Horak, Miller and Rosa's methods the following result is presented.

LEMMA 10 *(Brankovic, Horak, Miller and Rosa, 1998,* [4]*) Let n, h and g be positive integers such that $2 \leq h \leq g \leq n - h$. Then there exists a premature partial latin square of order n and size $hg + n - h - g + 1$.*

Proof: Take an $n \times n$ array L and partition it into four subarrays of dimension $h \times g$, $h \times (n - g)$, $(n - h) \times g$ and $(n - h) \times (n - g)$. Denote the $h \times g$ subarray by R. Fill the cells of R with the entries $\{1, \ldots, g\}$, in such a way that R is row latin and column latin. Next fill each cell $(h + \delta, g + \delta)$, for $\delta = 1, \ldots, n - h - g + 1$, of L with the symbol $g + 1$. Leave all other cells empty. Then L takes the form:

$$L = \begin{array}{|c|c|}\hline \quad R \quad & \qquad\qquad\qquad \\ \hline & \begin{matrix} g+1 & & \\ & \ddots & \\ & & g+1 \\ & & \\ & & \end{matrix} \\ \hline \end{array}$$

Then L is a partial latin square containing $(gh) + n - h - g + 1$ filled cells. Now consider the placement of the h copies of the symbol $g + 1$ in rows 1 to h. Columns 1 to g are filled in these rows and the symbol $g + 1$ already occurs in columns $g + 1$ to $n - h + 1$. Hence the symbol $g + 1$ can only be placed in columns $n - h + 2$ to n of rows 1 to h. That is, in at most $h - 1$ columns. Thus it is not possible to complete L. However, if any entry from L is removed, then it can be verified, using Ryser's result, that the reduced partial latin square is completable. □

Through this and a number of other constructions Brankovic, Horak, Miller and Rosa were able to show:

THEOREM 11 *(Brankovic, Horak, Miller and Rosa, 1998,* [4]*) There exist premature partial latin squares of order n and size s where*

- *if n is even, $s \in [n + 1, \frac{n^2}{4} + n - 2]$,*
- *if n is odd, $s \in [n + 1, \frac{n^2}{4} + \frac{3}{4}]$.*

However this work by no means settles the question of the spectrum of premature partial latin squares, and so there is still the open question:

Open Problem: Determine fully the spectrum of premature partial latin squares.

6 Critical Sets

A critical set is a partial latin square that has a unique completion, but for which removal of any entry destroys this property. In order to study critical sets one must first understand the concept of a latin interchange.

Let I and I' be two partial latin squares, of the same order, size and with corresponding filled cells. They are said to be *mutually balanced* if the symbols in each row (and column) of I are the same as those in the corresponding row (and column) of I', and they are said to be *disjoint* if no cell in I' contains the same symbol as the corresponding cell in I. Partial latin squares, I and I', which satisfy the above conditions are said to be *latin interchanges.* The connection between critical sets and latin interchanges is summarised in the next lemma.

LEMMA 12 *A partial latin square $C \subset L$, of size s and order n, is a critical set for the latin square L if and only if the following hold:*

1. *C contains an element of every latin interchange that occurs in L;*
2. *for each $(i, j; k) \in C$, there exists a latin interchange I_r in L such that $I_r \cap C = \{(i, j; k)\}$.*

Proof.

1. If C does not contain an element from some latin interchange I, where I has disjoint mate I', then C is also a partial latin square contained in $L' = (L \setminus I) \cup I'$. Hence C is not uniquely completable.

2. If no such latin interchange I_r can be found, then the position $(i,\ j;\ k)$ may be deleted from C and $C \setminus \{(i,\ j;\ k)\}$ will still have unique completion and thus a critical set for L.

Since 1977, there has been a considerable amount of research into the properties of critical sets, with much of this research pertaining to the spectrum of critical sets. That is, the study of

$$S = \{s \mid \text{ there exists a critical set of order } n \text{ and size } s\}.$$

Let $lcs(n)$ denote the size of the largest critical set in any latin square, of order n, and $scs(n)$ denote the size of the smallest critical set in any latin square, of order n. It is easily shown that for latin squares, of order n, $lcs(n) \leq n^2 - n$ and $scs(n) \geq n - 1$, but small case analysis indicates that these bounds are far from sharp. In 1977, Nelder conjectured that $lcs(n) = (n^2 - n)/2$ and $scs(n) = \lfloor n^2/4 \rfloor$. However in 1982, the bound for $lcs(n)$ was shown to be false. At that time, Stinson and van Rees **[19]** exhibited examples of critical sets for which $lcs(n) \geq (n^2 - n)/2$.

As for $scs(n)$, the evidence tends to suggests that Nelder's conjecture may be right, though to date an exact lower bound has not been established. In 1978, Curran and van Rees **[6]** proved that for all n there exists critcal sets, of order n, and size $\lfloor \frac{n^2}{4} \rfloor$. So an upper bound on $scs(n)$ is $\lfloor \frac{n^2}{4} \rfloor$. More recently, Bate and van Rees **[2]** have established this as an exact lower bound for semi–strong critical sets. If L is a latin square, of order n, and C a critical set for L, then the set C is said to be a *semi–strong* critical set if there exists a set $\{P_1, \ldots, P_m\}$ of $m = n^2 - |C|$ partial latin squares of order n, which satisfies the following properties:

1. $C = P_1 \subset P_2 \subset \ldots \subset P_m \subset L$;

2. for any i, $1 \leq i \leq m-1$, $P_i \cup \{(r_i, c_i; e_i)\} = P_{i+1}$ such that one of $P_i \cup \{(r_i, c_i; e)\}$, or $P_i \cup \{(r_i, c; e_i)\}$ or $P_i \cup \{(r, c_i; e_i)\}$ is not a partial latin square for any $e \in N \setminus \{e_i\}$ or $c \in N \setminus \{c_i\}$ or $r \in N \setminus \{r_i\}$, respectively.

Bate and van Rees proved:

LEMMA 13 *(Bate and van Rees, 1999,* **[2]***) For any latin square of order n, the size of the smallest strong critical set is $\lfloor n^2/4 \rfloor$.*

Fu, Fu and Rodger **[10]** have also contributed to this research by showing that $scs(n) \geq \lfloor \frac{7n-3}{6} \rfloor$, but exact values still need to be established for general critical sets.

Open Question: Given a positive integer n, what is the exact value of $lcs(n)$ and $scs(n)$?

Using the above as a guide, Donovan and Howse **[7]** showed that if $S^+ = \{s \mid \lfloor n^2/4 \rfloor \leq s \leq (n^2 - n)/2\}$, then

- if n is odd, $S^+ \subseteq S$, and

- if n is even, $(S^+ \setminus \{\frac{n^2}{4}+1\}) \subseteq S$.

Then in 1999, Bean and Donovan **[3]** established that when n is even there exists a critical set of order n and size $\frac{n^2}{4}+1$, thus proving the following result.

LEMMA 14 *(Donovan and Howse, 1999,* **[7]**, *Bean and Donovan, 1999,* **[3]***) There exist critical sets of order n and size s, whenever* $\lfloor n^2/4 \rfloor \le s \le (n^2-n)2$.

The proof of the above lemma was by direct construction, involving a large number of cases and reproducing the detail here will be of very little value. Instead the remainder of this section will focus on the unsolved problem of systematically constructing critical sets of order n and size greater than $\frac{n^2-n}{2}$.

Open Question: For what values of $s > (n^2-n)/2$ does there exist a critical set of order n and size s?

One possible approach is to use techniques similar to those used for maximal partial latin squares and premature partial latin squares. That is, to partition the array into subarrays and use a *product type* construction. To this end a definition of the direct product of two latin squares is given.

Let M and L be two latin squares (partial latin squares) of order m and n respectively. The *direct product* of M with L is the array given by the set

$$M \times L = \{((i_m, i_l), (j_m, j_l); (k_m, k_l)) \mid (i_m, j_m; k_m) \in M \wedge (i_l, j_l; k_l) \in L\}.$$

One may think of the direct product as an $mn \times mn$ array constructed by first identifying the cells of M which contain symbol i, for $i = 1, \dots, m$, and then replacing these cells by copies of L; that is, subarrays with the same structure as L on the symbols mi to $(m+1)i-1$. It is easily verified that the array $M \times L$ is also a latin square (partial latin square).

Let P be a latin interchange. Define $R(P) = \{r \mid (r,c;e) \in P\}$, $C(P) = \{c \mid (r,c;e) \in P\}$, $E(P) = \{e \mid (r,c;e) \in P\}$. Let P_1 and P_2 be two latin interchanges. A latin interchange P_1 is said to be *equivalent (isotopic) to* P_2, if there exists an ordered triple (α, β, γ) of one-to-one functions such that $\alpha(R(P_1)) = R(P_2)$, $\beta(C(P_1)) = C(P_2)$, $\gamma(E(P_1)) = E(P_2)$ and $P_2 = \{(\alpha(i), \beta(j); \gamma(k)) \mid (i,j;k) \in P_1\}$. Donovan, Gower and Khodkar **[8]** have shown that for any latin squares M and L and latin interchanges $PM \subset M$ and $PL \subset L$, if PM and PL are equivalent, then $PM \times PL$ is a latin interchange in $M \times L$.

Using this information they proved that under certain circumstances critical sets can be constructed using direct product techniques. The essence of these ideas lies in taking two critical sets CM and CL in the latin squares M and L. For each filled cell in CM replace it by an appropriate copy of L. Then for each empty cell in CM replace it by an appropriate copy of CL. Under certain strong conditions the resulting partial latin square in $M \times L$ is a critical set. (This result is a generalisation of an earlier result proved by Stinson and van Rees, **[19]**.) A formal statement of this result is given in the following theorem and is followed by an example.

THEOREM 15 *Let CL and CM be critical sets in L and M respectively, where either CL or CM is semi–strong. Assume that for each $(x_m, y_m; z_m) \in CM$ and each $(u_l, v_l; w_l) \in CL$ there exist latin interchanges $IM \subseteq M$ and $IL \subseteq L$ which satisfy the following conditions:*

1. $IM \cap CM = \{(x_m, y_m; z_m)\}$;
2. $IL \cap CL = \{(u_l, v_l; w_l)\}$;
3. *there exists an ordered triple of one-to-one functions* (α, β, γ) *such that* $(R(IM))\alpha = R(IL)$, $(C(IM))\beta = C(IL)$, $(E(IM))\gamma = E(IL)$ *and*

$$IL = \{(i_m\alpha, j_m\beta; k_m\gamma) \mid (i_m, j_m; k_m) \in IM\}; \text{ and}$$

4. $x_m\alpha = u_l$, $y_m\beta = v_l$, *and* $z_m\gamma = w_l$.

Then the partial latin square

$$\{((i_m, i_l), (j_m, j_l); (k_m, k_l)) \mid (i_m, j_m; k_m) \in CM, (i_l, j_l; k_l) \in L\} \ \cup$$
$$\{((i_m, i_l), (j_m, j_l); (k_m, k_l)) \mid (i_m, j_m; k_m) \in M \setminus CM, (i_l, j_l; k_l) \in CL\}$$

is a critical set in $M \times L$.

EXAMPLE 16 *Let M and L be latin squares as presented below, with CM and CL as shown.*

M

1	2	3
2	3	1
3	1	2

CM

1		
		2

L

1	2	3	4	5	6
2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	1	2	3	4	5

CL

1	2	3			
2	3				
3					
					4
				4	5

The following partial latin square is a critical set in $M \times L$.

1	2	3	4	5	6	7	8	9				13	14	15			
2	3	4	5	6	1	8	9					14	15				
3	4	5	6	1	2	9						15					
4	5	6	1	2	3												
5	6	1	2	3	4						10						16
6	1	2	3	4	5					10	11					16	17
7	8	9				13	14	15				1	2	3			
8	9					14	15					2	3				
9						15						3					
					10						16						4
				10	11					16	17					4	5
13	14	15				1	2	3				7	8	9	10	11	12
14	15					2	3					8	9	10	11	12	7
15						3						9	10	11	12	7	8
												10	11	12	7	8	9
					16						4	11	12	7	8	9	10
				16	17					4	5	12	7	8	9	10	11

So this paper concludes with the suggestion that a systematic application of a variant of Theorem 15 may yield sequential families of critical sets of sizes greater than $\frac{n^2-n}{2}$.

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