ABSTRACT

DEMIR, ISMAIL. Classification of 5−Dimensional Complex Nilpotent Leibniz Algebras. (Under the direction of Dr. Kailash Misra and Dr. Ernest Stitzinger.)

Leibniz algebras are certain generalization of Lie algebras. They were introduced by Bloh (1965) who called them D-algebras. Then it was popularized by Loday (1993) and the subject has been studied since then. Lie algebras are known to have many applications in mathematical areas including algebraic geometry, differential geometry, differential equations, number theory and also in physical areas such as general relativity, quantum mechanics, quantum field theory, string theory, particle physics and nuclear physics. The classification problem is one of the fundamental and important problems in Lie algebras. The famous Levi-Malcev (1905, 1950) theorem reduce the problem of classifying Lie algebras to classifying semisimple and solvable Lie algebras over a field of characteristic 0. The semisimple Lie algebras was classified by Cartan (1894) and later refined by Dynkin (1947). Malcev (1950) showed that the problem of classifying solvable Lie algebras can be reduced to classifying nilpotent Lie algebras. So far the complete classification of complex nilpotent Lie algebras of dimension $n \leq 7$ is known and the classification problem of complex nilpotent Lie algebras is wild in higher dimensions. The lack of antisymmetry property in Leibniz algebras makes the classification problem more difficult for Leibniz algebras.

In this work, we give the classification of complex nilpotent non-split non-Lie Leibniz algebras of dimension $n \leq 5$. A Leibniz algebras is called non-split if it doesn't have nontrivial ideals as a summand. We introduce the technique involving bilinear forms to obtain the classification of complex nilpotent non-split non-Lie Leibniz algebras with one dimensional derived algebra. The remaining cases are done by using some algebraic invariants.

Classification of 5–Dimensional Complex Nilpotent Leibniz Algebras

by
Ismail Demir

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2016

APPROVED BY:

Dr. Bojko Bakalov	Dr. Mohan Putcha
Dr. Kailash Misra Co-chair of Advisory Committee	Dr. Ernest Stitzinger Co-chair of Advisory Committee

DEDICATION

In memory of my father Bekir Demir.

To my mother, Behiye Demir, and all the others who believed in me.

BIOGRAPHY

Ismail was born in Izmir, Turkey on October 1, 1987. He graduated from Ege University with a B.S. degree in mathematics in 2009. He was granted full scholarship by the Ministry of National Education of Turkey to pursue graduate study in the USA. Then he attended North Carolina State University where he obtained his M.S. degree in mathematics in Fall 2012. He began to work on Leibniz algebras for his Master's project under the advisement of Dr. Ernest Stitzinger. He started his Ph.D. in Spring 2013. He was awarded NCSU Mathematics Department Winton-Rose Graduate Scholarship Award for his works on Ph.D. research.

ACKNOWLEDGEMENTS

I would like to thank my advisors, Dr. Kailash Misra and Dr. Ernest Stitzinger, for their guidance and encouragement throughout my graduate studies. They spent enormous amount of time and effort guiding me on this research. I was able to overcome many research challenges with their support.

I would also like to thank my committee members, Dr. Bojko Bakalov and Dr. Mohan Putcha. They are admirable instructors and I consider myself lucky to learn from them.

I would also like to thank my family for their unconditional love and support during my graduate career. Especially, I am grateful for my brother, Ahmet Demir, who is my first math teacher and is the person who let me to love mathematics. A special thanks to the Ministry of National Education of Turkey for their financial support during my graduate studies.

TABLE OF CONTENTS

LIST OF TABLES

Chapter 1

Introduction

Leibniz algebras are nonantisymmetric generalization of Lie algebras. Such algebras had been first considered by Bloh who called them D-algebras [7], considering their connections with derivations. While studying the properties of the homology of Lie algebras Loday noticed that the classical Chevalley-Eilenberg boundary map in the exterior module of a Lie algebra can be lifted to the tensor module which yields a new chain complex. For this chain complex to be well-defined the only property needed is the Leibniz identity. This was the motivation for Loday to introduce Leibniz algebras [20], [21], [22].

It is always an interesting and fundamental problem to give the classification of any kind of algebras. The problem of classifying all Lie algebras is still unsolved and it is very complicated. One of the immediate applications of Levi-Malcev Theorem is to reduce the problem of classifying Lie algebras over a field of characteristic 0 to classifying semisimple and solvable Lie algebras over a field of characteristic 0 [19], [24]. The classification of complex semisimple Lie algebras was completely given by Cartan [8] and later revisited by Dynkin [17]. The problem of the classification of complex solvable Lie algebras can be reduced to the classification of complex nilpotent Lie algebras due to Malcev [24].

Therefore, it has been of interest of many researchers to give the classification of complex nilpotent Lie algebras. The first work on this problem was given by Umlauf [42], in which he classified complex nilpotent Lie algebras of dimension $n \leq 6$. By applying the method of nilpotent elements of semisimple Lie algebras, Morozov classified complex nilpotent Lie algebras up to dimension 6 [25]. Later Safiullina [37] presented new results on the classification of 7−dimensional complex nilpotent Lie algebras using Morozov's method. Major stepforward was made by Vergne who introduced filiform Lie algebras as well as giving the complete classification of complex nilpotent Lie algebras of dimension $n \leq 7$ [43]. Skjelbred and Sund tackled the problem by studying the orbits under the action of a group on the second degree cohomology space of a smaller Lie algebra with coefficients in a trivial module [40]. In 1989 Romdhani obtained a classification of 7−dimensional complex nilpotent Lie algebras using only basic Linear algebra techniques such as Jordan forms of matrices and classification of bilinear forms [36]. Ancochea and Goze [6] and Seeley [38] are the other researchers who attacked the problem by following different approach, but they were later adjusted in [39] and [23]. In [11] De Graaf got the classification 6−dimensional complex nilpotent Lie algebras by using Gröbner bases and he compared it with the classifications of 6−dimensional complex nilpotent Lie algebras given before.

The classification problem of complex nilpotent Leibniz algebras were first studied by Loday himself. In [21] he obtained the complete classification of complex nilpotent Leibniz algebras of dimension $n \leq 2$. Later Ayupov and Omirov classified 3−dimensional complex nilpotent Leibniz algebras in [3] and [4].

The classification of complex nilpotent Lie algebras is already a complicated problem. Due to lack of antisymmetry the problem of classifying complex nilpotent Leibniz algebras is more difficult. The problem is especially difficult for $n \geq 3$ because that requires to solve a system of n^4 equations in n^3 unknowns. This difficulty led some researchers to work on a special subclass of nilpotent Leibniz algebras, namely filiform Leibniz algebras which is introduced by Ayupov and Omirov [5]. They extended the concept of filiform Lie algebras to Leibniz algebras and classified them for Leibniz algebras [5]. Using this result along with the classification of 5−dimensional associative algebras, Albeverio, Omirov and Rakhimov obtained the classification of 4−dimensional complex nilpotent Leibniz algebras [2]. In [27] Rakhimov and Bekbaev proposed an algortihm for classification of complex nilpotent Leibniz algebras deriving from naturally graded filiform Leibniz algebras with regard to invariant functions which allowed them to find isomorphism criterion for each class. In particular, they gave the classification of complex filiform Leibniz algebras of dimension 5 and 6. Following [27] many researchers worked on the classification of complex filiform Leibniz algebras of dimension $n \leq 9$ [26], [34], [33], [32], [29], [28], [16], [1].

As stated above one of the techniques to classify nilpotent Lie algebras was introduced by Skjelbred and Sund. Rakhimov and Langari were the first researchers who used Skjelbred-Sund method in Leibniz algebras [31]. They also applied this technique to obtain the classification of 3−dimensional complex nilpotent Leibniz algebras [30]. Rikhsiboev and Rakhimov [35] presented another classification of 3−dimensional Leibniz algebras by considering some invariants. It was not surprising that the technique involving Gröbner bases also was introduced for Leibniz algebras [18]. Casas, Insua and Ladra first introduced an algorithm for testing whether a given algebra actually corresponds to a Leibniz algebra [9]. Then they proposed another algorithm deciding whether any given two Leibniz algebras are isomorphic [10]. In their work [10] they gave the complete classification 3−dimensional Leibniz algebras and compared it with the one given in [3]. Demir, Misra and Stitzinger [13] used another approach involving the canonical forms for the congruence classes of matrices for bilinear forms to classify complex nilpotent Leibniz algebras of dimension $n \leq 3$. Their technique allowed them to classify complex nilpotent Leibniz algebras with one dimensional derived algebra of dimension $n \leq 8$ [14]. In fact this technique can be applied to complex nilpotent Leibniz algebra of any fixed dimension n. Using this technique and some invariants they also gave the complete classification of 4−dimensional complex nilpotent Leibniz algebras [15].

In Chapter 2, we recall basic notions for Leibniz algebras. We introduce the technique involving bilinear forms to give the classification of complex nilpotent non-Lie Leibniz algebras with one dimensional derived algebra. In Chapter 3, we also include the results on classification of complex nilpotent non-split non-Lie Leibniz algebras of dimension $n \leq 4$ that we obtained in [13] and [15].

There has been no attempt to give the complete classification of 5−dimensional complex nilpotent Leibniz algebras. As stated above there exists only partial results for classification of nilpotent Leibniz algebras of dimension $n \geq 5$. In Chapter 4, by applying the bilinear form technique introduced in Chapter 3 and using some algebraic invariants we classify 5−dimensional complex nilpotent non-split non-Lie Leibniz algebras. Throughout this work, all algebras are over the field of complex numbers. We restrict our attention to give isomorphism classes of non-split non-Lie nilpotent Leibniz algebras because split ones can always be obtained by non-split isomorphism classes. We use Mathematica program implementing Algorithm 2.6 given in [10] to check that the classes we obtained are indeed pairwise nonisomorphic.

Chapter 2

Preliminaries

In this section we give the basic definitions and properties for Leibniz algebras.

Definition 2.0.1. A (left) Leibniz algebra A is a $\mathbb{F}-$vector space equipped with a bilinear map $[\ ,\ ] : A \times A \to A$ satisfying the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \tag{2.1}$$

for all $a, b, c \in A$.

For a Leibniz algebra A and $a \in A$, we define the left multiplication operator $L_a : A \to A$ and the right multiplication operator $R_a : A \to A$ by $L_a(b) = [a,b]$ and $R_a(b) = [b,a]$ respectively for all $b \in A$. Note that by equation (2.1), the operator L_a is a derivation, but R_a is not a derivation. A (right) Leibniz algebra is a vector space equipped with a bilinear map such that the right multiplication is a derivation. Throughout this work, Leibniz algebra always refers to (left) Leibniz algebra. A (left) Leibniz algebra is not necessarily a (right) Leibniz algebra, as the following example shows.

Example 2.0.2. Let A be a 2-dimensional algebra with the following products:

$$[x,x] = 0, [x,y] = 0, [y,x] = x, [y,y] = x.$$

A is a (left) Leibniz algebra, but it is not a (right) Leibniz algebra, since $[[y,y],y] \neq [y,[y,y]] + [[y,y],y]$.

Any Lie algebra is clearly a Leibniz algebra. A Leibniz algebra A satisfying the condition that $[a,a] = a^2 = 0$ for all $a \in A$, is a Lie algebra since in this case the Leibniz identity becomes the Jacobi identity. A Leibniz algebra which is not a Lie algebra is called a non-Lie Leibniz algebra.

For any element $a \in A$ and $n \in \mathbb{Z}_{>1}$ we define $a^n \in A$ inductively by defining $a^1 = a$ and $a^{k+1} = [a, a^k]$. Similarly, we define A^n by $A^1 = A$ and $A^{k+1} = [A, A^k]$. The Leibniz algebra A is said to be abelian if $A^2 = 0$. Furthermore, it follows from (2.1) that $L_{a^n} = 0$ for $n \in \mathbb{Z}_{>1}$.

Example 2.0.3. Let A be a $n-$dimensional Leibniz algebra generated by a single element a. Then $A = \text{span}\{\mathrm{a}, \mathrm{a}^2, \dots, \mathrm{a}^\mathrm{n}\}$ and we have $[a, a^n] = \alpha_1 a + \dots + \alpha_n a^n$ for some $\alpha_1, \dots, \alpha_n \in \mathbb{F}$. By Leibniz identity we have $0 = [a, [a^n, a]] = [[a, a^n], a] + [[a^n, a], a] = [\alpha_1 a + \dots + \alpha_n a^n, a] = \alpha_1 [a, a]$ which implies that $\alpha_1 = 0$. Hence $A^2 = \text{span}\{\mathrm{a}^2, \dots, \mathrm{a}^\mathrm{n}\}$. The Leibniz algebra A is called a $n-$dimensional cyclic Leibniz algebra.

Definition 2.0.4. Let I be a subspace of a Leibniz algebra A. Then I is a subalgebra if $[I, I] \subseteq I$, an ideal if $[A, I], [I, A] \subseteq I$.

A^2 is called derived algebra of A. Given any Leibniz algebra A we denote $Leib(A) = \{a^2 \mid a \in A\}$. In particular, $Leib(A)$ is an abelian ideal of A. $Leib(A)$ is a right ideal by definition. The fact that $Leib(A)$ is a left ideal follows from the identity $[a, [b, b]] = [a + [b, b], a + [b, b]] - [a, a]$. For any ideal I of A we define the quotient Leibniz algebra in the usual way. It can be seen that $A/Leib(A)$ is a Lie algebra.

Definition 2.0.5. The left center of A is denoted by $Z^l(A)$=$\{x \in A \mid [x, a] = 0$ for all $a \in A\}$ and the right center of A is denoted by $Z^r(A)$=$\{x \in A \mid [a, x] = 0$ for all $a \in A\}$. The center of A is $Z(A) = Z^l(A) \cap Z^r(A)$.

Let A be a Leibniz algebra. Then the series of ideals

$$A^{(0)} = A \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \dots \text{ where } A^{(1)} = [A, A], A^{(i+1)} = [A^{(i)}, A^{(i)}]$$

is called the derived series of A.

Definition 2.0.6. A Leibniz algebra A is solvable if $A^{(m)} = 0$ for some integer $m \geq 0$.

Definition 2.0.7. A Leibniz algebra A is nilpotent of class c if $A^{c+1} = 0$ but $A^c \neq 0$.

Definition 2.0.8. A Leibniz algebra A is said to be split if it can be written as a direct sum of two nontrivial ideals. Otherwise, A is called non-split.

Definition 2.0.9. A $n-$dimensional Leibniz algebra A is said to be filiform Leibniz algebra if $\dim(A^i) = n - i$, for $2 \leq i \leq n$.

Chapter 3

Classification of Nilpotent Leibniz Algebras of $\dim \leq 4$

3.1 Classification of Nilpotent Leibniz Algebras with One Dimensional Derived Algebra

Let A be a n-dimensional non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 1$. Then $A^2 = Leib(A) = \text{span}\{x_n\}$ for some $0 \neq x_n \in A$. Let V be a complementary subspace to A^2 in A such that $A = A^2 \oplus V$. Then for any $u, v \in V$, we have $[u, v] = cx_n$ for some $c \in \mathbb{C}$. Define the bilinear form $f(\ ,\ ) : V \times V \to \mathbb{C}$ by $f(u, v) = c$ for all $u, v \in V$. The canonical forms for the congruence classes of matrices associated with the bilinear form $g(\ ,\ )$ on a vector space W given in [41], [12] is as follows. We denote

$$[A \backslash B] := \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$$

Theorem 3.1.1. *[12] The matrix of the bilinear form $g(\ ,\ ) : W \times W \to \mathbb{C}$ is congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the following types:*

1. $$A_{2k+1} = \left[\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \backslash \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \right]_{(2k+1)\times(2k+1)}$$

2. $$B_{2k}(c) = \begin{bmatrix} \begin{bmatrix} 0 & & & c \\ & & c & 1 \\ & \iddots & \iddots & \\ c & 1 & & 0 \end{bmatrix} \backslash \begin{bmatrix} & & & 1 \\ & & 1 & c \\ & \iddots & \iddots & \\ 1 & c & & 0 \end{bmatrix} \end{bmatrix}_{2k \times 2k}, \quad c \neq \pm 1.$$

3. $$C_{2k+1} = \begin{bmatrix} 0 & & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & \iddots & \iddots & \\ & & & 1 & 1 & & \\ & & 1 & -1 & & & \\ & \iddots & \iddots & & & & \\ 1 & -1 & & & & & 0 \end{bmatrix}_{(2k+1) \times (2k+1)}$$

4. $$D_{2k} = \begin{bmatrix} \begin{bmatrix} 0 & & & 1 \\ & & 1 & -1 \\ & \iddots & \iddots & \\ 1 & -1 & & 0 \end{bmatrix} \backslash \begin{bmatrix} & & & 1 \\ & & 1 & 1 \\ & \iddots & \iddots & \\ 1 & 1 & & 0 \end{bmatrix} \end{bmatrix}_{2k \times 2k} \quad (k \text{ even})$$

5. $$E_{2k} = \begin{bmatrix} 0 & & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & \iddots & \iddots & \\ & & & 1 & 1 & & \\ & & -1 & 1 & & & \\ & \iddots & \iddots & & & & \\ -1 & 1 & & & & & 0 \end{bmatrix}_{2k \times 2k}$$

6. $$F_{2k} = \begin{bmatrix} \begin{bmatrix} 0 & & & -1 \\ & & -1 & 1 \\ & \iddots & \iddots & \\ -1 & 1 & & 0 \end{bmatrix} \backslash \begin{bmatrix} & & & 1 \\ & & 1 & 1 \\ & \iddots & \iddots & \\ 1 & 1 & & 0 \end{bmatrix} \end{bmatrix}_{2k \times 2k} \quad (k \text{ odd})$$

Using Theorem 3.1.1, we choose a basis $\{x_1, x_2, \ldots, x_{n-1}\}$ for V so that the matrix of the bilinear form $f(\ ,\ ) : V \times V \to \mathbb{C}$ is the $(n-1) \times (n-1)$ matrix N given in Theorem 3.1.1. Then A has basis $\{x_1, x_2, \ldots, x_{n-1}, x_n\}$ and the multiplication among the basis vectors is completely determined by the matrix N since $A^2 \subseteq Z(A)$. Hence the resulting Leibniz algebras corresponding to distinct congruence class of matrices are pairwise nonisomorphic.

Lemma 3.1.2. *Let A be a non-Lie nilpotent Leibniz algebra with* $\dim(A^2) = 1$ *and* $A = A^2 \oplus V$*. The matrix of the bilinear form* $f(\ ,\ ) : V \times V \to \mathbb{C}$ *defined above is of the form* $N = K \oplus 0$ *if and only if A is split.*

Proof. By Theorem 3.1.1, the matrix of the bilinear form $f(\ ,\ )$ is of the form $N = K \oplus 0$ with respect to the basis $\{x_1, x_2, \ldots, x_{n-1}\}$ for V where K is a $k \times k$ matrix. Recall that here $A^2 = Leib(A) = \text{span}\{x_n\}$, hence A has basis $\{x_1, x_2, \ldots, x_{n-1}, x_n\}$. Set $I_1 = \text{span}\{x_1, \ldots, x_k, x_n\}$ and $I_2 = \text{span}\{x_{k+1}, \ldots, x_{n-1}\}$. Then I_1 and I_2 are ideals of A and $A = I_1 \oplus I_2$. So A is split.

Conversely, suppose A is split. Then $A = I_1 \oplus I_2$ where I_1, I_2 are ideals of A. Without loss of generality we can assume that $A^2 = Leib(A) = \text{span}\{x_n\}$ is contained in I_1. Then $[I_2, I_2] \subseteq A^2 \cap I_2 = \{0\}$ which implies that I_2 is abelian. Hence the matrix $N = K \oplus 0$ for some $k \times k$ matrix K, $k < n - 1$. □

It can be seen that by using Theorem 3.1.1 and Lemma 3.1.2, we can give the complete classification of non-split non-Lie nilpotent Leibniz algebras with one dimensional derived algebra of any fixed dimension n. In fact we obtained the complete classification up to dimension 8 in [14].

3.2 Classification of Nilpotent Leibniz Algebras of $\dim(A) \leq 4$

We gave the complete classification of complex nilpotent Leibniz algebras of dimension $n \leq 3$ in [13]. We list our results here.

Theorem 3.2.1. *[13] Let A be a non-split non-Lie nilpotent Leibniz algebra of* $\dim(A) = 2$*. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2\}$ with the nonzero products given by the following:*

$\mathcal{A}_1$**:** $[x_1, x_1] = x_2$.

Theorem 3.2.2. *[13] Let A be a non-split non-Lie nilpotent Leibniz algebra of* $\dim(A) = 3$*. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3\}$ with the nonzero products given by the following:*

$\mathcal{A}_1$**:** $[x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_3.$

$\mathcal{A}_2(\alpha)$**:** $[x_1, x_2] = x_3, [x_2, x_1] = \alpha x_3, \quad \alpha \in \mathbb{C}\backslash\{-1, 1\}.$

$\mathcal{A}_3$**:** $[x_1, x_1] = x_3, [x_2, x_2] = x_3.$

$\mathcal{A}_4$**:** $[x_1, x_1] = x_2, [x_1, x_2] = x_3.$

Remark. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1, 1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_2(\alpha_1)$ and $\mathcal{A}_2(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

We also obtained the classification of 4−dimensional non-split non-Lie nilpotent Leibniz algebras in [15]. By comparing our classification with classification given in [2] we realized that one isomorphism class was missed in their list.

Theorem 3.2.3. *[15] Let A be a non-split non-Lie nilpotent Leibniz algebra of* $\dim(A) = 4$*. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4\}$ with the nonzero products given by the following:*

$\mathcal{A}_1$**:** $[x_1, x_3] = x_4, [x_3, x_2] = x_4.$

$\mathcal{A}_2$**:** $[x_1, x_3] = x_4, [x_2, x_2] = x_4, [x_2, x_3] = x_4 = -[x_3, x_2], [x_3, x_1] = x_4.$

$\mathcal{A}_3$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_3, x_3] = x_4.$

$\mathcal{A}_4$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = x_4, [x_3, x_3] = x_4.$

$\mathcal{A}_5(\alpha)$**:** $[x_1, x_2] = x_4, [x_2, x_1] = \alpha x_4, [x_3, x_3] = x_4, \quad \alpha \in \mathbb{C}\backslash\{-1, 1\}.$

$\mathcal{A}_6$**:** $[x_1, x_1] = x_4, [x_2, x_2] = x_4, [x_3, x_3] = x_4.$

$\mathcal{A}_7$**:** $[x_1, x_1] = x_2, [x_1, x_2] = x_3, [x_1, x_3] = x_4.$

$\mathcal{A}_8$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1].$

$\mathcal{A}_9$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_4.$

$\mathcal{A}_{10}$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_1, x_3] = x_4 = -[x_3, x_1].$

$\mathcal{A}_{11}$**:** $[x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_4, [x_1, x_3] = x_4 = -[x_3, x_1].$

$\mathcal{A}_{12}$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_1, x_3] = x_4 = -[x_3, x_1].$

$\mathcal{A}_{13}$**:** $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_4, [x_1, x_3] = x_4 = -[x_3, x_1].$

$\mathcal{A}_{14}$**:** $[x_1, x_1] = x_3, [x_1, x_2] = x_4.$

$\mathcal{A}_{15}$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4.$

$\mathcal{A}_{16}$**:** $[x_1, x_2] = x_4, [x_2, x_1] = x_3, [x_2, x_2] = -x_3.$

$\mathcal{A}_{17}(\alpha)$**:** $[x_1, x_1] = x_3, [x_1, x_2] = x_4, [x_2, x_1] = \alpha x_4, \quad \alpha \in \mathbb{C}\backslash\{-1, 0\}.$

$\mathcal{A}_{18}(\alpha)$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_1, x_2] = \alpha x_3, [x_2, x_2] = -x_4, \quad \alpha \in \mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{19}$**:** $[x_1, x_1] = x_3, [x_1, x_2] = x_3, [x_2, x_1] = x_3 + x_4, [x_2, x_2] = x_4.$

$\mathcal{A}_{20}$**:** $[x_1, x_2] = x_3, [x_1, x_3] = x_4.$

$\mathcal{A}_{21}$**:** $[x_1, x_2] = x_3, [x_2, x_2] = x_4, [x_1, x_3] = x_4.$

$\mathcal{A}_{22}$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_4, [x_1, x_3] = x_4.$

$\mathcal{A}_{23}$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_4, [x_2, x_2] = x_4, [x_1, x_3] = x_4.$

$\mathcal{A}_{24}$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_1, x_3] = x_4.$

$\mathcal{A}_{25}$**:** $[x_1, x_1] = x_3, [x_2, x_2] = x_4, [x_1, x_3] = x_4.$

Remark.

1. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1, 1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_5(\alpha_1)$ and $\mathcal{A}_5(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1, 0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{17}(\alpha_1)$ and $\mathcal{A}_{17}(\alpha_2)$ are not isomorphic.

3. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{18}(\alpha_1)$ and $\mathcal{A}_{18}(\alpha_2)$ are not isomorphic.

Chapter 4

Classification of 5–Dimensional Nilpotent Leibniz Algebras

Let A be a 5–dimensional non-split non-Lie nilpotent Leibniz algebra. Then since A is non-Lie we have $\dim(A^2) = 1, 2, 3$ or 4. The case $\dim(A^2) = 4$ can be done using Lemma 1 in [5].

Theorem 4.0.1. *Let A be a 5–dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 4$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by the following:*

$\mathcal{A}_1 :$ $[x_1, x_1] = x_2, [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

We give the following Lemmas which are very useful. The following Lemma is a direct consequence of Proposition 4.2 in [13].

Lemma 4.0.2. *If A is a nilpotent Leibniz algebra of class c then $A^c \subseteq Z(A)$*

Lemma 4.0.3. *Let A be a non-split Leibniz algebra then $Z(A) \subseteq A^2$.*

Proof. Let A be a non-split Leibniz algebra. Assume $Z(A) \nsubseteq A^2$. Take a complementary subspace W to A^2 in A such that $A = W \oplus A^2$. Let V be a complementary subspace to $Z(A) \cap W$ in A such that $A = V \oplus (Z(A) \cap W)$. Choose $I_1 = Z(A) \cap W$ and $I_2 = V$. Note that $Z(A) \cap W \subseteq Z(A)$ hence it is an ideal. Also V is an ideal since V contains A^2. Therefore, $A = I_1 \oplus I_2$ where I_1 and I_2 are nontrivial ideals of A. Then A is split, which is a contradiction. $\square$

Lemma 4.0.4. *Let A be a nilpotent Leibniz algebra and* $\dim(Leib(A)) = 1$. *Then* $Leib(A) \subseteq Z(A)$.

Proof. If $[A, Leib(A)] = 0$ then $Leib(A) \subseteq Z(A)$. Assume $[A, Leib(A)] \neq 0$. Then using $Leib(A)$ is an ideal we get $Leib(A) = [A, Leib(A)]$. So
$Leib(A) = [A, Leib(A)] \subseteq [A, A^2] = A^3 \Rightarrow Leib(A) \subseteq A^3$.
$Leib(A) = [A, Leib(A)] \subseteq [A, A^3] = A^4 \Rightarrow Leib(A) \subseteq A^4$. By doing this repetitively we see that $Leib(A) \subseteq A^n$ for any natural number n. This implies A is not nilpotent which is a contradiction. Hence $Leib(A) \subseteq Z(A)$. □

Lemma 4.0.5. *Let A be $n-$dimensional nilpotent Leibniz algebra and* $\dim(Z(A)) = n-k$. *If* $\dim(Leib(A)) = 1$ *then* $\dim(A^2) \leq \frac{k^2-k+2}{2}$.

Proof. By Lemma 4.0.4 we have $Leib(A) \subseteq Z(A)$. Let $Leib(A) = \mathrm{span}\{\mathrm{e_n}\}$. Extend this to a basis $\{e_{k+1}, e_{k+2}, \ldots, e_{n-1}, e_n\}$ for $Z(A)$. Then the nonzero products in $A = \mathrm{span}\{\mathrm{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_n}\}$ are given by:

$$[e_r, e_r] = \theta_r e_n, [e_i, e_j] = \sum_{t=1}^{n-1} \alpha_{ij}^t e_t + \beta_{ij} e_n, [e_j, e_i] = -\sum_{t=1}^{n-1} \alpha_{ij}^t e_t + \gamma_{ji} e_n.$$

for $1 \leq r, i, j \leq k, i \neq j$. Then $dim(A^2) \leq \{\text{number of } (i,j)\text{'s where } 1 \leq i < j \leq k\} + 1$. Note that the number of (i,j)'s where $1 \leq i < j \leq k$ is equal to $\frac{k^2-k}{2}$. Hence $\dim(A^2) \leq \frac{k^2-k+2}{2}$. □

Lemma 4.0.6. *Let A be $n-$dimensional nilpotent Leibniz algebra and* $\dim(A^2) = n-k$. *If* $\dim(Leib(A)) = 1$ *and* $A^3 = Leib(A)$ *then* $n \leq \frac{k^2+k+2}{2}$.

Proof. Let $Leib(A) = A^3 = \mathrm{span}\{\mathrm{e_n}\}$. Extend this to a basis $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ for A^2. Then the nonzero products in $A = \mathrm{span}\{\mathrm{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_n}\}$ are the following:

$$[e_i, e_j] = \sum_{t=k+1}^{n-1} \alpha_{ij}^t e_t + \beta_{ij} e_n, [e_j, e_i] = -\sum_{t=k+1}^{n-1} \alpha_{ij}^t e_t + \gamma_{ij} e_n.$$

for $1 \leq i, j \leq k$ where $i \neq j$ and other products are in $Leib(A)$. Then $dim(A^2) \leq \{\text{number of } (i,j)\text{'s where } 1 \leq i < j \leq k\} + 1$. Hence $n \leq \frac{k^2+k+2}{2}$. □

4.1 Classification of Nilpotent Leibniz Algebras of Dimension 5 with $\dim(A^2) = 1$

We apply the bilinear form technique given in Chapter 3 to get the following result.

Theorem 4.1.1. *Let A be a non-split non-Lie nilpotent Leibniz algebra of* $\dim(A) = 5$ *with* $\dim(A^2) = 1$. *Then A is isomorphic to a Leibniz algebra spanned by* $\{x_1, x_2, x_3, x_4, x_5\}$ *with the nonzero products given by one of the following:*

$\mathcal{A}_1(\alpha)$: $[x_1, x_4] = x_5, [x_2, x_3] = x_5, [x_2, x_4] = \alpha x_5, [x_3, x_2] = \alpha x_5, [x_4, x_1] = \alpha x_5, [x_4, x_2] = x_5, \quad \alpha \in \mathbb{C}\backslash\{-1, 1\}$.

$\mathcal{A}_2$: $[x_1, x_4] = x_5, [x_2, x_3] = x_5, [x_2, x_4] = x_5 = -[x_4, x_2], [x_3, x_2] = x_5, [x_4, x_1] = x_5$.

$\mathcal{A}_3$: $[x_1, x_4] = x_5 = -[x_4, x_1], [x_2, x_3] = x_5 = -[x_3, x_2], [x_2, x_4] = x_5, [x_3, x_3] = x_5, [x_4, x_2] = x_5$.

$\mathcal{A}_4$: $[x_1, x_3] = x_5, [x_3, x_2] = x_5, [x_4, x_4] = x_5$.

$\mathcal{A}_5$: $[x_1, x_3] = x_5, [x_2, x_2] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_3, x_1] = x_5, [x_4, x_4] = x_5$.

$\mathcal{A}_6$: $[x_1, x_2] = x_5 = -[x_2, x_1], [x_3, x_4] = x_5 = -[x_4, x_3], [x_4, x_4] = x_5$.

$\mathcal{A}_7(\alpha)$: $[x_1, x_2] = x_5 = -[x_2, x_1], [x_3, x_4] = x_5, [x_4, x_3] = \alpha x_5, \quad \alpha \in \mathbb{C}\backslash\{-1, 1\}$.

$\mathcal{A}_8$: $[x_1, x_2] = x_5 = -[x_2, x_1], [x_2, x_2] = x_5, [x_3, x_4] = x_5 = -[x_4, x_3], [x_4, x_4] = x_5$

$\mathcal{A}_9(\alpha)$: $[x_1, x_2] = x_5 = -[x_2, x_1], [x_2, x_2] = x_5, [x_3, x_4] = x_5, [x_4, x_3] = \alpha x_5, \quad \alpha \in \mathbb{C}\backslash\{-1, 1\}$.

$\mathcal{A}_{10}(\alpha, \beta)$: $[x_1, x_2] = x_5, [x_2, x_1] = \alpha x_5, [x_3, x_4] = x_5, [x_4, x_3] = \beta x_5, \quad \alpha, \beta \in \mathbb{C}\backslash\{-1, 1\}$.

$\mathcal{A}_{11}$: $[x_1, x_2] = x_5 = -[x_2, x_1], [x_3, x_3] = x_5, [x_4, x_4] = x_5$.

$\mathcal{A}_{12}$: $[x_1, x_2] = x_5 = -[x_2, x_1], [x_2, x_2] = x_5, [x_3, x_3] = x_5, [x_4, x_4] = x_5$.

$\mathcal{A}_{13}(\alpha)$: $[x_1, x_2] = x_5, [x_2, x_1] = \alpha x_5, [x_3, x_3] = x_5, [x_4, x_4] = x_5 \quad \alpha \in \mathbb{C}\backslash\{-1, 1\}$.

$\mathcal{A}_{14}$: $[x_1, x_1] = x_5, [x_2, x_2] = x_5, [x_3, x_3] = x_5, [x_4, x_4] = x_5$.

Proof. Let $A^2 = Leib(A) = \text{span}\{x_5\}$. Let V be a complementary subspace to A^2 in A such that $A = V \oplus A^2$. Then by Theorem 3.1.1, there exists an ordered basis $\{x_1, x_2, x_3, x_4\}$ of V such that the matrix of the bilinear form $f(\ ,\ )$ on V is one of the following listed below. Here we group the matrices corresponding to each partition of 4.

Partition	4×4 matrices
4	B_4, D_4, E_4
$3+1$	$A_3 \oplus C_1, C_3 \oplus C_1$
$2+2$	$F_2 \oplus F_2, F_2 \oplus E_2, F_2 \oplus B_2, E_2 \oplus E_2, E_2 \oplus B_2, B_2 \oplus B_2$
$2+1+1$	$F_2 \oplus C_1 \oplus C_1, E_2 \oplus C_1 \oplus C_1, B_2 \oplus C_1 \oplus C_1$
$1+1+1+1$	$C_1 \oplus C_1 \oplus C_1 \oplus C_1$

Table 4.1: 4×4 matrices of the bilinear form $f(\ ,\ )$

Now $\{x_1, x_2, x_3, x_4, x_5\}$ is an ordered basis for A and we have an isomorphism class corresponding to each matrix of the bilinear form $f(\ ,\ )$ on V listed above. Thus we have 14 isomorphism classes with the nonzero multiplications among basis vectors given in the statement of this theorem since the algebra corresponding to $F_2 \oplus F_2$ is a Lie algebra. $\square$

Remark.

1. If $\alpha_1, \alpha_2 \in \mathbb{C} \backslash \{-1, 1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_1(\alpha_1)$ and $\mathcal{A}_1(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

2. If $\alpha_1, \alpha_2 \in \mathbb{C} \backslash \{-1, 1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_7(\alpha_1)$ and $\mathcal{A}_7(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

3. If $\alpha_1, \alpha_2 \in \mathbb{C} \backslash \{-1, 1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_9(\alpha_1)$ and $\mathcal{A}_9(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

4. If $\alpha_1, \beta_1 \in \mathbb{C} \backslash \{-1, 1\}$ then we have the following isomorphism criterions in the family $A_{10}(\alpha, \beta)$: $A_{10}(\alpha_1, \beta_1) \cong A_{10}(\alpha_1, \beta_1), A_{10}(\alpha_1, \beta_1) \cong A_{10}(\alpha_1, \frac{1}{\beta_1})$, $A_{10}(\alpha_1, \beta_1) \cong A_{10}(\frac{1}{\alpha_1}, \beta_1), A_{10}(\alpha_1, \beta_1) \cong A_{10}(\frac{1}{\alpha_1}, \frac{1}{\beta_1}), A_{10}(\alpha_1, \beta_1) \cong A_{10}(\beta_1, \alpha_1), A_{10}(\alpha_1, \beta_1) \cong A_{10}(\beta_1, \frac{1}{\alpha_1})$, $A_{10}(\alpha_1, \beta_1) \cong A_{10}(\frac{1}{\beta_1}, \alpha_1)$ and $A_{10}(\alpha_1, \beta_1) \cong A_{10}(\frac{1}{\beta_1}, \frac{1}{\alpha_1})$.

5. If $\alpha_1, \alpha_2 \in \mathbb{C} \backslash \{-1, 1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{13}(\alpha_1)$ and $\mathcal{A}_{13}(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

4.2 Classification of Nilpotent Leibniz Algebras of Dimension 5 with $\dim(A^2) = 3$

Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3$. Then $\dim(A^3) = 0, 1$ or 2.

4.2.1 $\dim(A^3) = 0$

Let $\dim(A^2) = 3$ and $A^3 = 0$. Then by Lemma 4.0.2 and Lemma 4.0.3 we have $A^2 = Z(A)$. We get $\dim(Leib(A)) \neq 1$ from Lemma 4.0.5. Then since $Leib(A) \subseteq A^2$ we have $2 \leq \dim(Leib(A)) \leq 3$.

Theorem 4.2.1. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3$, $\dim(A^3) = 0$ and $\dim(Leib(A)) = 2$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4 + x_5, [x_2, x_2] = x_5$.

$\mathcal{A}_2$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5$.

Proof. Let $Leib(A) = \text{span}\{\text{e}_4, \text{e}_5\}$. Extend this to a basis $\{e_3, e_4, e_5\}$ for $A^2 = Z(A)$. Choose $e_1, e_2 \in A$ such that $[e_1, e_1] = e_4, [e_2, e_2] = e_5$. Then the nonzero products in $A = \text{span}\{\text{e}_1, \text{e}_2, \text{e}_3, \text{e}_4, \text{e}_5\}$ are given by:

$$[e_1, e_1] = e_4, [e_1, e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_1 e_3 + \beta_1 e_4 + \beta_2 e_5, [e_2, e_2] = e_5.$$

Take $\theta = (\alpha_2 + \beta_1)(\alpha_3 + \beta_2) - 1$.

- If $\theta = 0$ then the base change $x_1 = (\alpha_2 + \beta_1)e_1, x_2 = e_2, x_3 = (\alpha_2 + \beta_1)(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), x_4 = (\alpha_2 + \beta_1)^2 e_4, x_5 = e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\theta \neq 0, \alpha_3 + \beta_2 = 0$ and $\alpha_2 + \beta_1 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\theta \neq 0, \alpha_3 + \beta_2 = 0$ and $\alpha_2 + \beta_1 \neq 0$ then the base change $x_1 = \frac{i(\alpha_2+\beta_1)}{2} e_1, x_2 = i e_1 - \frac{2i}{\alpha_2+\beta_1} e_2, x_3 = \alpha_1 e_3 + \frac{\alpha_2-\beta_1}{2} e_4 + \alpha_3 e_5, x_4 = -\frac{(\alpha_2+\beta_1)^2}{4} e_4, x_5 = e_4 - \frac{4}{(\alpha_2+\beta_1)^2} e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- $\theta \neq 0$ and $\alpha_3 + \beta_2 \neq 0$ then the base change $x_1 = -\frac{\theta^2+\sqrt{-\theta^3}}{(\alpha_3+\beta_2)\theta^2\sqrt{\frac{\theta}{\sqrt{-\theta^3}}}} e_1 - \sqrt{\frac{\theta}{\sqrt{-\theta^3}}} e_2, x_2 = \frac{-\theta^2+\sqrt{-\theta^3}}{2(-\theta^3)^{5/8}} e_1 - \frac{\alpha_3+\beta_2}{2(-\theta^3)^{1/8}} e_2, x_3 = -\alpha_1(-\theta^3)^{3/8}(\frac{\theta}{\sqrt{-\theta^3}})^{5/2} e_3 - (\frac{\alpha_2-\beta_1}{2})(-\theta^3)^{3/8}(\frac{\theta}{\sqrt{-\theta^3}})^{5/2} e_4 - (\frac{\alpha_3-\beta_2}{2})(-\theta^3)^{3/8}(\frac{\theta}{\sqrt{-\theta^3}})^{5/2} e_5, x_4 = \frac{-2\theta^2+(\theta-1)\sqrt{-\theta^3}}{(\alpha_3+\beta_2)^2\sqrt{-\theta^3}} e_4 + e_5, x_5 = \frac{\theta(\theta^2-\theta-2\sqrt{-\theta^3})}{4(-\theta^3)^{3/4}} e_4 + \frac{(\alpha_3+\beta_2)^2\theta^2}{4(-\theta^3)^{3/4}} e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

$\square$

Theorem 4.2.2. *Let A be a $5-$dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3$, $\dim(A^3) = 0$ and $\dim(Leib(A)) = 3$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1(\alpha)$: $[x_1, x_1] = x_4, [x_1, x_2] = \alpha x_4 + x_5, [x_2, x_1] = x_3, [x_2, x_2] = x_5, \quad \alpha \in \mathbb{C}.$

$\mathcal{A}_2$: $[x_1, x_1] = x_4, [x_1, x_2] = x_3, [x_2, x_1] = x_3, [x_2, x_2] = x_5.$

Proof. Let $Leib(A) = A^2 = Z(A) = \mathrm{span}\{e_3, e_4, e_5\}$. Choose $e_1, e_2 \in A$ such that $[e_1, e_1] = e_4, [e_2, e_2] = e_5$. Then the nonzero products in $A = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by:

$$[e_1, e_1] = e_4, [e_1, e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5, [e_2, e_2] = e_5.$$

Case 1: Let $\alpha_1 = 0$. Then $\beta_1 \neq 0$ since $\dim(A^2) = 3$.

$$[e_1, e_1] = e_4, [e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5, [e_2, e_2] = e_5. \tag{4.1}$$

- If $\alpha_3 = 0$ then the base change $x_1 = (\alpha_2 - 1)e_1 - e_2, x_2 = \alpha_2 e_1 - e_2, x_3 = (1 - \alpha_2)\beta_1 e_3 + ((1 - \alpha_2)\beta_2 - \alpha_2)e_4 + ((1 - \alpha_2)\beta_3 + 1)e_5, x_4 = (1 - \alpha_2)\beta_1 e_3 + (1 - \alpha_2)(\beta_2 + 1)e_4 + ((1 - \alpha_2)\beta_3 + 1)e_5, x_5 = -\alpha_2\beta_1 e_3 - \alpha_2\beta_2 e_4 + (1 - \alpha_2\beta_3)e_5$ shows that A is isomorphic to $\mathcal{A}_1(0)$.

- If $\alpha_3 \neq 0$ then the base change $x_1 = e_1, x_2 = \alpha_3 e_2, x_3 = \alpha_3(\beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5), x_4 = e_4, x_5 = \alpha_3^2 e_5$ shows that A is isomorphic to $\mathcal{A}_1(\alpha)$.

Case 2: Let $\alpha_1 \neq 0$.

Case 2.1: Let $\beta_1 = 0$. Then the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_5, x_5 = e_4$ shows that A is isomorphic to an algebra with nonzero products given by (4.1). Hence A is isomorphic to $\mathcal{A}_1(\alpha)$.

Case 2.2: Let $\beta_1 \neq 0$. Take $\theta_1 = \beta_1\alpha_2 - \beta_2\alpha_1$ and $\theta_2 = \beta_1\alpha_3 - \beta_3\alpha_1$. Note that $\alpha_1 + \beta_1 \neq 0$ since $\dim(Leib(A)) = 3$.

- $\theta_1 = 0 = \theta_2$ and $\alpha_1 = \beta_1$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\theta_1 = 0 = \theta_2$ and $\alpha_1 \neq \beta_1$(taking $k = \frac{\alpha_1}{\beta_1}$) then the base change $x_1 = -\frac{i\sqrt[4]{(k-1)^4(k+1)^5}\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}}{k^2}e_1 - \frac{ik(k+1)}{\sqrt[4]{(k-1)^4(k+1)^5}}e_2, x_2 = \frac{ik(k+1)}{\sqrt[4]{(k-1)^4(k+1)^5}\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}}e_1 - \frac{ik}{\sqrt[4]{(k-1)^4(k+1)^5}}e_2, x_3 = -\beta_1(\frac{\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}(k^2+1)}{k})e_3 + (-\beta_2(\frac{\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}(k^2+1)}{k}) + \frac{1}{k} + 1)e_4 + (-\beta_3(\frac{\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}(k^2+1)}{k}) - \frac{k^2(k+1)}{\sqrt{(k-1)^4(k+1)^5}})e_5, x_4 = -\frac{\beta_1(k+1)^2\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}}{k}e_3 + (-\frac{\beta_2(k+1)^2\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}}{k} + \frac{(k+1)^2}{k^2})e_4 + (-\frac{\beta_3(k+1)^2\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}}{k} - \frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}})e_5, x_5 = -\beta_1\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}}e_3 + (1 - \beta_2\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}})e_4 + (-\beta_1\sqrt{-\frac{k^2(k+1)^2}{\sqrt{(k-1)^4(k+1)^5}}} - \frac{k^2}{\sqrt{(k-1)^4(k+1)^5}})e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.1). Hence A is isomorphic to $\mathcal{A}_1(\alpha)$.

- If $\theta_1 = 0$ and $\theta_2 \neq 0$(taking $k = \frac{\alpha_1}{\beta_1}$) then the base change $x_1 = \frac{(k+1)^{3/8}}{\sqrt{k}}e_1 - \frac{(k+1)^{3/8}\theta_2}{\sqrt{k}\beta_1}e_2, x_2 = \frac{\sqrt{k}}{(k+1)^{5/8}}e_1, x_3 = -\frac{k\theta_2}{\sqrt[4]{k+1}}e_3 + (-\frac{k\theta_2\beta_2}{\sqrt[4]{k+1}\beta_1} + \frac{1}{\sqrt[4]{k+1}})e_4 + (-\frac{k\theta_2\beta_3}{\sqrt[4]{k+1}\beta_1} - \frac{\theta_2^2}{\sqrt[4]{k+1}\beta_1^2})e_5, x_4 = -\frac{(k+1)^{7/4}\theta_2}{k}e_3 + (-\frac{(k+1)^{7/4}\theta_2\beta_2}{k\beta_1} + \frac{(k+1)^{3/4}}{k})e_4 - \frac{(k+1)^{7/4}\theta_2\beta_3}{k\beta_1}e_5, x_5 = \frac{k}{(k+1)^{5/4}}e_4 + (\frac{\theta_2}{\beta_1})^2 e_5$ isomorphic to an algebra with nonzero products given by (4.1). Hence A is isomorphic to $\mathcal{A}_1(\alpha)$.

- If $\theta_1 \neq 0$ then the base change $x_1 = \frac{\theta_1}{\alpha_1}e_1 + e_2, x_2 = \frac{\alpha_1+\beta_1}{\alpha_1}e_2, x_3 = \frac{(\alpha_1+\beta_1)\theta_1}{\alpha_1^2}(\beta_1 e_3 + \beta_2 e_4) + (\frac{(\alpha_1+\beta_1)\theta_1\beta_3}{\alpha_1^2} + \frac{\alpha_1+\beta_1}{\alpha_1})e_5, x_4 = \frac{\theta_1}{\alpha_1}(\alpha_1+\beta_1)e_3 + (\frac{\theta_1(\alpha_2+\beta_2)}{\alpha_1} + (\frac{\theta_1}{\alpha_1})^2)e_4 + (1 + \frac{\theta_1(\alpha_3+\beta_3)}{\alpha_1})e_5, x_5 = (\frac{\alpha_1+\beta_1}{\alpha_1})^2 e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.1). Hence A is isomorphic to $\mathcal{A}_1(\alpha)$.

$\square$

Remark. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_1(\alpha_1)$ and $\mathcal{A}_1(\alpha_2)$ are not isomorphic.

4.2.2 $\dim(A^3) = 1$

Let $\dim(A^2) = 3$ and $\dim(A^3) = 1$. Then by Lemma 4.0.2 and Lemma 4.0.3 we have $A^3 \subseteq Z(A) \subseteq A^2$. Note that $A^2 \neq Z(A)$ since $A^3 \neq 0$. Hence $\dim(Z(A)) = 1$ or 2.

First we consider the case $\dim(Z(A)) = 2$. Note that since $Leib(A) \subseteq A^2$ we have $1 \leq \dim(Leib(A)) \leq 3$.

Theorem 4.2.3. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3$, $\dim(A^3) = 1$, $\dim(Z(A)) = 2$ and $\dim(Leib(A)) = 1$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_1, x_3] = x_4 = -[x_3, x_1]$.

$\mathcal{A}_2$: $[x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_3] = x_4 = -[x_3, x_1]$.

$\mathcal{A}_3$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_2, x_2] = x_5, [x_1, x_3] = x_4 = -[x_3, x_1]$.

$\mathcal{A}_4$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3 = -[x_2, x_1], [x_1, x_3] = x_4 = -[x_3, x_1]$.

Proof. Using Lemma 4.0.6 we see that $A^3 \neq Leib(A)$. Also from 4.0.4 we have $Leib(A) \subseteq Z(A)$. Let $Leib(A) = \text{span}\{e_5\}$ and $A^3 = \text{span}\{e_4\}$. Then $Z(A) = \text{span}\{e_4, e_5\}$. Extend this to a basis $\{e_3, e_4, e_5\}$ of A^2. Then the nonzero products in $A = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_5, [e_1, e_2] = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_2 e_3 - \alpha_3 e_4 + \alpha_5 e_5, [e_2, e_2] = \alpha_6 e_5,$$
$$[e_1, e_3] = \beta_1 e_4 = -[e_3, e_1], [e_2, e_3] = \beta_2 e_4 = -[e_3, e_2].$$

Without loss of generality, we can assume $\beta_2 = 0$ because if $\beta_2 \neq 0$ and $\beta_1 = 0$(resp. $\beta_1 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \beta_2 e_1 - \beta_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_2 = 0$. Then $\beta_1 \neq 0$ since $A^3 \neq 0$. Hence the products in A are given by the following:

$$[e_1, e_1] = \alpha_1 e_5, [e_1, e_2] = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_2 e_3 - \alpha_3 e_4 + \alpha_5 e_5, [e_2, e_2] = \alpha_6 e_5,$$
$$[e_1, e_3] = \beta_1 e_4 = -[e_3, e_1].$$

Case 1: Let $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_2 e_3 - \alpha_3 e_4 + \alpha_5 e_5, [e_2, e_2] = \alpha_6 e_5,$$
$$[e_1, e_3] = \beta_1 e_4 = -[e_3, e_1]. \quad (4.2)$$

- If $\alpha_6 = 0$ then $\alpha_4 + \alpha_5 \neq 0$ since $\dim(A^2) = 3$. Then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, x_4 = \alpha_2\beta_1 e_4, x_5 = (\alpha_4 + \alpha_5)e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\alpha_6 \neq 0$ and $\alpha_4 + \alpha_5 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, x_4 = \alpha_2\beta_1 e_4, x_5 = \alpha_6 e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\alpha_6 \neq 0$ and $\alpha_4+\alpha_5 \neq 0$ then with the following change of basis $x_1 = \alpha_6 e_1, x_2 = (\alpha_4 + \alpha_5)e_2, x_3 = \alpha_6(\alpha_4+\alpha_5)(\alpha_2 e_3+\alpha_3 e_4+\alpha_4 e_5), x_4 = \alpha_6^2\alpha_2\beta_1(\alpha_4+\alpha_5)e_4, x_5 = \alpha_6(\alpha_4+\alpha_5)^2 e_5$ A is isomorphic to $\mathcal{A}_3$.

Case 2: Let $\alpha_1 \neq 0$. If $(\alpha_4 + \alpha_5, \alpha_6) \neq (0,0)$ then the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ (where $\alpha_1 x^2 + (\alpha_4+\alpha_5)x + \alpha_6 = 0$) shows that A is isomorphic to an algebra with nonzero products given by (4.2). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2$ or $\mathcal{A}_3$. So let $\alpha_6 = 0 = \alpha_4 + \alpha_5$. Then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, x_4 = \alpha_2\beta_1 e_4, x_5 = \alpha_1 e_5$ shows that A is isomorphic to $\mathcal{A}_4$. □

Next we consider the case $\dim(Z(A)) = 2 = \dim(Leib(A))$. It can be seen that we have either $Leib(A) \neq Z(A)$ or $Leib(A) = Z(A)$. We start with the case $Leib(A) \neq Z(A)$.

Theorem 4.2.4. *Let A be a $5-$dimensional non-split non-Lie nilpotent Leibniz algebra with* $\dim(A^2) = 3$, $\dim(A^3) = 1$, $\dim(Z(A)) = 2 = \dim(Leib(A))$ *and* $Leib(A) \neq Z(A)$. *Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_2] = x_3 + x_4, [x_2, x_1] = -x_3, [x_1, x_4] = x_5.$

$\mathcal{A}_2$**:** $[x_1, x_2] = x_3 + x_4, [x_2, x_1] = -x_3, [x_2, x_2] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_3$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_1, x_4] = x_5.$

$\mathcal{A}_4$**:** $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_4] = x_5.$

Proof. Let $A^3 = \mathrm{span}\{\mathrm{e}_5\}$. Extend this to bases of $\{e_4, e_5\}$, $\{e_3, e_5\}$ of $Leib(A)$ and $Z(A)$, respectively. Then $A^2 = \mathrm{span}\{\mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5\}$ and the nonzero products in $A = \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5,$$
$$[e_2, e_2] = \beta_3 e_4 + \beta_4 e_5, [e_1, e_4] = \beta_5 e_5, [e_2, e_4] = \beta_6 e_5.$$

From the Leibniz identities $[e_1,[e_2,e_1]] = [[e_1,e_2],e_1] + [e_2,[e_1,e_1]]$ and $[e_1,[e_2,e_2]] = [[e_1,e_2],e_2] + [e_2,[e_1,e_2]]$ we get the following equations:

$$\begin{cases} \beta_1\beta_5 = \alpha_1\beta_6 \\ \alpha_4\beta_6 = \beta_3\beta_5 \end{cases} \tag{4.3}$$

If $\beta_6 \neq 0$ and $\beta_5 = 0$ then by (4.3) we have $\alpha_1 = 0 = \alpha_4$. Then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_6 = 0$. Also if $\beta_6 \neq 0$ and $\beta_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_6 e_1 - \beta_5 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_6 = 0$. So we can assume $\beta_6 = 0$. Then $\beta_5 \neq 0$ since $A^3 \neq 0$. Using (4.3) we get $\beta_1 = 0 = \beta_3$. Hence we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2,e_1] = -\alpha_3 e_3 + \beta_2 e_5,$$
$$[e_2,e_2] = \beta_4 e_5, [e_1,e_4] = \beta_5 e_5.$$

Case 1: Let $\alpha_1 = 0$. Then $\alpha_4 \neq 0$ since $\dim(A^2) = 3$. Hence we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2,e_1] = -\alpha_3 e_3 + \beta_2 e_5,$$
$$[e_2,e_2] = \beta_4 e_5, [e_1,e_4] = \beta_5 e_5. \tag{4.4}$$

Without loss of generality we can assume $\alpha_2 = 0$. Otherwise with the base change $x_1 = \beta_5 e_1 - \alpha_2 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1,e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2,e_1] = -\alpha_3 e_3 + \beta_2 e_5, [e_2,e_2] = \beta_4 e_5, [e_1,e_4] = \beta_5 e_5.$$

- If $\beta_4 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_3 e_3 - \beta_2 e_5, x_4 = \alpha_4 e_4 + (\alpha_5 + \beta_2)e_5, x_5 = \alpha_4\beta_5 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.
- If $\beta_4 \neq 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_4\beta_5}{\beta_4}e_2, x_3 = \frac{\alpha_4\beta_5}{\beta_4}(\alpha_3 e_3 - \beta_2 e_5), x_4 = \frac{\alpha_4\beta_5}{\beta_4}(\alpha_4 e_4 + (\alpha_5 + \beta_2)e_5), x_5 = \frac{(\alpha_4\beta_5)^2}{\beta_4}e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

Case 2: Let $\alpha_1 \neq 0$.

- If $\alpha_4 = 0$ and $\beta_4 = 0$ then the base change $x_1 = e_1, x_2 = \beta_5 e_2 - (\alpha_5 + \beta_2)e_4, x_3 = \beta_5(\alpha_3 e_3 - \beta_2 e_5), x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1\beta_5 e_5$ shows that A is isomorphic to $\mathcal{A}_3$.
- If $\alpha_4 = 0$ and $\beta_4 \neq 0$ then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_1}{\beta_5}}(\beta_5 e_2 - (\alpha_5 + \beta_2)e_4), x_3 = \sqrt{\frac{\alpha_1}{\beta_5}}\beta_5(\alpha_3 e_3 - \beta_2 e_5), x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1\beta_5 e_5$ shows that A is isomorphic to $\mathcal{A}_4$.
- If $\alpha_4 \neq 0$ then the base change $x_1 = \alpha_4 e_1 - \alpha_1 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.4). Hence A is isomorphic to $\mathcal{A}_1$ or $\mathcal{A}_2$.

□

Theorem 4.2.5. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3$, $\dim(A^3) = 1$, $\dim(Z(A)) = 2 = \dim(Leib(A))$ and $Leib(A) = Z(A)$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_2$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_5, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_3$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_5, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_4$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_5$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_4, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_6$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_7(\alpha)$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4 + \alpha x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_5 = -[x_3, x_1], \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_8$: $[x_1, x_1] = x_4, [x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_1, x_3] = x_5 = -[x_3, x_1]$.

$\mathcal{A}_9$: $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_3] = x_5 = -[x_3, x_1]$.

Proof. Let $A^3 = \text{span}\{e_5\}$. Extend this to bases of $\{e_4, e_5\}$, $\{e_3, e_4, e_5\}$ of $Leib(A)$ and A^2, respectively. Then the nonzero products in $A = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by the following:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5, [e_1,e_2]=\alpha_3e_3+\alpha_4e_4+\alpha_5e_5, [e_2,e_1]=-\alpha_3e_3+\beta_1e_4+\beta_2e_5,$$
$$[e_2,e_2]=\beta_3e_4+\beta_4e_5, [e_1,e_3]=\gamma_1e_5, [e_2,e_3]=\gamma_2e_5, [e_3,e_1]=\gamma_3e_4+\gamma_4e_5,$$
$$[e_3,e_2]=\gamma_5e_4+\gamma_6e_5, [e_3,e_3]=\gamma_7e_4+\gamma_8e_5.$$

From the Leibniz identities $[e_1,[e_2,e_1]]=[[e_1,e_2],e_1]+[e_2,[e_1,e_1]], [e_1,[e_2,e_2]]=[[e_1,e_2],e_2]+[e_2,[e_1,e_2]]$ and $[e_1,[e_2,e_3]]=[[e_1,e_2],e_3]+[e_2,[e_1,e_3]]$ we get the following equations:

$$\begin{cases}\gamma_3=\gamma_5=\gamma_7=\gamma_8=0\\ \gamma_4=-\gamma_1\\ \gamma_6=-\gamma_2\end{cases}\tag{4.5}$$

Note that if $\gamma_2\neq 0$ and $\gamma_1=0$(resp. $\gamma_1\neq 0$) then with the base change $x_1=e_2, x_2=e_1, x_3=e_3, x_4=e_4, x_5=e_5$(resp. $x_1=e_1, x_2=\gamma_2e_1-\gamma_1e_2, x_3=e_3, x_4=e_4, x_5=e_5$) we can make $\gamma_2=0$. So we can assume $\gamma_2=0$. Then by (4.5) $\gamma_6=0$, and so $\gamma_1,\gamma_4\neq 0$ since $A^3\neq 0$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5, [e_1,e_2]=\alpha_3e_3+\alpha_4e_4+\alpha_5e_5, [e_2,e_1]=-\alpha_3e_3+\beta_1e_4+\beta_2e_5,$$
$$[e_2,e_2]=\beta_3e_4+\beta_4e_5, [e_1,e_3]=\gamma_1e_5=-[e_3,e_1].$$

Case 1: Let $\alpha_1=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_2e_5, [e_1,e_2]=\alpha_3e_3+\alpha_4e_4+\alpha_5e_5, [e_2,e_1]=-\alpha_3e_3+\beta_1e_4+\beta_2e_5,$$
$$[e_2,e_2]=\beta_3e_4+\beta_4e_5, [e_1,e_3]=\gamma_1e_5=-[e_3,e_1]. \tag{4.6}$$

Case 1.1: Let $\beta_3=0$. Then $\alpha_4+\beta_1\neq 0$ since $\dim(Leib(A))=2$.

- If $\beta_4=0$ then $\alpha_2\neq 0$ since $\dim(Leib(A))=2$. Then the base change $x_1=e_1, x_2=\frac{\alpha_2}{\alpha_3\gamma_1}e_2, x_3=\frac{\alpha_2}{\alpha_3\gamma_1}(\alpha_3e_3+\alpha_4e_4+\alpha_5e_5), x_4=\frac{\alpha_2}{\alpha_3\gamma_1}((\alpha_4+\beta_1)e_4+(\alpha_5+\beta_2)e_5), x_5=\alpha_2e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\beta_4 \neq 0$ and $\alpha_2 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_3\gamma_1}{\beta_4}e_2, x_3 = \frac{\alpha_3\gamma_1}{\beta_4}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = \frac{\alpha_3\gamma_1}{\beta_4}((\alpha_4+\beta_1)e_4 + (\alpha_5+\beta_2)e_5), x_5 = \frac{(\alpha_3\gamma_1)^2}{\beta_4}e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\beta_4 \neq 0$ and $\alpha_2 \neq 0$ then the base change $x_1 = \frac{\sqrt{\alpha_2\beta_4}}{\alpha_3\gamma_1}e_1, x_2 = \frac{\alpha_2}{\alpha_3\gamma_1}e_2, x_3 = \frac{\alpha_2\sqrt{\alpha_2\beta_4}}{(\alpha_3\gamma_1)^2}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = \frac{\alpha_2\sqrt{\alpha_2\beta_4}}{(\alpha_3\gamma_1)^2}((\alpha_4+\beta_1)e_4 + (\alpha_5+\beta_2)e_5), x_5 = \frac{\alpha_2^2\beta_4}{(\alpha_3\gamma_1)^2}e_5$ shows that A is isomorphic to $\mathcal{A}_3$.

Case 1.2: Let $\beta_3 \neq 0$.

Case 1.2.1: Let $\alpha_2 = 0$. Take $\theta = (\alpha_5+\beta_2)\beta_3 - (\alpha_4+\beta_1)\beta_4$. Note that $\theta \neq 0$ because otherwise $\dim(Leib(A)) = 1$, which contradicts with our claim.

- If $\alpha_4 + \beta_1 = 0$ and $\theta \neq 0$ then the base change $x_1 = \frac{\alpha_5+\beta_2}{\alpha_3\gamma_1}e_1, x_2 = e_2, x_3 = \frac{\alpha_5+\beta_2}{\alpha_3\gamma_1}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = \beta_3 e_4 + \beta_4 e_5, x_5 = \frac{(\alpha_5+\beta_2)^2}{\alpha_3\gamma_1}e_5$ shows that A is isomorphic to $\mathcal{A}_4$.

- If $\alpha_4 + \beta_1 \neq 0$ and $\theta \neq 0$ then the base change $x_1 = \frac{\theta}{\alpha_3\beta_3\gamma_1}e_1 - \frac{\theta(\alpha_4+\beta_1)}{\alpha_3\beta_3^2\gamma_1}e_2, x_2 = -\frac{\theta(\alpha_4+\beta_1)}{\alpha_3\beta_3^2\gamma_1}e_2, x_3 = -\frac{\alpha_3\theta^2(\alpha_4+\beta_1)}{\alpha_3^2\beta_3^3\gamma_1^2}e_3 + \frac{\beta_1\theta^2(\alpha_4+\beta_1)}{\alpha_3^2\beta_3^3\gamma_1^2}e_4 + \frac{(\beta_2\beta_3-\theta)\theta^2(\alpha_4+\beta_1)}{\alpha_3^2\beta_3^4\gamma_1^2}e_5, x_4 = \frac{\theta^2(\alpha_4+\beta_1)^2}{(\alpha_3\gamma_1\beta_3^2)^2}(\beta_3 e_4 + \beta_4 e_5), x_5 = -\frac{\theta^3(\alpha_4+\beta_1)}{\alpha_3^2\beta_3^4\gamma_1^2}e_5$ shows that A is isomorphic to $\mathcal{A}_7(1)$.

Case 1.2.2: Let $\alpha_2 \neq 0$.

- If $\alpha_4 + \beta_1 = 0 = \alpha_5 + \beta_2$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_2}{\alpha_3\gamma_1}e_2, x_3 = \frac{\alpha_2}{\alpha_3\gamma_1}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = (\frac{\alpha_2}{\alpha_3\gamma_1})^2(\beta_3 e_4 + \beta_4 e_5), x_5 = \alpha_2 e_5$ shows that A is isomorphic to $\mathcal{A}_5$.

- If $\alpha_4 + \beta_1 = 0$ and $\alpha_5 + \beta_2 \neq 0$ then the base change $x_1 = \frac{\alpha_5+\beta_2}{\alpha_3\gamma_1}e_1, x_2 = \frac{\alpha_2}{\alpha_3\gamma_1}e_2, x_3 = \frac{\alpha_2(\alpha_5+\beta_2)}{(\alpha_3\gamma_1)^2}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = (\frac{\alpha_2}{\alpha_3\gamma_1})^2(\beta_3 e_4 + \beta_4 e_5), x_5 = \frac{\alpha_2(\alpha_5+\beta_2)^2}{(\alpha_3\gamma_1)^2}e_5$ shows that A is isomorphic to $\mathcal{A}_6$.

- If $\alpha_4 + \beta_1 \neq 0$ then the base change $x_1 = \frac{\alpha_2\beta_3}{\alpha_3\gamma_1(\alpha_4+\beta_1)}e_1, x_2 = \frac{\alpha_2}{\alpha_3\gamma_1}e_2, x_3 = \frac{\alpha_3^2\beta_3}{(\alpha_3\gamma_1)^2(\alpha_4+\beta_1)}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = (\frac{\alpha_2}{\alpha_3\gamma_1})^2(\beta_3 e_4 + \beta_4 e_5), x_5 = \frac{\alpha_2^3\beta_3^2}{(\alpha_3\gamma_1)^2(\alpha_4+\beta_1)^2}e_5$ shows that A is isomorphic to $\mathcal{A}_7(\alpha)$.

Case 2: Let $\alpha_1 \neq 0$. If $(\beta_3, \alpha_4 + \beta_1) \neq (0,0)$ then the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\alpha_1 x^2 + (\alpha_4+\beta_1)x + \beta_3 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.6). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ or $\mathcal{A}_7(\alpha)$. So let $\beta_3 = 0 = \alpha_4 + \beta_1$.

- If $\beta_4 = 0$ then $\alpha_5 + \beta_2 \neq 0$ since $\dim(Leib(A)) = 2$. Then the base change $x_1 = \frac{\alpha_5+\beta_2}{\alpha_3\gamma_1}e_1, x_2 = e_2, x_3 = \frac{\alpha_5+\beta_2}{\alpha_3\gamma_1}(\alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5), x_4 = (\frac{\alpha_5+\beta_2}{\alpha_3\gamma_1})^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \frac{(\alpha_5+\beta_2)^2}{\alpha_3\gamma_1}e_5$ shows that A is isomorphic to $\mathcal{A}_8$.
- If $\beta_4 \neq 0$ then the base change $x_1 = e_1 - \frac{\alpha_5+\beta_2}{2\beta_4}e_2, x_2 = \frac{\alpha_3\gamma_1}{\beta_4}e_2, x_3 = \frac{\alpha_3^2\gamma_1}{\beta_4}e_3 + \frac{\alpha_3\alpha_4\gamma_1}{\beta_4}e_4 + (\frac{\alpha_3\alpha_5\gamma_1}{\beta_4} - \frac{\alpha_3(\alpha_5+\beta_2)\gamma_1}{2\beta_4})e_5, x_4 = \alpha_1 e_4 + (\alpha_2 - \frac{(\alpha_5+\beta_2)^2}{2\beta_4} + \frac{(\alpha_5+\beta_2)^2}{4\beta_4})e_5, x_5 = \frac{(\alpha_3\gamma_1)^2}{\beta_4}e_5$ shows that A is isomorphic to $\mathcal{A}_9$.

□

Remark. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_7(\alpha_1)$ and $\mathcal{A}_7(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{\alpha_1}{\alpha_1 - 1}$.

Theorem 4.2.6. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3 = \dim(Leib(A))$, $\dim(A^3) = 1$ and $\dim(Z(A)) = 2$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_2$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_2, x_2] = x_5, [x_1, x_3] = x_5.$

$\mathcal{A}_3$**:** $[x_1, x_1] = x_3, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_4$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_5$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_6$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4 + x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_7$**:** $[x_1, x_1] = x_3, [x_1, x_2] = x_4, [x_2, x_1] = x_5, [x_1, x_3] = x_5.$

$\mathcal{A}_8(\alpha)$**:** $[x_1, x_1] = x_3, [x_1, x_2] = x_4, [x_2, x_1] = \alpha x_4, [x_1, x_3] = x_5, \quad \alpha \in \mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_9(\alpha)$**:** $[x_1, x_1] = x_3, [x_1, x_2] = x_4, [x_2, x_1] = \alpha x_4, [x_2, x_2] = x_5, [x_1, x_3] = x_5, \quad \alpha \in \mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{10}$**:** $[x_1, x_2] = x_3, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_{11}$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_{12}$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_4, [x_2, x_2] = x_4, [x_1, x_3] = x_5.$

$\mathcal{A}_{13}$: $[x_1,x_2]=x_3,[x_2,x_1]=x_4+x_5,[x_2,x_2]=x_4,[x_1,x_3]=x_5.$

$\mathcal{A}_{14}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_1,x_3]=x_5.$

$\mathcal{A}_{15}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_1]=x_5,[x_1,x_3]=x_5.$

$\mathcal{A}_{16}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_2]=x_5,[x_1,x_3]=x_5.$

$\mathcal{A}_{17}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_1]=x_5,[x_2,x_2]=x_5,[x_1,x_3]=x_5.$

$\mathcal{A}_{18}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_1]=x_4,[x_1,x_3]=x_5.$

$\mathcal{A}_{19}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_1]=x_4,[x_2,x_2]=x_5,[x_1,x_3]=x_5.$

$\mathcal{A}_{20}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_1]=\alpha x_4,[x_2,x_2]=x_4,[x_1,x_3]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{21}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_3,[x_2,x_1]=\alpha x_4+x_5,[x_2,x_2]=x_4,[x_1,x_3]=x_5,\quad \alpha\in\mathbb{C}.$

Proof. Let $A^3=\text{span}\{e_5\}$. Extend this to bases $\{e_4,e_5\}$, $\{e_3,e_4,e_5\}$ of $Z(A)$ and $Leib(A)=A^2$, respectively. Then the nonzero products in $A=\text{span}\{e_1,e_2,e_3,e_4,e_5\}$ are given by:

$$[e_1,e_1]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5,[e_1,e_2]=\alpha_4e_3+\alpha_5e_4+\alpha_6e_5,[e_2,e_1]=\beta_1e_3+\beta_2e_4+\beta_3e_5,$$
$$[e_2,e_2]=\beta_4e_3+\beta_5e_4+\beta_6e_5,[e_1,e_3]=\gamma_1e_5,[e_2,e_3]=\gamma_2e_5.$$

From the Leibniz identities $[e_1,[e_2,e_1]]=[[e_1,e_2],e_1]+[e_2,[e_1,e_1]]$ and $[e_1,[e_2,e_2]]=[[e_1,e_2],e_2]+[e_2,[e_1,e_2]]$ we get the following equations:

$$\begin{cases}\beta_1\gamma_1=\alpha_1\gamma_2\\ \alpha_4\gamma_2=\beta_4\gamma_1\end{cases}\tag{4.7}$$

If $\gamma_2\neq0$ and $\gamma_1=0$(resp. $\gamma_1\neq0$) then with the base change $x_1=e_2,x_2=e_1,x_3=e_3,x_4=e_4,x_5=e_5$(resp. $x_1=e_1,x_2=\gamma_2e_1-\gamma_1e_2,x_3=e_3,x_4=e_4,x_5=e_5$) we can make $\gamma_2=0$. So we can assume $\gamma_2=0$. Then $\gamma_1\neq0$ since $A^3\neq0$. By (4.7) we have $\beta_1=0=\beta_4$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5,[e_1,e_2]=\alpha_4e_3+\alpha_5e_4+\alpha_6e_5,[e_2,e_1]=\beta_2e_4+\beta_3e_5,$$
$$[e_2,e_2]=\beta_5e_4+\beta_6e_5,[e_1,e_3]=\gamma_1e_5.$$

Case 1: Let $\alpha_4 = 0$. Then $\alpha_1 \neq 0$ since $\dim(A^2) = 3$.

Case 1.1: Let $\alpha_5 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_6 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2, e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5. \quad (4.8)$$

We can assume $\alpha_6 = 0$ because if $\alpha_6 \neq 0$ with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \alpha_6 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5, [e_2, e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5.$$

Case 1.1.1: Let $\beta_5 = 0$. Then $\beta_2 \neq 0$ since $\dim(A^2) = 3$.

- If $\beta_6 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \beta_2 e_4 + \beta_3 e_5, x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.
- If $\beta_6 \neq 0$ then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_1\gamma_1}{\beta_6}} e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \sqrt{\frac{\alpha_1\gamma_1}{\beta_6}}(\beta_2 e_4 + \beta_3 e_5), x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

Case 1.1.2: Let $\beta_5 \neq 0$. Take $\theta = \beta_3 \beta_5 - \beta_2 \beta_6$.

- If $\beta_2 = 0$ and $\theta = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \beta_5 e_4 + \beta_6 e_5, x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_3$.
- If $\beta_2 = 0$ and $\theta \neq 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_1\gamma_1}{\beta_3} e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = (\frac{\alpha_1\gamma_1}{\beta_3})^2(\beta_5 e_4 + \beta_6 e_5), x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_4$.
- If $\beta_2 \neq 0$ and $\theta = 0$ then the base change $x_1 = \beta_5 e_1, x_2 = \beta_2 e_2, x_3 = \beta_5^2(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), x_4 = \beta_2^2(\beta_5 e_4 + \beta_6 e_5), x_5 = \beta_5^3 \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_5$.
- If $\beta_2 \neq 0$ and $\theta \neq 0$ then the base change $x_1 = \frac{\beta_2\theta}{\alpha_1\beta_5^2\gamma_1} e_1, x_2 = \frac{\beta_2^2\theta}{\alpha_1\beta_5^3\gamma_1} e_2, x_3 = (\frac{\beta_2\theta}{\alpha_1\beta_5^2\gamma_1})^2(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), x_4 = (\frac{\beta_2^2\theta}{\alpha_1\beta_5^3\gamma_1})^2(\beta_5 e_4 + \beta_6 e_5), x_5 = \frac{(\beta_2\theta)^3}{\beta_5^6(\alpha_1\gamma_1)^2} e_5$ shows that A is isomorphic to $\mathcal{A}_6$.

Case 1.2: Let $\alpha_5 \neq 0$. If $\beta_5 \neq 0$ then the base change $x_1 = \beta_5 e_1 - \alpha_5 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.8). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5$ or $\mathcal{A}_6$. Then let $\beta_5 = 0$ which implies $\alpha_5 + \beta_2 \neq 0$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2, e_2] = \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5.$$

Case 1.2.1: Let $\beta_6 = 0$.

- If $\beta_2 = 0 = \beta_3$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \alpha_5 e_4 + \alpha_6 e_5, x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_8(0)$.
- If $\beta_2 = 0$ and $\beta_3 \neq 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_1 \gamma_1}{\beta_3} e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \frac{\alpha_1 \gamma_1}{\beta_3}(\alpha_5 e_4 + \alpha_6 e_5), x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_7$.
- If $\beta_2 \neq 0$ then the base change $x_1 = e_1, x_2 = e_2 + \frac{\alpha_5 \beta_3 - \alpha_6 \beta_2}{\beta_2 \gamma_1} e_3, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \alpha_5 e_4 + \frac{\alpha_5 \beta_3}{\beta_2} e_5, x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_8(\alpha) (\alpha \in \mathbb{C} \backslash \{-1, 0\})$.

Case 1.2.2: Let $\beta_6 \neq 0$. If $\beta_3 \neq 0$ then with the base change $x_1 = \beta_6 e_1 - \beta_3 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_3 = 0$. So we can assume $\beta_3 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_2 e_4, [e_2, e_2] = \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5.$$

Without loss of generality, we can assume $\alpha_6 = 0$ because if $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \alpha_6 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_5 e_4, [e_2, e_1] = \beta_2 e_4, [e_2, e_2] = \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5.$$

Then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_1 \gamma_1}{\beta_6}} e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \alpha_5 \sqrt{\frac{\alpha_1 \gamma_1}{\beta_6}} e_4, x_5 = \alpha_1 \gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_9(\alpha)$.

Case 2: Let $\alpha_4 \neq 0$. If $\alpha_1 \neq 0$ then with the base change $x_1 = \alpha_4 e_1 - \alpha_1 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_1 = 0$. Then assume $\alpha_1 = 0$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2, e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5.$$

Case 2.1: Let $\alpha_2 = 0$. If $\alpha_3 \neq 0$ then with the base change $x_1 = \gamma_1 e_1 - \alpha_3 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_3 = 0$. So we can assume $\alpha_3 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5, [e_2, e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5.$$

Note that if $\beta_5 = 0$ then $\dim(Leib(A)) = 2$ which is a contradiction. Suppose $\beta_5 \neq 0$. Take $\theta = \beta_3\beta_5 - \beta_2\beta_6$.

- If $\beta_2 = 0 = \theta$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, x_4 = \beta_5 e_4 + \beta_6 e_5, x_5 = \alpha_4\gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{10}$.
- If $\beta_2 = 0$ and $\theta \neq 0$ then the base change $x_1 = \frac{\beta_3}{\alpha_4\gamma_1} e_1, x_2 = e_2, x_3 = \frac{\beta_3}{\alpha_4\gamma_1}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \beta_5 e_4 + \beta_6 e_5, x_5 = \frac{\beta_3^2}{\alpha_4\gamma_1} e_5$ shows that A is isomorphic to $\mathcal{A}_{11}$.
- If $\beta_2 \neq 0$ and $\theta = 0$ then the base change $x_1 = \beta_5 e_1, x_2 = \beta_2 e_2, x_3 = \beta_2\beta_5(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \beta_2^2(\beta_5 e_4 + \beta_6 e_5), x_5 = \alpha_4\beta_2\beta_5^2\gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{12}$.
- If $\beta_2 \neq 0$ and $\theta \neq 0$ then the base change $x_1 = \frac{\theta}{\alpha_4\beta_5\gamma_1} e_1, x_2 = \frac{\beta_2\theta}{\alpha_4\beta_5^2\gamma_1} e_2, x_3 = \frac{\beta_2\theta^2}{\alpha_4^2\beta_5^3\gamma_1^2}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = (\frac{\beta_2\theta}{\alpha_4\beta_5^2\gamma_1})^2(\beta_5 e_4 + \beta_6 e_5), x_5 = \frac{\beta_2\theta^3}{\alpha_4^2\beta_5^4\gamma_1^2} e_5$ shows that A is isomorphic to $\mathcal{A}_{13}$.

Case 2.2: Let $\alpha_2 \neq 0$.
Case 2.2.1: Let $\beta_5 = 0$.
Case 2.2.1.1: Let $\beta_2 = 0$.

- If $\beta_3 = 0 = \beta_6$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, x_4 = \alpha_2 e_4 + \alpha_3 e_5, x_5 = \alpha_4\gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{14}$.
- If $\beta_6 = 0$ and $\beta_3 \neq 0$ then the base change $x_1 = \frac{\beta_3}{\alpha_4\gamma_1} e_1, x_2 = e_2, x_3 = \frac{\beta_3}{\alpha_4\gamma_1}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = (\frac{\beta_3}{\alpha_4\gamma_1})^2(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{\beta_3^2}{\alpha_4\gamma_1} e_5$ shows that A is isomorphic to $\mathcal{A}_{15}$.
- If $\beta_6 \neq 0$ and $\beta_3 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_4\gamma_1}{\beta_6} e_2, x_3 = \frac{\alpha_4\gamma_1}{\beta_6}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \alpha_2 e_4 + \alpha_3 e_5, x_5 = \frac{(\alpha_4\gamma_1)^2}{\beta_6} e_5$ shows that A is isomorphic to $\mathcal{A}_{16}$.
- If $\beta_6 \neq 0$ and $\beta_3 \neq 0$ then the base change $x_1 = \frac{\beta_3}{\alpha_4\gamma_1} e_1, x_2 = \frac{\beta_3^2}{\alpha_4\beta_6\gamma_1} e_2, x_3 = \frac{\beta_3^3}{\alpha_4^2\beta_6\gamma_1^2}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = (\frac{\beta_3}{\alpha_4\gamma_1})^2(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{\beta_3^4}{\alpha_4^2\beta_6\gamma_1^2} e_5$ shows that A is isomorphic to $\mathcal{A}_{17}$.

Case 2.2.1.2: Let $\beta_2 \neq 0$.

- If $\beta_6 = 0$ then the base change $x_1 = \frac{\beta_2}{\alpha_2}e_1 + \frac{\alpha_2\beta_3 - \alpha_3\beta_2}{\alpha_2\gamma_1}e_3, x_2 = e_2, x_3 = \frac{\beta_2}{\alpha_2}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \frac{\beta_2^2}{\alpha_2}e_4 + \frac{\beta_2\beta_3}{\alpha_2}e_5, x_5 = \frac{\alpha_4\beta_2^2\gamma_1}{\alpha_2^2}e_5$ shows that A is isomorphic to $\mathcal{A}_{18}$.

- If $\beta_6 \neq 0$ then the base change $x_1 = \frac{\alpha_2\beta_6}{\alpha_4\beta_2}e_1 + \frac{(\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_2\beta_6}{\alpha_4\beta_2^2\gamma_1}e_3, x_2 = \frac{\alpha_2^2\beta_6}{\alpha_4\beta_2^2}e_2, x_3 = \frac{\alpha_2^3\beta_6^2}{\alpha_4^2\beta_2^3}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \frac{\alpha_2^3\beta_6^2}{\alpha_4^2\beta_2^2}e_4 + \frac{\alpha_2^3\beta_3\beta_6^2}{\alpha_4^2\beta_2^3}e_5, x_5 = \frac{\alpha_2^4\beta_6^3}{\alpha_4^2\beta_2^4}e_5$ shows that A is isomorphic to $\mathcal{A}_{19}$.

Case 2.2.2: Let $\beta_5 \neq 0$. Take $\theta_1 = \frac{\beta_2}{(\alpha_2\beta_5)^{1/2}}, \theta_2 = \frac{\alpha_2\beta_3 - \alpha_3\beta_2}{\alpha_2\alpha_4\gamma_1}, \theta_3 = \frac{\alpha_2\beta_6 - \alpha_3\beta_5}{\alpha_4\gamma_1(\alpha_2\beta_5)^{1/2}}$. Then the base change $y_1 = e_1, y_2 = (\frac{\alpha_2}{\beta_5})^{1/2}e_2, y_3 = (\frac{\alpha_2}{\beta_5})^{1/2}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), y_4 = \alpha_2 e_4 + \alpha_3 e_5, y_5 = \alpha_4\gamma_1(\frac{\alpha_2}{\beta_5})^{1/2}e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = y_3, [y_2, y_1] = \theta_1 y_4 + \theta_2 y_5, [y_2, y_2] = y_4 + \theta_3 y_5, [y_1, y_3] = y_5.$$

Without loss of generality, we can assume $\theta_3 = 0$, because if $\theta_3 \neq 0$ then with the base change $x_1 = y_1 + \theta_3 y_3, x_2 = y_2 + y_3, x_3 = y_3 + y_5, x_4 = y_4 + \theta_3 y_5, x_5 = y_5$ we can make $\theta_3 = 0$. Then we have the following products in A:

$$[y_1, y_1] = y_4, [y_1, y_2] = y_3, [y_2, y_1] = \theta_1 y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_1, y_3] = y_5.$$

- If $\theta_2 = 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{20}(\alpha)$.

- If $\theta_2 \neq 0$ then the base change $x_1 = \theta_2 y_1, x_2 = \theta_2 y_2, x_3 = \theta_2^2 y_3, x_4 = \theta_2^2 y_4, x_5 = \theta_2^3 y_5$ shows that A is isomorphic to $\mathcal{A}_{21}(\alpha)$.

□

Remark.

1. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_8(\alpha_1)$ and $\mathcal{A}_8(\alpha_2)$ are not isomorphic.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_9(\alpha_1)$ and $\mathcal{A}_9(\alpha_2)$ are not isomorphic.

3. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{20}(\alpha_1)$ and $\mathcal{A}_{20}(\alpha_2)$ are isomorphic if and only if $\alpha_2 = -\alpha_1$.

4. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{21}(\alpha_1)$ and $\mathcal{A}_{21}(\alpha_2)$ are isomorphic if and only if $\alpha_2 = -\alpha_1$.

Now suppose $\dim(Z(A)) = 1$. If $\dim(Leib(A)) = 1$ then from Lemma 4.0.2 and Lemma 4.0.4 we have $A^3 = Z(A) = Leib(A)$. Then using Lemma 4.0.6 we see that $\dim(A) \leq 4$ which is a contradiction. Therefore $\dim(Leib(A)) = 2$ or 3.

Theorem 4.2.7. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $\dim(A^2) = 3$, $\dim(A^3) = 1 = \dim(Z(A))$ *and* $\dim(Leib(A)) = 2$. *Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$: $[x_1, x_2] = -x_3 + x_4, [x_2, x_1] = x_3, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5$.

$\mathcal{A}_2$: $[x_1, x_2] = -x_3 + x_4, [x_2, x_1] = x_3, [x_2, x_2] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5$.

$\mathcal{A}_3(\alpha)$: $[x_1, x_2] = -x_3 = [x_2, x_1], [x_2, x_2] = x_4, [x_2, x_3] = -\alpha x_5, [x_3, x_2] = (\alpha - 1)x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_4(\alpha)$: $[x_1, x_2] = -x_3 + x_4, [x_2, x_1] = x_3, [x_2, x_2] = x_4, [x_2, x_3] = -\alpha x_5, [x_3, x_2] = (\alpha - 1)x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_5$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_3, x_1] = x_5, [x_1, x_4] = x_5$.

$\mathcal{A}_6$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_5, [x_3, x_1] = x_5, [x_1, x_4] = x_5$.

$\mathcal{A}_7$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_3, x_1] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5$.

$\mathcal{A}_8$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_5, [x_3, x_1] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5$.

$\mathcal{A}_9(\alpha)$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_4, [x_3, x_1] = x_5, [x_2, x_3] = \alpha x_5, [x_3, x_2] = (1 - \alpha)x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}\backslash\{0\}$.

$\mathcal{A}_{10}$: $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5$.

$\mathcal{A}_{11}$: $[x_1, x_1] = x_4, [x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5$.

Proof. Let $A^3 = Z(A) = \text{span}\{e_5\}$. Extend this to bases $\{e_4, e_5\}$, $\{e_3, e_4, e_5\}$ of $Leib(A)$ and A^2, respectively. Then the nonzero products in $A = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by:

$$\begin{aligned}[]
[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5,\\
[e_2, e_2] = \beta_3 e_4 + \beta_4 e_5, [e_1, e_3] = \gamma_1 e_5, [e_3, e_1] = \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_5, [e_3, e_2] = \gamma_4 e_5,\\
[e_3, e_3] = \gamma_5 e_5, [e_1, e_4] = \gamma_6 e_5, [e_2, e_4] = \gamma_7 e_5, [e_3, e_4] = \gamma_8 e_5.
\end{aligned}$$

From the Leibniz identity $[e_1, [e_2, e_1]] = [[e_1, e_2], e_1] + [e_2, [e_1, e_1]]$ we get the following equation:

$$\alpha_3(\gamma_1 + \gamma_2) + \alpha_1 \gamma_7 - \beta_1 \gamma_6 = 0 \tag{4.9}$$

Furthermore, from $[e_1, [e_2, e_2]] = [[e_1, e_2], e_2] + [e_2, [e_1, e_2]]$ we get the following equation:

$$\alpha_3(\gamma_3 + \gamma_4) + \alpha_4 \gamma_7 - \beta_3 \gamma_6 = 0 \tag{4.10}$$

Note that $\alpha_3 \neq 0$ since $\dim(A^2) = 3$. From the Leibniz identities $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$ and $[e_1, [e_2, e_4]] = [[e_1, e_2], e_4] + [e_2, [e_1, e_4]]$ and using $\alpha_3 \neq 0$ we get $\gamma_5 = 0 = \gamma_8$. Without loss of generality, we can assume $\gamma_7 = 0$. This is because if $\gamma_7 \neq 0$ and $\gamma_6 = 0$(resp. $\gamma_6 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \gamma_7 e_1 - \gamma_6 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\gamma_7 = 0$. Then $\gamma_6 \neq 0$ since $\dim(Z(A)) = 1$. Hence we have the following products in A:

$$\begin{aligned}[]
[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5,\\
[e_2, e_2] = \beta_3 e_4 + \beta_4 e_5, [e_1, e_3] = \gamma_1 e_5, [e_3, e_1] = \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_5, [e_3, e_2] = \gamma_4 e_5,\\
[e_1, e_4] = \gamma_6 e_5.
\end{aligned}$$

Note that if $\gamma_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = e_2, x_3 = \gamma_6 e_3 - \gamma_1 e_4, x_4 = e_4, x_5 = e_5$ we can make $\gamma_1 = 0$. So let $\gamma_1 = 0$. Then we have the following products in A:

$$\begin{aligned}[]
[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5,\\
[e_2, e_2] = \beta_3 e_4 + \beta_4 e_5, [e_3, e_1] = \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_5, [e_3, e_2] = \gamma_4 e_5, [e_1, e_4] = \gamma_6 e_5.
\end{aligned}$$

Case 1: Let $\alpha_1 = 0$. Then the products in A are the following:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5, [e_2, e_2] = \beta_3 e_4 + \beta_4 e_5,$$
$$[e_3, e_1] = \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_5, [e_3, e_2] = \gamma_4 e_5, [e_1, e_4] = \gamma_6 e_5. \quad (4.11)$$

We can assume $\alpha_2 = 0$, because otherwise with the base change $x_1 = \gamma_6 e_1 - \alpha_2 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5, [e_2, e_2] = \beta_3 e_4 + \beta_4 e_5,$$
$$[e_3, e_1] = \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_5, [e_3, e_2] = \gamma_4 e_5, [e_1, e_4] = \gamma_6 e_5.$$

Case 1.1: Let $\gamma_2 = 0$. Then by (4.9) we get $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_2 e_5, [e_2, e_2] = \beta_3 e_4 + \beta_4 e_5,$$
$$[e_2, e_3] = \gamma_3 e_5, [e_3, e_2] = \gamma_4 e_5, [e_1, e_4] = \gamma_6 e_5. \quad (4.12)$$

Case 1.1.1: Let $\beta_3 = 0$. Then by (4.10) we get $\gamma_4 = -\gamma_3$. Also we have $\alpha_4 \neq 0$ since $\dim(A^2) = 3$. Note that $\gamma_3 \neq 0$ since $\dim(Z(A)) = 1$.

- If $\beta_4 = 0$ then the base change $x_1 = -\alpha_3\gamma_3 e_1, x_2 = \alpha_4\gamma_6 e_2, x_3 = -\alpha_3\alpha_4\gamma_3\gamma_6(-\alpha_3 e_3 + \beta_2 e_5), x_4 = -\alpha_3\alpha_4\gamma_3\gamma_6[\alpha_4 e_4 + (\alpha_5 + \beta_2)e_5], x_5 = (\alpha_3\alpha_4\gamma_3\gamma_6)^2 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.
- If $\beta_4 \neq 0$ then the base change $x_1 = -\frac{\beta_4}{\alpha_3\gamma_3}e_1, x_2 = \frac{\alpha_4\beta_4\gamma_6}{\alpha_3^2\gamma_3^2}e_2, x_3 = -\frac{\alpha_4\beta_4^2\gamma_6}{\alpha_3^3\gamma_3^3}(-\alpha_3 e_3 + \beta_2 e_5), x_4 = -\frac{\alpha_4\beta_4^2\gamma_6}{\alpha_3^3\gamma_3^3}[\alpha_4 e_4 + (\alpha_5 + \beta_2)e_5], x_5 = \frac{\alpha_4^2\beta_4^3\gamma_6^2}{\alpha_3^4\gamma_3^4}e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

Case 1.1.2: Let $\beta_3 \neq 0$. Let $\theta_1 = \frac{(\alpha_5+\beta_2)\beta_3 - \alpha_4\beta_4}{\beta_3^2\gamma_6}$ and $\theta_2 = \frac{\alpha_3\gamma_3}{\beta_3\gamma_6}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = -\alpha_3 e_3 + \beta_2 e_5, y_4 = \beta_3 e_4 + \beta_4 e_5, y_5 = \beta_3\gamma_6 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = -y_3 + \frac{\alpha_4}{\beta_3}y_4 + \theta_1 y_5, [y_2, y_1] = y_3, [y_2, y_2] = y_4, [y_2, y_3] = -\theta_2 y_5, [y_3, y_2] = (\theta_2 - 1)y_5, [y_1, y_4] = y_5.$$

If $\theta_1 \neq 0$ then with the base change $x_1 = y_1, x_2 = y_2 - \theta_1 y_4, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\theta_1 = 0$. So we can assume $\theta_1 = 0$. Hence we have the following products in A:

$$[y_1, y_2] = -y_3 + \frac{\alpha_4}{\beta_3} y_4, [y_2, y_1] = y_3, [y_2, y_2] = y_4, [y_2, y_3] = -\theta_2 y_5, [y_3, y_2] = (\theta_2 - 1) y_5, [y_1, y_4] = y_5.$$

- If $\alpha_4 = 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_3(\alpha)$.

- If $\alpha_4 \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{\alpha_4}{\beta_3} y_2, x_3 = \frac{\alpha_4}{\beta_3} y_3, x_4 = (\frac{\alpha_4}{\beta_3})^2 y_4, x_5 = (\frac{\alpha_4}{\beta_3})^2 y_5$ shows that A is isomorphic to $\mathcal{A}_4(\alpha)$.

Case 1.2: Let $\gamma_2 \neq 0$.

Case 1.2.1: Let $\beta_3 = 0$. Then by (4.10) we get $\gamma_4 = -\gamma_3$. Also $\alpha_4 + \beta_1 \neq 0$ since $\dim(A^2) = 3$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, y_4 = (\alpha_4 + \beta_1) e_4 + (\alpha_5 + \beta_2) e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = y_3, [y_2, y_1] = -y_3 + y_4, [y_2, y_2] = \beta_4 y_5, [y_3, y_1] = \alpha_3 \gamma_2 y_5, [y_2, y_3] = \alpha_3 \gamma_3 y_5 = -[y_3, y_2],$$
$$[y_1, y_4] = (\alpha_4 + \beta_1) \gamma_6 y_5.$$

Then the Leibniz identity $[y_1, [y_2, y_1]] = [[y_1, y_2], y_1] + [y_2, [y_1, y_1]]$ gives $\alpha_3 \gamma_2 = (\alpha_4 + \beta_1) \gamma_6$.

- If $\gamma_3 = 0 = \beta_4$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = (\alpha_4 + \beta_1) \gamma_6 y_5$ shows that A is isomorphic to $\mathcal{A}_5$.

- If $\gamma_3 = 0$ and $\beta_4 \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{(\alpha_4+\beta_1)\gamma_6}{\beta_4} y_2, x_3 = \frac{(\alpha_4+\beta_1)\gamma_6}{\beta_4} y_3, x_4 = \frac{(\alpha_4+\beta_1)\gamma_6}{\beta_4} y_4, x_5 = \frac{(\alpha_4+\beta_1)^2\gamma_6^2}{\beta_4} y_5$ shows that A is isomorphic to $\mathcal{A}_6$.

- If $\gamma_3 \neq 0$ and $\beta_4 = 0$ then the base change $x_1 = y_1, x_2 = \frac{(\alpha_4+\beta_1)\gamma_6}{\alpha_3\gamma_3} y_2, x_3 = \frac{(\alpha_4+\beta_1)\gamma_6}{\alpha_3\gamma_3} y_3, x_4 = \frac{(\alpha_4+\beta_1)\gamma_6}{\alpha_3\gamma_3} y_4, x_5 = \frac{(\alpha_4+\beta_1)^2\gamma_6^2}{\alpha_3\gamma_3} y_5$ shows that A is isomorphic to $\mathcal{A}_7$.

- If $\gamma_3 \neq 0$ and $\beta_4 \neq 0$ then the base change $x_1 = \frac{\beta_4}{\alpha_3\gamma_3} y_1, x_2 = \frac{(\alpha_4+\beta_1)\beta_4\gamma_6}{(\alpha_3\gamma_3)^2} y_2, x_3 = \frac{(\alpha_4+\beta_1)\beta_4^2\gamma_6}{(\alpha_3\gamma_3)^3} y_3, x_4 = \frac{(\alpha_4+\beta_1)\beta_4^2\gamma_6}{(\alpha_3\gamma_3)^3} y_4, x_5 = \frac{(\alpha_4+\beta_1)^2\beta_4^3\gamma_6^2}{(\alpha_3\gamma_3)^4} y_5$ shows that A is isomorphic to $\mathcal{A}_8$.

Case 1.2.2: Let $\beta_3 \neq 0$. Then the base change $y_1 = e_1, y_2 = e_2, y_3 = \alpha_3 e_3 + \alpha_4 e_4 +$

$\alpha_5e_5, y_4 = \beta_3e_4 + \beta_4e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = y_3, [y_2, y_1] = -y_3 + \frac{\alpha_4 + \beta_1}{\beta_3}y_4 + \frac{(\alpha_5 + \beta_2)\beta_3 - (\alpha_4 + \beta_1)\beta_4}{\beta_3}y_5, [y_2, y_2] = y_4,$$
$$[y_3, y_1] = \alpha_3\gamma_2y_5, [y_2, y_3] = \alpha_3\gamma_3y_5, [y_3, y_2] = \alpha_3\gamma_4y_5, [y_1, y_4] = \beta_3\gamma_6y_5.$$

Then the Leibniz identity $[y_1, [y_2, y_1]] = [[y_1, y_2], y_1] + [y_2, [y_1, y_1]]$ gives $\alpha_3\gamma_2 = (\alpha_4+\beta_1)\gamma_6$. This implies that $\alpha_4 + \beta_1 \neq 0$.

- If $\gamma_3 = 0$ then the Leibniz identity $[y_2, [y_1, y_2]] = [[y_2, y_1], y_2] + [y_1, [y_2, y_2]]$ gives the equation $\alpha_3\gamma_4 = \beta_3\gamma_6$, and so $\gamma_4 \neq 0$. Then the base change $x_1 = \gamma_4y_1 - \gamma_2y_2 + \frac{((\alpha_5+\beta_2)\beta_3-(\alpha_4+\beta_1)\beta_4)\gamma_2}{\beta_3^2\gamma_6}y_4, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.12). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3(\alpha)$ or $\mathcal{A}_4(\alpha)$.

- If $\gamma_3 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_9(\alpha)$.

Case 2: Let $\alpha_1 \neq 0$. First suppose $(\alpha_4 + \beta_1, \beta_3) \neq (0, 0)$. Take $x \in \mathbb{C}$ such that $\alpha_1x^2 + (\alpha_4 + \beta_1)x + \beta_3 = 0$. Then if $\gamma_3 = 0$(resp. $\gamma_3 \neq 0$) the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = -\frac{\gamma_6}{\gamma_3}e_3 + \frac{1}{x}e_4, x_4 = e_4, x_5 = e_5$) shows that A is isomorphic to an algebra with the nonzero products given by (4.11). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3(\alpha), \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8$ or $\mathcal{A}_9(\alpha)$. Now suppose $\alpha_4 + \beta_1 = 0 = \beta_3$. Then by (4.10) we have $\gamma_4 = -\gamma_3$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \alpha_3e_3 + \alpha_4e_4 + \alpha_5e_5, y_4 = \alpha_1e_4 + \alpha_2e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = y_3, [y_2, y_1] = -y_3 + (\alpha_5 + \beta_2)y_5, [y_2, y_2] = \beta_4y_5,$$
$$[y_3, y_1] = \alpha_3\gamma_2y_5, [y_2, y_3] = \alpha_3\gamma_3y_5 = -[y_3, y_2], [y_1, y_4] = \alpha_1\gamma_6y_5.$$

Note that from the Leibniz identity $[y_1, [y_2, y_1]] = [[y_1, y_2], y_1] + [y_2, [y_1, y_1]]$ we get $\gamma_2 = 0$. So $\gamma_3 \neq 0$ since $\dim(Z(A)) = 1$.

- If $\beta_4 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{10}$.

- If $\beta_4 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{11}$.

□

Remark. 1. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_3(\alpha_1)$ and $\mathcal{A}_3(\alpha_2)$ are not isomorphic.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_4(\alpha_1)$ and $\mathcal{A}_4(\alpha_2)$ are not isomorphic.

3. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_9(\alpha_1)$ and $\mathcal{A}_9(\alpha_2)$ are not isomorphic.

Theorem 4.2.8. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3 = \dim(Leib(A))$ and $\dim(A^3) = 1 = \dim(Z(A))$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1(\alpha)$: $[x_1, x_1] = x_4, [x_1, x_2] = \alpha x_4, [x_2, x_1] = x_3, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_2$: $[x_1, x_1] = x_4, [x_1, x_2] = -x_4, [x_2, x_1] = -x_3, [x_2, x_2] = x_3, [x_2, x_3] = x_5, [x_1, x_4] = x_5$.

$\mathcal{A}_3(\alpha,\beta)$: $[x_1, x_1] = x_4, [x_1, x_2] = \alpha x_4, [x_2, x_1] = \beta x_3, [x_2, x_2] = x_3, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha, \beta \in \mathbb{C}, \alpha\beta \neq 1$.

$\mathcal{A}_4(\alpha)$: $[x_1, x_2] = x_3, [x_2, x_1] = \alpha x_3, [x_2, x_2] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}\backslash\{-1, 0\}$.

$\mathcal{A}_5(\alpha,\beta)$: $[x_1, x_1] = x_4, [x_1, x_2] = x_3+\alpha x_4, [x_2, x_1] = \beta x_3, [x_2, x_2] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}, \beta \in \mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_6(\alpha,\beta,\gamma)$: $[x_1, x_1] = \alpha x_4, [x_1, x_2] = x_3 + \beta x_4, [x_2, x_1] = \gamma x_3, [x_2, x_2] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha, \beta, \gamma \in \mathbb{C}$.

$\mathcal{A}_7(\alpha,\beta)$: $[x_1, x_1] = x_3 + \alpha x_4, [x_1, x_2] = x_3 + \beta x_4, [x_2, x_1] = -x_3 + x_4, [x_2, x_2] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha, \beta \in \mathbb{C}$.

Proof. Let $A^3 = Z(A) = \text{span}\{e_5\}$. Extend this to a basis $\{e_3, e_4, e_5\}$ of $Leib(A) = A^2$. Then the nonzero products in $A = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2, e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_5, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5, [e_2, e_4] = \gamma_4 e_5.$$

From the Leibniz identity $[e_1, [e_2, e_1]] = [[e_1, e_2], e_1] + [e_2, [e_1, e_1]]$ we get the following equation:

$$\beta_1\gamma_1 + \beta_2\gamma_3 = \alpha_1\gamma_2 + \alpha_2\gamma_4 \tag{4.13}$$

Furthermore, from $[e_1,[e_2,e_2]] = [[e_1,e_2],e_2]+[e_2,[e_1,e_2]]$ we get the following equation:

$$\beta_4\gamma_1 + \beta_5\gamma_3 = \alpha_4\gamma_2 + \alpha_5\gamma_4 \tag{4.14}$$

Note that if $\gamma_4 \neq 0$ and $\gamma_3 = 0$(resp. $\gamma_3 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \gamma_4 e_1 - \gamma_3 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\gamma_4 = 0$. So we can assume $\gamma_4 = 0$. Then $\gamma_2, \gamma_3 \neq 0$ since $\dim(Z(A)) = 1$. Hence we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2,e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5, [e_1,e_3] = \gamma_1 e_5, [e_2,e_3] = \gamma_2 e_5, [e_1,e_4] = \gamma_3 e_5.$$

We can assume $\alpha_3 = 0$, because if $\alpha_3 \neq 0$ then with the base change $x_1 = \gamma_3 e_1 - \alpha_3 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_3 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2,e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5, [e_1,e_3] = \gamma_1 e_5, [e_2,e_3] = \gamma_2 e_5, [e_1,e_4] = \gamma_3 e_5.$$

If $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \alpha_6 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. So let $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2,e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2,e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5, [e_1,e_3] = \gamma_1 e_5, [e_2,e_3] = \gamma_2 e_5, [e_1,e_4] = \gamma_3 e_5.$$

Furthermore we can assume $\beta_3 = 0$, because if $\beta_3 \neq 0$ then with the base change $x_1 = \gamma_2 e_1 - \beta_3 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_3 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2,e_1] = \beta_1 e_3 + \beta_2 e_4, [e_2,e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5,$$
$$[e_1,e_3] = \gamma_1 e_5, [e_2,e_3] = \gamma_2 e_5, [e_1,e_4] = \gamma_3 e_5.$$

If $\beta_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_2 e_2 - \beta_6 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_6 = 0$. So we can assume $\beta_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4, [e_2, e_2] = \beta_4 e_3 + \beta_5 e_4,$$
$$[e_1, e_3] = \gamma_1 e_5, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1: Let $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_4, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4, [e_2, e_2] = \beta_4 e_3 + \beta_5 e_4,$$
$$[e_1, e_3] = \gamma_1 e_5, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5. \quad (4.15)$$

Note that if $\gamma_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = e_2, x_3 = \gamma_3 e_3 - \gamma_1 e_4, x_4 = e_4, x_5 = e_5$ we can make $\gamma_1 = 0$. So let $\gamma_1 = 0$. Then by (4.13) we have $\beta_2 = 0$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_4, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2, e_1] = \beta_1 e_3, [e_2, e_2] = \beta_4 e_3 + \beta_5 e_4, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1.1: Let $\alpha_4 = 0$. Then by (4.14) we have $\beta_5 = 0$. Hence the products in A are given by:

$$[e_1, e_1] = \alpha_2 e_4, [e_1, e_2] = \alpha_5 e_4, [e_2, e_1] = \beta_1 e_3, [e_2, e_2] = \beta_4 e_3, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1.1.1: Let $\beta_4 = 0$. Then $\beta_1 \neq 0$ since $\dim(A^2) = 3$. Also $\alpha_2 \neq 0$ since $\dim(Leib(A)) = 3$. Then the products in A are given by:

$$[e_1, e_1] = \alpha_2 e_4, [e_1, e_2] = \alpha_5 e_4, [e_2, e_1] = \beta_1 e_3, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5. \quad (4.16)$$

The base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_2\gamma_3}{\beta_1\gamma_2}} e_2, x_3 = \sqrt{\frac{\alpha_2\beta_1\gamma_3}{\gamma_2}} e_3, x_4 = \alpha_2 e_4, x_5 = \alpha_2\gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_1(\alpha)$.

Case 1.1.2: Let $\beta_4 \neq 0$. If $\alpha_2 = 0$ then the base change $x_1 = e_2, x_2 = e_1, x_3 = e_4, x_4 = e_3, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.16). Hence A is isomorphic to $\mathcal{A}_1(\alpha)$. Now let $\alpha_2 \neq 0$. Take $\theta = (\frac{\alpha_2\gamma_3}{\beta_4\gamma_2})^{1/3}$. The base change $y_1 = e_1, y_2 = \theta e_2, y_3 = \beta_4\theta^2 e_3, y_4 = \alpha_2 e_4, y_5 = \alpha_2\gamma_3 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = \frac{\alpha_5\theta}{\alpha_2} y_4, [y_2, y_1] = \frac{\beta_1}{\beta_4\theta} y_3, [y_2, y_2] = y_3, [y_2, y_3] = y_5, [y_1, y_4] = y_5.$$

- If $\alpha_5\beta_1 = \alpha_2\beta_4$ and $(\frac{\alpha_5\theta}{\alpha_2})^3 + 1 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2$.
- If $\alpha_5\beta_1 = \alpha_2\beta_4$ and $(\frac{\alpha_5\theta}{\alpha_2})^3 + 1 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_1(0)$.
- If $\alpha_5\beta_1 \neq \alpha_2\beta_4$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_3(\alpha, \beta)$.

Case 1.2: Let $\alpha_4 \neq 0$. Then by (4.14) we have $\beta_5 = \frac{\alpha_4\gamma_2}{\gamma_3}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \alpha_4 e_3, y_4 = \beta_5 e_4, y_5 = \beta_5\gamma_3 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \frac{\alpha_2}{\beta_5} y_4, [y_1, y_2] = y_3 + \frac{\alpha_5}{\beta_5} y_4, [y_2, y_1] = \frac{\beta_1}{\alpha_4} y_3, [y_2, y_2] = \frac{\beta_4}{\alpha_4} y_3 + y_4, [y_2, y_3] = y_5, [y_1, y_4] = y_5.$$

Note that if $\beta_4 = 0$ then $\alpha_4 + \beta_1 \neq 0$ since $\dim(Leib(A)) = 3$.

- If $\beta_4 = 0, \alpha_2 = 0, \alpha_5 = 0$ and $\frac{\beta_1}{\alpha_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2$.
- If $\beta_4 = 0, \alpha_2 = 0, \alpha_5 = 0$ and $\frac{\beta_1}{\alpha_4} \in \mathbb{C}\backslash\{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\alpha)$.
- If $\beta_4 = 0, \alpha_2 = 0$ and $\alpha_5 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_1(\alpha)$.
- If $\beta_4 = 0$ and $\alpha_2 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_5(\alpha, \beta)$.
- If $\beta_4 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_6(\alpha, \beta, \gamma)$.

Case 2: Let $\alpha_1 \neq 0$. If $(\alpha_4 + \beta_1, \beta_4) \neq (0, 0)$ then the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ (where $\alpha_1 x^2 + (\alpha_4 + \beta_1)x + \beta_4 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.15). Hence A is isomorphic to $\mathcal{A}_1(\alpha), \mathcal{A}_2, \mathcal{A}_3(\alpha, \beta), \mathcal{A}_4(\alpha), \mathcal{A}_5(\alpha, \beta)$ or $\mathcal{A}_6(\alpha, \beta, \gamma)$. Then we can assume $\alpha_4 + \beta_1 = 0 = \beta_4$. Hence we have the following products:

$$\begin{aligned}[][e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2, e_1] = -\alpha_4 e_3 + \beta_2 e_4, [e_2, e_2] = \beta_5 e_4,\\ [e_1, e_3] = \gamma_1 e_5, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.\end{aligned}$$

Note that if $\gamma_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = e_2, x_3 = \gamma_3 e_3 - \gamma_1 e_4, x_4 = e_4, x_5 = e_5$ we can make $\gamma_1 = 0$. So let $\gamma_1 = 0$. Then we have the following products:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4, [e_2, e_1] = -\alpha_4 e_3 + \beta_2 e_4, [e_2, e_2] = \beta_5 e_4,$$
$$[e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 2.1: Let $\alpha_4 = 0$. Then by (4.14) we have $\beta_5 = 0$. Hence we have the following products:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4, [e_1, e_2] = \alpha_5 e_4, [e_2, e_1] = \beta_2 e_4, [e_2, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Note that if $\alpha_2 = 0$ then $\alpha_5 + \beta_2 \neq 0$ since $\dim(Leib(A)) = 3$.

- If $\frac{\alpha_5}{\beta_2} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2$.
- If $\frac{\alpha_5}{\beta_2} \in \mathbb{C} \backslash \{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\alpha)$.

Case 2.2: Let $\alpha_4 \neq 0$. Then by (4.13) and (4.14) we get $\beta_2 = \frac{\alpha_1 \gamma_2}{\gamma_3}$ and $\beta_5 = \frac{\alpha_4 \gamma_2}{\gamma_3}$. Then w.s.c.o.b. A is isomorphic to $\mathcal{A}_7(\alpha, \beta)$. □

Remark.

1. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_1(\alpha_1)$ and $\mathcal{A}_1(\alpha_2)$ are isomorphic if and only if $\alpha_2 = -\alpha_1$.
2. If $\alpha_1, \alpha_2 \in \mathbb{C} \backslash \{-1, 0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_4(\alpha_1)$ and $\mathcal{A}_4(\alpha_2)$ are not isomorphic.
3. Isomorphism conditions for the families $\mathcal{A}_3(\alpha, \beta), \mathcal{A}_5(\alpha, \beta), \mathcal{A}_6(\alpha, \beta, \gamma)$ and $\mathcal{A}_7(\alpha, \beta)$ are hard to compute.

4.2.3 $\dim(A^3) = 2$

Let $\dim(A^2) = 3, \dim(A^3) = 2$ and $A^4 = 0$. Then by Lemma 4.0.2 and Lemma 4.0.3 we have $A^3 = Z(A)$. Assume $\dim(Leib(A)) = 3$. Let $A^3 = Z(A) = \mathrm{span}\{\mathrm{e}_4, \mathrm{e}_5\}$. Extend this to a basis $\{e_3, e_4, e_5\}$ of $Leib(A) = A^2$. Then the nonzero products in $A = \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2, e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_4 + \gamma_2 e_4, [e_2, e_3] = \gamma_3 e_4 + \gamma_4 e_5.$$

From the Leibniz identity $[e_1,[e_2,e_1]] = [[e_1,e_2],e_1] + [e_2,[e_1,e_1]]$ we get the following equations:

$$\begin{cases} \beta_1\gamma_1 = \alpha_1\gamma_3 \\ \beta_1\gamma_2 = \alpha_1\gamma_4 \end{cases} \tag{4.17}$$

The Leibniz identity $[e_1,[e_2,e_2]] = [[e_1,e_2],e_2]+[e_2,[e_1,e_2]]$ gives the following equations:

$$\begin{cases} \beta_4\gamma_1 = \alpha_4\gamma_3 \\ \beta_4\gamma_2 = \alpha_4\gamma_4 \end{cases} \tag{4.18}$$

Suppose $\gamma_3 = 0$. Then $\gamma_1, \gamma_4 \neq 0$ since $\dim(A^3) = 2$. From (4.17) and (4.18) we have $\beta_1 = 0 = \beta_4 = \alpha_1 = \alpha_4$, which is a contradiction since $\dim(A^2) = 3$. Now suppose $\gamma_3 \neq 0$. If $\beta_1 = 0 = \beta_4$ then by (4.17) and (4.18) we get $\alpha_1 = 0 = \alpha_4$, contradiction. If $\beta_1 \neq 0$(resp. $\beta_4 \neq 0$) then by (4.17) (resp. by (4.18)) we have $\gamma_3\gamma_2 - \gamma_1\gamma_4 = 0$ that contradicts with the fact that $\dim(A^3) = 2$. Hence our assumption was wrong. Therefore $\dim(Leib(A)) = 1$ or $\dim(Leib(A)) = 2$.

Theorem 4.2.9. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3, \dim(A^3) = 2, A^4 = 0$ and $\dim(Leib(A)) = 1$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1,x_2,x_3,x_4,x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$: $[x_1,x_2] = x_3, [x_2,x_1] = -x_3 + x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2]$.

$\mathcal{A}_2$: $[x_1,x_2] = x_3 = -[x_2,x_1], [x_2,x_2] = x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2]$.

$\mathcal{A}_3$: $[x_1,x_2] = x_3, [x_2,x_1] = -x_3 + x_5, [x_2,x_2] = x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2]$.

$\mathcal{A}_4$: $[x_1,x_1] = x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2]$.

Proof. Let $Leib(A) = \text{span}\{e_5\}$. Extend this to bases $\{e_4,e_5\}$ and $\{e_3,e_4,e_5\}$ of $A^3 = Z(A)$ and A^2, respectively. Then the nonzero products in $A = \text{span}\{e_1,e_2,e_3,e_4,e_5\}$ are given by:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_3+\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=-\alpha_2e_3-\alpha_3e_4+\alpha_5e_5,[e_2,e_2]=\alpha_6e_5,$$
$$[e_1,e_3]=\beta_1e_4+\beta_2e_5,[e_2,e_3]=\beta_3e_4+\beta_4e_5,[e_3,e_1]=-\beta_1e_4+\beta_5e_5,[e_3,e_2]=-\beta_3e_4+\beta_6e_5,$$
$$[e_3,e_3]=\beta_7e_5.$$

From the Leibniz identities $[e_1,[e_2,e_1]]=[[e_1,e_2],e_1]+[e_2,[e_1,e_1]],[e_1,[e_2,e_2]]=[[e_1,e_2],e_2]+[e_2,[e_1,e_2]]$ and $[e_1,[e_2,e_3]]=[[e_1,e_2],e_3]+[e_2,[e_1,e_3]]$ we get the following equations:

$$\begin{cases}\beta_5=-\beta_2\\ \beta_6=-\beta_4\\ \beta_7=0\end{cases}\tag{4.19}$$

Note that if $\beta_3\neq 0$ and $\beta_1=0$(resp. $\beta_1\neq 0$) then with the base change $x_1=e_2,x_2=e_1,x_3=e_3,x_4=e_4,x_5=e_5$(resp. $x_1=e_1,x_2=\beta_3e_1-\beta_1e_2,x_3=e_3,x_4=e_4,x_5=e_5$) we can make $\beta_3=0$. So let $\beta_3=0$. Then $\beta_1,\beta_4\neq 0$ since $\dim(A^3)=2$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_3+\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=-\alpha_2e_3-\alpha_3e_4+\alpha_5e_5,[e_2,e_2]=\alpha_6e_5,$$
$$[e_1,e_3]=\beta_1e_4+\beta_2e_5=-[e_3,e_1],[e_2,e_3]=\beta_4e_5=-[e_3,e_2].$$

Case 1: Let $\alpha_1=0$. Then we have the following products in A:

$$[e_1,e_2]=\alpha_2e_3+\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=-\alpha_2e_3-\alpha_3e_4+\alpha_5e_5,[e_2,e_2]=\alpha_6e_5,$$
$$[e_1,e_3]=\beta_1e_4+\beta_2e_5=-[e_3,e_1],[e_2,e_3]=\beta_4e_5=-[e_3,e_2].\tag{4.20}$$

- If $\alpha_6=0$ then $\alpha_4+\alpha_5\neq 0$ since $Leib(A)\neq 0$. Then the base change $x_1=e_1,x_2=\frac{\alpha_4+\alpha_5}{\alpha_2\beta_4}e_2,x_3=\frac{\alpha_4+\alpha_5}{\alpha_2\beta_4}(\alpha_2e_3+\alpha_3e_4+\alpha_4e_5),x_4=\frac{\alpha_4+\alpha_5}{\beta_4}(\beta_1e_4+\beta_2e_5),x_5=\frac{(\alpha_4+\alpha_5)^2}{\alpha_2\beta_4}e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\alpha_6\neq 0$ and $\alpha_4+\alpha_5=0$ then the base change $x_1=\frac{\alpha_6}{\alpha_2\beta_4}e_1,x_2=e_2,x_3=\frac{\alpha_6}{\alpha_2\beta_4}(\alpha_2e_3+\alpha_3e_4+\alpha_4e_5),x_4=\frac{\alpha_6^2}{\alpha_2\beta_4^2}(\beta_1e_4+\beta_2e_5),x_5=\alpha_6e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\alpha_6\neq 0$ and $\alpha_4+\alpha_5\neq 0$ then the base change $x_1=\frac{\alpha_6}{\alpha_2\beta_4}e_1,x_2=\frac{\alpha_4+\alpha_5}{\alpha_2\beta_4}e_2,x_3=$

$\frac{\alpha_6(\alpha_4+\alpha_5)}{\alpha_2^2\beta_4^2}(\alpha_2e_3+\alpha_3e_4+\alpha_4e_5), x_4 = \frac{\alpha_6^2(\alpha_4+\alpha_5)}{\alpha_2^2\beta_4^3}(\beta_1e_4+\beta_2e_5), x_5 = \frac{\alpha_6(\alpha_4+\alpha_5)^2}{\alpha_2^2\beta_4^2}e_5$ shows that A is isomorphic to $\mathcal{A}_3$.

Case 2: Let $\alpha_1 \neq 0$. If $(\alpha_4+\alpha_5, \alpha_6) \neq (0,0)$ then the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ (where $\alpha_1 x^2 + (\alpha_4+\alpha_5)x + \alpha_6 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.20). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2$ or $\mathcal{A}_3$. Now let $\alpha_4+\alpha_5 = 0 = \alpha_6$. Then the base change $x_1 = \frac{\alpha_2\beta_4}{\alpha_1}e_1, x_2 = e_2, x_3 = \frac{\alpha_2\beta_4}{\alpha_1}(\alpha_2e_3+\alpha_3e_4+\alpha_4e_5), x_4 = \frac{\alpha_2^3\beta_4^2}{\alpha_1^2}(\beta_1e_4+\beta_2e_5), x_5 = \frac{\alpha_2^2\beta_4^2}{\alpha_1}e_5$ shows that A is isomorphic to $\mathcal{A}_4$. □

Theorem 4.2.10. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3, \dim(A^3) = 2 = \dim(Leib(A))$ and $A^4 = 0$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1(\alpha)$**:** $[x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_4, [x_2,x_2] = \alpha x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}\backslash\{0\}$.

$\mathcal{A}_2(\alpha)$**:** $[x_1,x_1] = x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_4, [x_2,x_2] = \alpha x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_3(\alpha)$**:** $[x_1,x_1] = \alpha x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_4+x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}\backslash\{0\}$.

$\mathcal{A}_4(\alpha)$**:** $[x_1,x_1] = \alpha x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_4+x_5, [x_2,x_2] = x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}\backslash\{0\}$.

$\mathcal{A}_5(\alpha)$**:** $[x_1,x_1] = \alpha x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_4+x_5, [x_2,x_2] = -\frac{1}{2}x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}\backslash\{-\frac{1}{6}, 0\}$.

$\mathcal{A}_6(\alpha)$**:** $[x_1,x_1] = \alpha x_5, [x_1,x_2] = x_3 = -[x_2,x_1], [x_2,x_2] = x_4+x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}\backslash\{-\frac{4}{27}, 0\}$.

$\mathcal{A}_7(\alpha,\beta)$**:** $[x_1,x_1] = \alpha x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_5, [x_2,x_2] = x_4+\beta x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha \in \mathbb{C}\backslash\{0\}, \beta \in \mathbb{C}, 4\alpha\beta \neq 1, 8\alpha\beta^3 - 2\beta^2 + 1 \neq 0, 16\alpha\beta^3 \neq 1 + 6\beta^2 \pm \sqrt{4\beta^2+12\beta+1}, -27\alpha\beta \neq 9\beta^2 + 2\beta^4 \pm 2\sqrt{\beta^2(3+\beta^2)^3}$.

$\mathcal{A}_8(\alpha,\beta,\gamma)$**:** $[x_1,x_1] = \alpha x_5, [x_1,x_2] = x_3, [x_2,x_1] = -x_3+x_4+\beta x_5, [x_2,x_2] = x_4+\gamma x_5, [x_1,x_3] = x_4 = -[x_3,x_1], [x_2,x_3] = x_5 = -[x_3,x_2], \quad \alpha,\beta,\gamma \in \mathbb{C}$.

$\mathcal{A}_9(\alpha,\beta)$**:** $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=x_3, [x_2,x_1]=-x_3+\beta x_5, [x_2,x_2]=x_5, [x_1,x_3]=x_4=-[x_3,x_1], [x_2,x_3]=x_5=-[x_3,x_2], \quad \alpha,\beta\in\mathbb{C}.$

Proof. Assume $Leib(A)\neq Z(A)$. Using A is nilpotent and $Leib(A)\neq Z(A)$ we see that $\dim(A^3)=1$, which is a contradiction. Hence $Leib(A)=Z(A)=A^3$. Let $Leib(A)=Z(A)=A^3=\mathrm{span}\{\mathrm{e}_4,\mathrm{e}_5\}$. Extend this to a basis $\{e_3,e_4,e_5\}$ of A^2. Then the nonzero products in $A=\mathrm{span}\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4,\mathrm{e}_5\}$ are given by:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5, [e_1,e_2]=\alpha_3e_3+\alpha_4e_4+\alpha_5e_5, [e_2,e_1]=-\alpha_3e_3+\beta_1e_4+\beta_2e_5,$$
$$[e_2,e_2]=\beta_3e_4+\beta_4e_5, [e_1,e_3]=\beta_5e_4+\beta_6e_5, [e_2,e_3]=\gamma_1e_4+\gamma_2e_5,$$
$$[e_3,e_1]=\gamma_3e_4+\gamma_4e_5, [e_3,e_2]=\gamma_5e_4+\gamma_6e_5, [e_3,e_3]=\gamma_7e_4+\gamma_8e_5.$$

From the Leibniz identities $[e_1,[e_2,e_1]]=[[e_1,e_2],e_1]+[e_2,[e_1,e_1]], [e_1,[e_2,e_2]]=[[e_1,e_2],e_2]+[e_2,[e_1,e_2]]$ and $[e_1,[e_2,e_3]]=[[e_1,e_2],e_3]+[e_2,[e_1,e_3]]$ we get the following equations:

$$\begin{cases}\gamma_3=-\beta_5\\ \gamma_4=-\beta_6\\ \gamma_5=-\gamma_1\\ \gamma_6=-\gamma_2\\ \gamma_7=0=\gamma_8\end{cases}\tag{4.21}$$

Note that if $\gamma_1\neq0$ and $\beta_5=0$(resp. $\beta_5\neq0$) then with the base change $x_1=e_2,x_2=e_1,x_3=e_3,x_4=e_4,x_5=e_5$(resp. $x_1=e_1,x_2=\gamma_1e_1-\beta_5e_2,x_3=e_3,x_4=e_4,x_5=e_5$) we can make $\gamma_1=0$. So let $\gamma_1=0$. Then $\gamma_5=0$ by (4.21) and $\beta_5,\gamma_2\neq0$ since $\dim(A^3)=2$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5, [e_1,e_2]=\alpha_3e_3+\alpha_4e_4+\alpha_5e_5, [e_2,e_1]=-\alpha_3e_3+\beta_1e_4+\beta_2e_5,$$
$$[e_2,e_2]=\beta_3e_4+\beta_4e_5, [e_1,e_3]=\beta_5e_4+\beta_6e_5=-[e_3,e_1], [e_2,e_3]=\gamma_2e_5=-[e_3,e_2].$$

Case 1: Let $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_3+\alpha_4 e_4+\alpha_5 e_5, [e_2,e_1] = -\alpha_3 e_3+\beta_1 e_4+\beta_2 e_5, [e_2,e_2] = \beta_3 e_4+\beta_4 e_5,$$
$$[e_1,e_3] = \beta_5 e_4 + \beta_6 e_5 = -[e_3,e_1], [e_2,e_3] = \gamma_2 e_5 = -[e_3,e_2]. \quad (4.22)$$

Take $\theta_1 = \frac{\alpha_2}{\alpha_3\gamma_2}, \theta_2 = \frac{\alpha_4+\beta_1}{\alpha_3\beta_5}, \theta_3 = \frac{(\alpha_5+\beta_2)\beta_5-(\alpha_4+\beta_1)\beta_6}{\alpha_3\beta_5\gamma_2}, \theta_4 = \frac{\beta_3}{\alpha_3\beta_5}$ and $\theta_5 = \frac{\beta_4\beta_5-\beta_3\beta_6}{\beta_5}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, y_4 = \alpha_3(\beta_5 e_4 + \beta_6 e_5), y_5 = \alpha_3\gamma_2 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1] = \theta_1 y_5, [y_1,y_2] = y_3, [y_2,y_1] = -y_3 + \theta_2 y_4 + \theta_3 y_5, [y_2,y_2] = \theta_4 y_4 + \theta_5 y_5,$$
$$[y_1,y_3] = y_4 = -[y_3,y_1], [y_2,y_3] = y_5 = -[y_3,y_2].$$

Note that $(\theta_2,\theta_4) \neq (0,0)$ and $(\theta_1,\theta_3,\theta_5) \neq (0,0,0)$ since $\dim(Leib(A)) = 2$. Take $\theta_6 = \frac{\theta_1\sqrt{\theta_3\theta_4}}{\theta_3^2}$ and $\theta_7 = \frac{\theta_5}{\sqrt{\theta_3\theta_4}}$.

- If $\theta_4 = 0, \theta_3 = 0$ and $\theta_1 = 0$ then $\theta_2, \theta_5 \neq 0$. Then the base change $x_1 = \theta_2 y_1, x_2 = y_2, x_3 = \theta_2 y_3, x_4 = \theta_2^2 y_4, x_5 = \theta_2 y_5$ shows that A is isomorphic to $\mathcal{A}_1(\alpha)$.
- If $\theta_4 = 0, \theta_3 = 0$ and $\theta_1 \neq 0$ then $\theta_2 \neq 0$. Then the base change $x_1 = \theta_2 y_1, x_2 = \sqrt{\theta_1\theta_2}y_2, x_3 = \theta_2\sqrt{\theta_1\theta_2}y_3, x_4 = \theta_2^2\sqrt{\theta_1\theta_2}y_4, x_5 = \theta_1\theta_2^2 y_5$ shows that A is isomorphic to $\mathcal{A}_2(\alpha)$.
- If $\theta_4 = 0, \theta_3 \neq 0$ and $\frac{\theta_5}{\theta_2} = 0$ then $\theta_2 \neq 0$. Then the base change $x_1 = \theta_2 y_1, x_2 = \theta_3 y_2, x_3 = \theta_2\theta_3 y_3, x_4 = \theta_2^2\theta_3 y_4, x_5 = \theta_2\theta_3^2 y_5$ shows that A is isomorphic to $\mathcal{A}_3(\alpha)$.
- If $\theta_4 = 0, \theta_3 \neq 0$ and $\frac{\theta_5}{\theta_2} = 0$ then $\theta_1, \theta_2 \neq 0$ since $\dim(Leib(A)) = 2$. Then the base change $x_1 = \theta_2 y_1, x_2 = \theta_3 y_2, x_3 = \theta_2\theta_3 y_3, x_4 = \theta_2^2\theta_3 y_4, x_5 = \theta_2\theta_3^2 y_5$ shows that A is isomorphic to $\mathcal{A}_3(\alpha)$.
- If $\theta_4 = 0, \theta_3 \neq 0, \frac{\theta_5}{\theta_2} = 1$ and $\theta_1 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_3(-\frac{1}{4})$.
- If $\theta_4 = 0, \theta_3 \neq 0, \frac{\theta_5}{\theta_2} = 1$ and $\theta_1 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\alpha)$.
- If $\theta_4 = 0, \theta_3 \neq 0, \frac{\theta_5}{\theta_2} = -\frac{1}{2}$ and $\theta_1 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_3(2)$.
- If $\theta_4 = 0, \theta_3 \neq 0, \frac{\theta_5}{\theta_2} = -\frac{1}{2}$ and $\theta_1 = -\frac{1}{6}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\frac{1}{3})$.

- If $\theta_4 = 0, \theta_3 \neq 0, \frac{\theta_5}{\theta_2} = -\frac{1}{2}$ and $\theta_1 \in \mathbb{C}\backslash\{-\frac{1}{6}, 0\}$ then the base change $x_1 = \theta_2 y_1, x_2 = \theta_3 y_2, x_3 = \theta_2\theta_3 y_3, x_4 = \theta_2^2\theta_3 y_4, x_5 = \theta_2\theta_3^2 y_5$ shows that A is isomorphic to $\mathcal{A}_5(\alpha)$.
- If $\theta_4 = 0, \theta_3 \neq 0$ and $\frac{\theta_5}{\theta_2} \in \mathbb{C}\backslash\{-\frac{1}{2}, 0, 1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2(\alpha)(\alpha \in \mathbb{C}\backslash\{-\frac{1}{2}, \frac{1}{2}, 1\})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 = 0, \theta_5 = 0$ then $\theta_1 \neq 0$. Then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2(-\frac{1}{2})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 = 0, \theta_5 \neq 0$ and $\frac{\theta_1\theta_4^2}{\theta_5^3} = -\frac{4}{27}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\frac{1}{9})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 = 0, \theta_5 \neq 0$ and $\frac{\theta_1\theta_4^2}{\theta_5^3} \neq -\frac{4}{27}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_6(\alpha)$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 = 1$ and $\theta_6^2 = -\frac{1}{54}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\frac{1}{9})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 = 1$ and $\theta_6^2 \neq -\frac{1}{54}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_6(\alpha)$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 = 0$ and $\theta_6 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_3(2)$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 = 0, \theta_6 = 0$ and $\theta_7^2 = -\frac{3}{2} \pm \sqrt{3}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\frac{1}{3})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 = 0, \theta_6 = 0$ and $\theta_7^2 \neq -\frac{3}{2} \pm \sqrt{3}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_5(\alpha)(\alpha \in \mathbb{C}\backslash\{-\frac{1}{6}, 0, \frac{1}{8}\})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 \neq 0, \theta_6 = 0$ and $\theta_7 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2(0)$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 \neq 0, \theta_6 = 0$ and $\theta_7 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_3(\alpha)(\alpha \in \mathbb{C}\backslash\{0, 2\})$.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 \neq 0, \theta_6 \neq 0$ and $16\theta_6\theta_7^3 = 1 + 6\theta_7^2 \pm \sqrt{4\theta_7^2 + 12\theta_7 + 1}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_2(\alpha)$ for some α values.
- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 \neq 0, \theta_6 \neq 0, 16\theta_6\theta_7^3 \neq 1 + 6\theta_7^2 \pm \sqrt{4\theta_7^2 + 12\theta_7 + 1}$ and $-27\theta_6\theta_7 = 9\theta_7^2 + 2\theta_7^4 \pm 2\sqrt{\theta_7^2(3 + \theta_7^2)^3}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_4(\alpha)$ for some α values.

- If $\theta_4 \neq 0, \theta_2 = 0, \theta_3 \neq 0, 4\theta_6\theta_7 \neq 1, 8\theta_6\theta_7^3 - 2\theta_7^2 + 1 \neq 0, \theta_6 \neq 0, 16\theta_6\theta_7^3 \neq 1 + 6\theta_7^2 \pm \sqrt{4\theta_7^2 + 12\theta_7 + 1}$ and $-27\theta_6\theta_7 \neq 9\theta_7^2 + 2\theta_7^4 \pm 2\sqrt{\theta_7^2(3+\theta_7^2)^3}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_7(\alpha, \beta)$.

- If $\theta_4 \neq 0$ and $\theta_2 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_8(\alpha, \beta, \gamma)$.

Case 2: Let $\alpha_1 \neq 0$. If $(\alpha_4 + \beta_1, \beta_3) \neq (0,0)$ then the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ (where $\alpha_1 x^2 + (\alpha_4 + \beta_1)x + \beta_3 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.22). Hence A is isomorphic to $\mathcal{A}_1(\alpha), \mathcal{A}_2(\alpha), \mathcal{A}_3(\alpha), \mathcal{A}_4(\alpha), \mathcal{A}_5(\alpha), \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha, \beta)$ or $\mathcal{A}_8(\alpha, \beta, \gamma)$. So we can assume $\alpha_4 + \beta_1 = 0 = \beta_3$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 - \alpha_4 e_4 + \beta_2 e_5,$$
$$[e_2, e_2] = \beta_4 e_5, [e_1, e_3] = \beta_5 e_4 + \beta_6 e_5 = -[e_3, e_1], [e_2, e_3] = \gamma_2 e_5 = -[e_3, e_2].$$

Note that if $\beta_4 = 0$ then $\alpha_5 + \beta_2 \neq 0$ since $\dim(Leib(A))$=2.

- If $\beta_4 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_1(\alpha)$.

- If $\beta_4 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_9(\alpha, \beta)$.

□

Remark.

1. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_1(\alpha_1)$ and $\mathcal{A}_1(\alpha_2)$ are not isomorphic.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_2(\alpha_1)$ and $\mathcal{A}_2(\alpha_2)$ are not isomorphic.

3. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_3(\alpha_1)$ and $\mathcal{A}_3(\alpha_2)$ are not isomorphic.

4. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_4(\alpha_1)$ and $\mathcal{A}_4(\alpha_2)$ are not isomorphic.

5. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-\frac{1}{6}, 0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_5(\alpha_1)$ and $\mathcal{A}_5(\alpha_2)$ are not isomorphic.

6. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-\frac{4}{27}, 0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_6(\alpha_1)$ and $\mathcal{A}_6(\alpha_2)$ are not isomorphic.

7. Isomorphism conditions for the families $\mathcal{A}_7(\alpha, \beta), \mathcal{A}_8(\alpha, \beta, \gamma)$ and $\mathcal{A}_9(\alpha, \beta)$ are hard to compute.

Let $\dim(A^2) = 3, \dim(A^3) = 2$ and $\dim(A^4) = 1$. Then by Lemma 4.0.2 and using $0 \neq A^4 \subseteq A^3$ we have $A^4 = Z(A)$. Assume $\dim(Leib(A)) = 2$. Take W such that $A^2 = Leib(A) \oplus W$. If $W = Z(A)$ then $A^3 = Leib(A)$ since $Leib(A)$ is an ideal. If $W \neq Z(A)$ and $W \notin A^3$ then $A^3 = Leib(A)$. Furthermore if $W \neq Z(A)$ and $W \subseteq A^3$ then $[A, W] \subseteq [A, A^3] = A^4 = Z(A) \subseteq Leib(A)$. Then $A^3 = [A, A^2] \subseteq Leib(A)$, so $A^3 = Leib(A)$. In all cases we get $A^3 = Leib(A)$. Hence $A^3 = Leib(A)$. Let $Z(A) = A^4 = \mathrm{span}\{e_5\}$. Extend this to bases of $\{e_4, e_5\}$ and $\{e_3, e_4, e_5\}$ of $Leib(A) = A^3$ and A^2, respectively. Then the nonzero products in $A = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5, [e_2, e_1] = -\alpha_3 e_3 + \beta_1 e_4 + \beta_2 e_5,$$
$$[e_2, e_2] = \beta_3 e_4 + \beta_4 e_5, [e_1, e_3] = \beta_5 e_4 + \beta_6 e_5, [e_3, e_1] = \gamma_1 e_4 + \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_4 + \gamma_4 e_5,$$
$$[e_3, e_2] = \gamma_5 e_4 + \gamma_6 e_5, [e_3, e_3] = \theta_1 e_4 + \theta_2 e_5, [e_1, e_4] = \theta_3 e_5, [e_2, e_4] = \theta_4 e_5, [e_3, e_4] = \theta_5 e_5.$$

Leibniz identities give the following equations:

$$\begin{cases} \gamma_1 = -\beta_5 \\ \alpha_3(\beta_6 + \gamma_2) + \alpha_1\theta_4 - \beta_1\theta_3 = 0 \\ \gamma_5 = -\gamma_3 \\ \alpha_3(\gamma_4 + \gamma_6) + \alpha_4\theta_4 - \beta_3\theta_3 = 0 \\ \theta_1 = 0 \\ \alpha_3\theta_2 + \beta_5\theta_4 - \gamma_3\theta_3 \\ \theta_5 = 0 \\ \gamma_1\theta_3 = 0 \\ \gamma_5\theta_3 = \alpha_3\theta_2 \\ \gamma_1\theta_4 = -\alpha_3\theta_2 \\ \gamma_5\theta_4 = 0 \end{cases} \tag{4.23}$$

Suppose $\theta_4 = 0$. Then $\theta_3 \neq 0$ since $\dim(Z(A)) = 1$. By (4.23) we have $\theta_2 = 0 = \gamma_5 = \gamma_3 = \gamma_1 = \beta_5$. Then $\dim(A^3) = 1$ which is a contradiction. Now suppose $\theta \neq 0$. If $\theta_3 = 0$ then by (4.23) we have $\theta_2 = 0 = \gamma_1 = \gamma_5 = \gamma_3 = \beta_5$. This implies that $\dim(A^3) = 1$, contradiction. If $\theta_3 \neq 0$ then by (4.23) we have $\gamma_1 = 0 = \gamma_5 = \gamma_3 = \beta_5$. Then again we get $\dim(A^3) = 1$, contradiction. Hence our assumption was wrong. Then $\dim(Leib(A)) = 1$

or $\dim(Leib(A)) = 3$.

Theorem 4.2.11. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 3, \dim(A^3) = 2$ and $\dim(A^4) = 1 = \dim(Leib(A))$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_1, x_3] = x_4 = -[x_3, x_1], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_2$: $[x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_3] = x_4 = -[x_3, x_1], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_3$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_2, x_2] = x_5, [x_1, x_3] = x_4 = -[x_3, x_1], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_4$: $[x_1, x_2] = x_3 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_3] = x_4 = -[x_3, x_1], [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_5(\alpha)$: $[x_1, x_2] = x_3, [x_2, x_1] = -x_3 + x_5, [x_2, x_2] = \alpha x_5, [x_1, x_3] = x_4 = -[x_3, x_1], [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5 = -[x_4, x_1], \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_6$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3 = -[x_2, x_1], [x_1, x_3] = x_4 = -[x_3, x_1], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_7$: $[x_1, x_1] = x_5, [x_1, x_2] = x_3 = -[x_2, x_1], [x_1, x_3] = x_4 = -[x_3, x_1], [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5 = -[x_4, x_1]$.

Proof. By Lemma 4.0.4, $Leib(A) = Z(A)$. Let $A^4 = Z(A) = Leib(A) = \mathrm{span}\{\mathrm{e}_5\}$. Extend this to bases $\{e_4, e_5\}$ and $\{e_3, e_4, e_5\}$ of A^3 and A^2, respectively. Then the nonzero products in $A = \mathrm{span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5\}$ are given by:

$$
\begin{aligned}
&[e_1, e_1] = \alpha_1 e_5, [e_1, e_2] = \alpha_2 e_3 + \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_2 e_3 - \alpha_3 e_4 + \alpha_5 e_5, [e_2, e_2] = \alpha_6 e_5,\\
&[e_1, e_3] = \beta_1 e_4 + \beta_2 e_5, [e_3, e_1] = -\beta_1 e_4 + \beta_3 e_5, [e_2, e_3] = \beta_4 e_4 + \beta_5 e_5, [e_3, e_2] = -\beta_4 e_4 + \beta_6 e_5,\\
&[e_3, e_3] = \gamma_1 e_5, [e_1, e_4] = \gamma_2 e_5, [e_4, e_1] = \gamma_3 e_5, [e_2, e_4] = \gamma_4 e_5, [e_4, e_2] = \gamma_5 e_5,\\
&[e_3, e_4] = \gamma_6 e_5, [e_4, e_3] = \gamma_7 e_5, [e_4, e_4] = \gamma_8 e_5.
\end{aligned}
$$

Leibniz identities give the following equations:

$$\begin{cases} \alpha_2(\beta_2+\beta_3)+\alpha_3(\gamma_2+\gamma_3)=0 \\ \alpha_2(\beta_5+\beta_6)+\alpha_3(\gamma_4+\gamma_5)=0 \\ \beta_4\gamma_2=\alpha_2\gamma_1+\alpha_3\gamma_7+\beta_1\gamma_4 \\ \alpha_2\gamma_6+\alpha_3\gamma_8=0 \\ \beta_1(\gamma_2+\gamma_3)=0 \\ \beta_1\gamma_5+\beta_4\gamma_2+\alpha_2\gamma_1+\alpha_3\gamma_6=0 \\ \beta_1(\gamma_6+\gamma_7)=0 \\ \beta_1\gamma_8=0 \\ \beta_1\gamma_4+\beta_4\gamma_3-\alpha_2\gamma_1-\alpha_3\gamma_6=0 \\ \beta_4(\gamma_4+\gamma_5)=0 \\ \beta_4(\gamma_6+\gamma_7)=0 \\ \beta_4\gamma_8=0 \end{cases} \tag{4.24}$$

Note that $\beta_1 \neq 0$ or $\beta_4 \neq 0$ since $\dim(A^3)=2$. Then by (4.24) we have $\gamma_6 = 0 = \gamma_7 = \gamma_8$. Note that if $\beta_4 \neq 0$ and $\beta_1 = 0$(resp. $\beta_1 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \beta_3 e_1 - \beta_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_4 = 0$. So let $\beta_4 = 0$. Then $\beta_1 \neq 0$. From (4.24) we have $\beta_3 = -\beta_2$ and $\gamma_3 = -\gamma_2$. Suppose $\gamma_5 \neq 0$. Then by (4.24) we get $\beta_1 = 0$, contradiction. Hence $\gamma_5 = 0$. Then by (4.24) we have $\gamma_1 = 0 = \gamma_4$ and $\beta_6 = -\beta_5$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1 e_5, [e_1,e_2]=\alpha_2 e_3+\alpha_3 e_4+\alpha_4 e_5, [e_2,e_1]=-\alpha_2 e_3-\alpha_3 e_4+\alpha_5 e_5, [e_2,e_2]=\alpha_6 e_5,$$
$$[e_1,e_3]=\beta_1 e_4+\beta_2 e_5=-[e_3,e_1], [e_2,e_3]=\beta_5 e_5=-[e_3,e_2], [e_1,e_4]=\gamma_2 e_5=-[e_4,e_1].$$

Case 1: Let $\alpha_1 = 0$. Then we have the following products in A:

$$\begin{aligned} &[e_1,e_2]=\alpha_2 e_3+\alpha_3 e_4+\alpha_4 e_5, [e_2,e_1]=-\alpha_2 e_3-\alpha_3 e_4+\alpha_5 e_5, [e_2,e_2]=\alpha_6 e_5, \\ &[e_1,e_3]=\beta_1 e_4+\beta_2 e_5=-[e_3,e_1], [e_2,e_3]=\beta_5 e_5=-[e_3,e_2], [e_1,e_4]=\gamma_2 e_5=-[e_4,e_1]. \end{aligned} \tag{4.25}$$

- If $\beta_5 = 0$ and $\alpha_6 = 0$ then $\alpha_4 + \alpha_5 \neq 0$ since $\dim(Leib(A)) = 1$. The base change $x_1 = (\frac{\alpha_4+\alpha_5}{\alpha_2\beta_1\gamma_2})^{1/2}e_1, x_2 = e_2, x_3 = (\frac{\alpha_4+\alpha_5}{\alpha_2\beta_1\gamma_2})^{1/2}(\alpha_2e_3 + \alpha_3e_4 + \alpha_4e_5), x_4 = \frac{\alpha_4+\alpha_5}{\beta_1\gamma_2}(\beta_1e_4 + \beta_2e_5), x_5 = \frac{(\alpha_4+\alpha_5)^{3/2}}{(\alpha_2\beta_1\gamma_2)^{1/2}}e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\beta_5 = 0, \alpha_6 \neq 0$ and $\alpha_4 + \alpha_5 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_2\beta_1\gamma_2}{\alpha_6}e_2, x_3 = \frac{\alpha_2\beta_1\gamma_2}{\alpha_6}(\alpha_2e_3 + \alpha_3e_4 + \alpha_4e_5), x_4 = \frac{\alpha_2^2\beta_1\gamma_2}{\alpha_6}(\beta_1e_4 + \beta_2e_5), x_5 = \frac{(\alpha_2\beta_1\gamma_2)^2}{\alpha_6}e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\beta_5 = 0, \alpha_6 \neq 0$ and $\alpha_4 + \alpha_5 \neq 0$ then the base change $x_1 = (\frac{\alpha_4+\alpha_5}{\alpha_2\beta_1\gamma_2})^{1/2}e_1, x_2 = \frac{(\alpha_4+\alpha_5)^{3/2}}{(\alpha_2\beta_1\gamma_2)^{1/2}\alpha_6}e_2, x_3 = \frac{(\alpha_4+\alpha_5)^2}{\alpha_2\beta_1\gamma_2\alpha_6}(\alpha_2e_3 + \alpha_3e_4 + \alpha_4e_5), x_4 = \frac{(\alpha_4+\alpha_5)^{5/2}\alpha_2}{(\alpha_2\beta_1\gamma_2)^{3/2}\alpha_6}(\beta_1e_4 + \beta_2e_5), x_5 = \frac{(\alpha_4+\alpha_5)^3}{\alpha_2\beta_1\gamma_2\alpha_6}e_5$ shows that A is isomorphic to $\mathcal{A}_3$.

- If $\beta_5 \neq 0$ and $\alpha_4 + \alpha_5 = 0$ then $\alpha_6 \neq 0$ since $\dim(Leib(A)) = 1$. Then the base change $x_1 = \frac{\alpha_6}{\alpha_2\beta_5}e_1, x_2 = \frac{\alpha_6^2\beta_1\gamma_2}{\alpha_2^2\beta_5^3}e_2, x_3 = \frac{\alpha_6^3\beta_1\gamma_2}{\alpha_2^3\beta_5^4}(\alpha_2e_3+\alpha_3e_4+\alpha_4e_5), x_4 = \frac{\alpha_6^4\beta_1\gamma_2}{\alpha_2^3\beta_5^5}(\beta_1e_4+\beta_2e_5), x_5 = \frac{\alpha_6^5\beta_1^2\gamma_2^2}{\alpha_2^4\beta_5^6}e_5$ shows that A is isomorphic to $\mathcal{A}_4$.

- If $\beta_5 \neq 0$ and $\alpha_4 + \alpha_5 \neq 0$ then the base change $x_1 = (\frac{\alpha_4+\alpha_5}{\alpha_2\beta_1\gamma_2})^{1/2}e_1, x_2 = \frac{\alpha_4+\alpha_5}{\alpha_2\beta_5}e_2, x_3 = \frac{(\alpha_4+\alpha_5)^{3/2}}{(\alpha_2\beta_1\gamma_2)^{1/2}\alpha_2\beta_5}(\alpha_2e_3 + \alpha_3e_4 + \alpha_4e_5), x_4 = \frac{(\alpha_4+\alpha_5)^2}{\alpha_2\beta_1\gamma_2\beta_5}(\beta_1e_4 + \beta_2e_5), x_5 = \frac{(\alpha_4+\alpha_5)^{5/2}}{(\alpha_2\beta_1\gamma_2)^{1/2}\alpha_2\beta_5}e_5$ shows that A is isomorphic to $\mathcal{A}_5(\alpha)$.

Case 2: Let $\alpha_1 \neq 0$. If $(\alpha_4 + \alpha_5, \alpha_6) \neq (0,0)$ then the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ (where $\alpha_1x^2 + (\alpha_4 + \alpha_5)x + \alpha_6 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.25). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ or $\mathcal{A}_5(\alpha)$. Then assume $\alpha_4 + \alpha_5 = 0 = \alpha_6$.

- If $\beta_5 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_1}{\alpha_2\beta_1\gamma_2}e_2, x_3 = \frac{\alpha_1}{\alpha_2\beta_1\gamma_2}(\alpha_2e_3 + \alpha_3e_4 + \alpha_4e_5), x_4 = \frac{\alpha_1}{\beta_1\gamma_2}(\beta_1e_4 + \beta_2e_5), x_5 = \alpha_1e_5$ shows that A is isomorphic to $\mathcal{A}_6$.

- If $\beta_5 \neq 0$ then the base change $x_1 = \frac{\alpha_1^{1/3}\beta_5^{1/3}}{\alpha_2^{1/3}\beta_1^{2/3}\gamma_2^{2/3}}e_1, x_2 = \frac{\alpha_1^{2/3}}{\alpha_2^{2/3}\beta_1^{1/3}\gamma_2^{1/3}\beta_5^{1/3}}e_2, x_3 = \frac{\alpha_1}{\alpha_2\beta_1\gamma_2}(\alpha_2e_3+\alpha_3e_4 + \alpha_4e_5), x_4 = \frac{\alpha_1^{4/3}\beta_5^{1/3}}{\alpha_2^{1/3}\beta_1^{5/3}\gamma_2^{5/3}}(\beta_1e_4 + \beta_2e_5), x_5 = \frac{\alpha_1^{5/3}\beta_5^{2/3}}{\alpha_2^{2/3}\beta_1^{4/3}\gamma_2^{4/3}}e_5$ shows that A is isomorphic to $\mathcal{A}_7$.

□

Remark. If $\alpha_1, \alpha_2 \in \mathbb{C}$ then $\mathcal{A}_5(\alpha_1)$ and $\mathcal{A}_5(\alpha_2)$ are isomorphic if and only if $\alpha_2^4 = \alpha_1^4$.

Theorem 4.2.12. *Let A be a $5-$dimensional non-split non-Lie nilpotent Leibniz algebra with* $\dim(A^2) = 3 = \dim(Leib(A)), \dim(A^3) = 2$ *and* $\dim(A^4) = 1$*. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

$\mathcal{A}_2$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_5, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

$\mathcal{A}_3$**:** $[x_1, x_2] = x_3, [x_2, x_2] = x_5, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

$\mathcal{A}_4$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_5, [x_2, x_2] = x_5, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

$\mathcal{A}_5$**:** $[x_1, x_2] = x_3, [x_2, x_2] = x_4, [x_1, x_3] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_6$**:** $[x_1, x_2] = x_3, [x_2, x_1] = x_5, [x_2, x_2] = x_4, [x_1, x_3] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_7(\alpha)$**:** $[x_1, x_2] = x_3, [x_2, x_1] = \alpha x_5, [x_2, x_2] = x_4 + x_5, [x_1, x_3] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}.$

$\mathcal{A}_8$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_5, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

$\mathcal{A}_9$**:** $[x_1, x_1] = x_3, [x_2, x_2] = x_5, [x_1, x_3] = x_4, [x_1, x_4] = x_5.$

$\mathcal{A}_{10}(\alpha)$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4, [x_2, x_2] = \alpha x_5, [x_1, x_3] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}.$

$\mathcal{A}_{11}$**:** $[x_1, x_1] = x_3, [x_2, x_1] = x_4 + x_5, [x_2, x_2] = 2x_5, [x_1, x_3] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5.$

Proof. Let $A^4 = Z(A) = \mathrm{span}\{e_5\}$. Extend this to bases $\{e_4, e_5\}$ and $\{e_3, e_4, e_5\}$ of A^3 and A^2, respectively. Then the nonzero products in $A = \mathrm{span}\{e_1, e_2, e_3, e_4, e_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_5 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_1 e_3 + \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2, e_2] = \beta_4 e_3 + \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_2, e_3] = \gamma_3 e_4 + \gamma_4 e_5,$$
$$[e_1, e_4] = \gamma_5 e_5, [e_2, e_4] = \gamma_6 e_5.$$

Leibniz identities give the following equations:

$$\begin{cases} \beta_1\gamma_1 = \alpha_1\gamma_3 \\ \beta_1\gamma_2 + \beta_2\gamma_5 = \alpha_1\gamma_4 + \alpha_2\gamma_6 \\ \beta_4\gamma_1 = \alpha_4\gamma_3 \\ \beta_4\gamma_2 + \beta_5\gamma_5 = \alpha_4\gamma_4 + \alpha_5\gamma_6 \\ \gamma_3\gamma_5 = \gamma_1\gamma_6 \end{cases} \tag{4.26}$$

We can assume $\gamma_3 = 0$, because if $\gamma_3 \neq 0$ and $\gamma_1 = 0$(resp. $\gamma_1 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \gamma_3 e_1 - \gamma_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\gamma_3 = 0$. Then $\gamma_1 \neq 0$ since $\dim(A^3) = 2$. So by (4.26) we have $\beta_1 = 0 = \beta_4 = \gamma_6$. Then $\gamma_5 \neq 0$ since $\dim(Z(A)) = 1$. Hence we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_5 + \alpha_2 e_4 + \alpha_3 e_5, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_1] = \beta_2 e_4 + \beta_3 e_5,$$
$$[e_2,e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1,e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_2,e_3] = \gamma_4 e_5, [e_1,e_4] = \gamma_5 e_5.$$

Case 1: Let $\alpha_1 = 0$. Then $\alpha_4 \neq 0$ since $\dim(A^2) = 3$. Then from (4.26) we get $\beta_2 = 0$. Hence we have the following products in A:

$$\begin{aligned} &[e_1,e_1] = \alpha_2 e_4 + \alpha_3 e_5, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_1] = \beta_3 e_5, \\ &[e_2,e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1,e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_2,e_3] = \gamma_4 e_5, [e_1,e_4] = \gamma_5 e_5. \end{aligned} \tag{4.27}$$

If $\alpha_2 \neq 0$ then with the base change $x_1 = \gamma_1 e_1 - \alpha_2 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. So let $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_3 e_5, [e_1,e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_1] = \beta_3 e_5,$$
$$[e_2,e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1,e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_2,e_3] = \gamma_4 e_5, [e_1,e_4] = \gamma_5 e_5.$$

Also if $\alpha_3 \neq 0$ then with the base change $x_1 = \gamma_5 e_1 - \gamma_3 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$

we can make $\alpha_3 = 0$. So we can assume $\alpha_3 = 0$. Then the products in A are the following:

$$[e_1, e_2] = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_1] = \beta_3 e_5, [e_2, e_2] = \beta_5 e_4 + \beta_6 e_5, [e_1, e_3] = \gamma_1 e_4 + \gamma_2 e_5,$$
$$[e_2, e_3] = \gamma_4 e_5, [e_1, e_4] = \gamma_5 e_5.$$

Case 1.1: Let $\gamma_4 = 0$. Then by (4.26) we have $\beta_5 = 0$.

- If $\beta_6 = 0 = \beta_3$ then $x_1 = e_1, x_2 = e_2, x_3 = \alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5, x_4 = \alpha_4(\gamma_1 e_4 + \gamma_2 e_5), x_5 = \alpha_4 \gamma_1 \gamma_5 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.
- If $\beta_6 = 0$ and $\beta_3 \neq 0$ then the base change $x_1 = \sqrt{\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5}} e_1, x_2 = e_2, x_3 = \sqrt{\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5}} (\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \frac{\beta_3}{\gamma_1 \gamma_5}(\gamma_1 e_4 + \gamma_2 e_5), x_5 = \beta_3 \sqrt{\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5}} e_5$ shows that A is isomorphic to $\mathcal{A}_2$.
- If $\beta_6 \neq 0$ and $\beta_3 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_4 \gamma_1 \gamma_5}{\beta_6} e_2, x_3 = \frac{\alpha_4 \gamma_1 \gamma_5}{\beta_6}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \frac{\alpha_4^2 \gamma_1 \gamma_5}{\beta_6}(\gamma_1 e_4 + \gamma_2 e_5), x_5 = \frac{(\alpha_4 \gamma_1 \gamma_5)^2}{\beta_6} e_5$ shows that A is isomorphic to $\mathcal{A}_3$.
- If $\beta_6 \neq 0$ and $\beta_3 \neq 0$ then the base change $x_1 = (\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5})^{1/2} e_1, x_2 = \frac{(\beta_3)^{3/2}}{\beta_6 (\alpha_4 \gamma_1 \gamma_5)^{1/2}} e_2, x_3 = \frac{\beta_3^2}{\alpha_4 \beta_6 \gamma_1 \gamma_5}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), x_4 = \frac{\beta_3^{5/2}}{\alpha_4^{1/2} \beta_6 \gamma_1^{3/2} \gamma_5^{3/2}}(\gamma_1 e_4 + \gamma_2 e_5), x_5 = \frac{\beta_3^3}{\alpha_4 \beta_6 \gamma_1 \gamma_5}$ shows that A is isomorphic to $\mathcal{A}_4$.

Case 1.2: Let $\gamma_4 \neq 0$. Then $\beta_5 = \frac{\alpha_4 \gamma_4}{\gamma_5}$ from (4.26). Take $\theta = \frac{\alpha_4^2 \gamma_1 (\beta_6 \gamma_1 - \beta_5 \gamma_2)}{\beta_5^3 \gamma_5}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_4 \gamma_1}{\beta_5} e_2, y_3 = \frac{\alpha_4 \gamma_1}{\beta_5}(\alpha_4 e_3 + \alpha_5 e_4 + \alpha_6 e_5), y_4 = \frac{\alpha_4^2 \gamma_1}{\beta_5}(\gamma_1 e_4 + \gamma_2 e_5), y_5 = \frac{\alpha_4^2 \gamma_1^2 \gamma_5}{\beta_5} e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = y_3, [y_2, y_1] = \frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5} y_5, [y_2, y_2] = y_4 + \theta y_5, [y_1, y_3] = y_4, [y_2, y_3] = y_5, [y_1, y_4] = y_5.$$

- If $\theta = 0$ and $\beta_3 = 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_5$.
- If $\theta = 0$ and $\beta_3 \neq 0$ then the base change $x_1 = \sqrt{\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5}} y_1, x_2 = \frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5} y_2, x_3 = (\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5})^{3/2} y_3, (\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5})^2 y_4, x_5 = (\frac{\beta_3}{\alpha_4 \gamma_1 \gamma_5})^{5/2} y_5$ shows that A is isomorphic to $\mathcal{A}_6$.
- If $\theta \neq 0$ then the base change $x_1 = \theta y_1, x_2 = \theta^2 y_2, x_3 = \theta^3 y_3, x_4 = \theta^4 y_4, x_5 = \theta^5 y_5$ shows that A is isomorphic to $\mathcal{A}_7(\alpha)$.

Case 2: Let $\alpha_1 \neq 0$. If $\alpha_4 \neq 0$ the base change $x_1 = \alpha_4 e_1 - \alpha_1 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.27). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ or $\mathcal{A}_7(\alpha)$. So let $\alpha_4 = 0$. Then from (4.26) we have $\beta_5 = 0$. Note that here if $\alpha_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \alpha_5 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Then we can assume $\alpha_5 = 0$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_1, e_2] = \alpha_6 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5, [e_2, e_2] = \beta_6 e_5,$$
$$[e_1, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_2, e_3] = \gamma_4 e_5, [e_1, e_4] = \gamma_5 e_5.$$

If $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_5 e_2 - \alpha_6 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we can assume $\alpha_6 = 0$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = \beta_2 e_4 + \beta_3 e_5, [e_2, e_2] = \beta_6 e_5, [e_1, e_3] = \gamma_1 e_4 + \gamma_2 e_5,$$
$$[e_2, e_3] = \gamma_4 e_5, [e_1, e_4] = \gamma_5 e_5.$$

Case 2.1: Let $\gamma_4 = 0$. Then by (4.26) we have $\beta_2 = 0$. Note that if $\beta_3 = 0 = \beta_6$ then A is split. So let $(\beta_3, \beta_6) \neq (0, 0)$. If $\beta_6 = 0$ then $\beta_3 \neq 0$. Then the base change $x_1 = e_1, x_2 = \frac{\alpha_1\gamma_1\gamma_5}{\beta_3} e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \alpha_1(\gamma_1 e_4 + \gamma_2 e_5), x_5 = \alpha_1\gamma_1\gamma_5 e_5$ shows that A is isomorphic to $\mathcal{A}_8$. Now suppose $\beta_6 \neq 0$. Without loss of generality we can assume $\beta_3 = 0$ because if $\beta_3 \neq 0$ then with the base change $x_1 = \beta_6 e_1 - \beta_3 e_2, x_2 = e_2 + \frac{\beta_3}{\gamma_5} e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_3 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_2] = \beta_6 e_5, [e_1, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_1, e_4] = \gamma_5 e_5.$$

Then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_1\gamma_1\gamma_5}{\beta_6}} e_2, x_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, x_4 = \alpha_1(\gamma_1 e_4 + \gamma_2 e_5), x_5 = \alpha_1\gamma_1\gamma_5 e_5$ shows that A is isomorphic to $\mathcal{A}_9$.

Case 2.2: Let $\gamma_4 \neq 0$. Then $\beta_2 = \frac{\alpha_1\gamma_4}{\gamma_5}$ from (4.26). Take $\theta = \frac{\beta_3\gamma_1 - \beta_2\gamma_2}{\beta_2\gamma_1\gamma_5}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_1\gamma_1}{\beta_2} e_2, y_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, y_4 = \alpha_1(\gamma_1 e_4 + \gamma_2 e_5), y_5 = \alpha_1\gamma_1\gamma_5 e_5$ shows that

A is isomorphic to the following algebra:

$$[y_1, y_1] = y_3, [y_2, y_1] = y_4 + \theta y_5, [y_2, y_2] = \frac{\alpha_1\beta_6\gamma_1}{\beta_2^2\gamma_5} y_5, [y_1, y_3] = y_4, [y_2, y_3] = y_5, [y_1, y_4] = y_5.$$

- If $\theta = 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{10}(\alpha)$.
- If $\theta \neq 0$ and $\frac{\alpha_1\beta_6\gamma_1}{\beta_2^2\gamma_5} \neq 2$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{10}(\alpha)$ again.
- If $\theta \neq 0$ and $\frac{\alpha_1\beta_6\gamma_1}{\beta_2^2\gamma_5} = 2$ then the base change $x_1 = \theta y_1, x_2 = \theta^2 y_2, x_3 = \theta^2 y_3, x_4 = \theta^3 y_4, x_6 = \theta^4 y_5$ shows that A is isomorphic to $\mathcal{A}_{11}$.

□

Remark. 1. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_7(\alpha_1)$ and $\mathcal{A}_7(\alpha_2)$ are not isomorphic.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{10}(\alpha_1)$ and $\mathcal{A}_{10}(\alpha_2)$ are not isomorphic.

Note that we classified 5−dimensional filiform Leibniz algebras in Theorem 4.2.11 and Theorem 4.2.12. We compare our classification with the classification given in [27]. They obtained the isomorphism classes in the classes FLb_5, SLb_5 and TLb_5. It can be seen that $\dim(Leib(FLb_5)) = 3 = \dim(Leib(SLb_5))$ and $\dim(Leib(TLb_5)) = 1$ or 0. The classification of FLb_5 and SLb_5 given in [27] completely agrees with Theorem 4.2.12. However we find some redundancy in the classification of TLb_5 since $L(2,1,0) \cong L(0,1,0)$ and $L(2,1,1) \cong L(0, \frac{1}{8}, \frac{1}{8})$. Also they missed the isomorphism classes $\mathcal{A}_2$ and $\mathcal{A}_4$ listed in Theorem 4.2.11.

4.3 Classification of Nilpotent Leibniz Algebras of Dimension 5 with $\dim(A^2) = 2$

Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 2$. Then $\dim(A^3) = 0$ or 1. Since $Leib(A) \subseteq A^2$ we have $\dim(Leib(A)) = 1$ or 2.

4.3.1 $\dim(A^3) = 1$

Let $\dim(A^2) = 2$ and $\dim(A^3) = 1$. Then by Lemma 4.0.2 and Lemma 4.0.3 we have $A^3 = Z(A)$.

Theorem 4.3.1. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 2$ and $\dim(A^3) = 1 = \dim(Leib(A))$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_2$**:** $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_3$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = x_5, [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_4$**:** $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_2, x_2] = x_5, [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_5$**:** $[x_1, x_1] = x_5, [x_1, x_2] = x_4 = -[x_2, x_1], [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_6$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_1, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_7$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_8$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_9$**:** $[x_1, x_1] = x_5, [x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_{10}(\alpha)$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = \alpha x_5, [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1], \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_{11}(\alpha)$**:** $[x_1, x_2] = x_4, [x_2, x_1] = -x_4+x_5, [x_2, x_2] = \alpha x_5, [x_2, x_3] = x_5, [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1], \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_{12}$**:** $[x_1, x_1] = x_5, [x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = \frac{1}{4}x_5, [x_2, x_3] = x_5, [x_3, x_3] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_{13}$**:** $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_3, x_2] = x_5 = -[x_2, x_3], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_{14}(\alpha)$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_3] = \alpha x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1], \quad \alpha \in \mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_{15}(\alpha)$: $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_1, x_3] = x_5, [x_2, x_3] = \alpha x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5 = -[x_4, x_1], \quad \alpha \in \mathbb{C}$.

$\mathcal{A}_{16}$: $[x_1, x_1] = x_5, [x_1, x_2] = x_4 = -[x_2, x_1], [x_3, x_2] = x_5 = -[x_2, x_3], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_{17}$: $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = x_5, [x_3, x_2] = x_5 = -[x_2, x_3], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_{18}$: $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_2, x_2] = x_5, [x_3, x_2] = x_5 = -[x_2, x_3], [x_1, x_4] = x_5 = -[x_4, x_1]$.

$\mathcal{A}_{19}$: $[x_1, x_2] = x_4 = -[x_2, x_1], [x_2, x_2] = x_5, [x_1, x_3] = x_5, [x_3, x_2] = x_5 = -[x_2, x_3], [x_1, x_4] = x_5 = -[x_4, x_1]$.

Proof. Note that by Lemma 4.0.4 we have $A^3 = Z(A) = Leib(A)$. Let $A^3 = Z(A) = Leib(A) = \text{span}\{\mathrm{e}_5\}$. Extend this to a basis $\{e_4, e_5\}$ of A^2. Then the nonzero products in $A = \text{span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_5, [e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_3, e_1] = -\beta_1 e_4 + \beta_3 e_5, [e_2, e_3] = \beta_4 e_4 + \beta_5 e_5, [e_3, e_2] = -\beta_4 e_4 + \beta_6 e_5, [e_3, e_3] = \beta_7 e_5, [e_1, e_4] = \gamma_1 e_5,$$
$$[e_4, e_1] = \gamma_2 e_5, [e_2, e_4] = \gamma_3 e_5, [e_4, e_2] = \gamma_4 e_5, [e_3, e_4] = \gamma_5 e_5, [e_4, e_3] = \gamma_6 e_5, [e_4, e_4] = \gamma_7 e_7.$$

Leibniz identities give the following equations:

$$\begin{cases} \alpha_2\gamma_2 = -\alpha_2\gamma_1 \\ \alpha_2\gamma_4 = -\alpha_2\gamma_3 \\ \beta_4\gamma_1 = \alpha_2\gamma_6 + \beta_1\gamma_3 \\ \alpha_2\gamma_7 = 0 \\ \beta_1\gamma_2 = -\beta_1\gamma_1 \\ \beta_4\gamma_1 + \beta_1\gamma_4 + \alpha_2\gamma_5 = 0 \\ \beta_1\gamma_6 = -\beta_1\gamma_5 \\ \beta_1\gamma_3 + \beta_4\gamma_2 - \alpha_2\gamma_5 = 0 \\ \beta_4\gamma_4 = -\beta_4\gamma_3 \\ \beta_4\gamma_6 = -\beta_4\gamma_5 \\ \beta_1\gamma_7 = 0 \\ \beta_4\gamma_7 = 0 \end{cases} \tag{4.28}$$

Note that $(\alpha_2, \beta_1, \beta_4) \neq (0,0,0)$ since $\dim(A^2) = 2$. Then by (4.28) we have $\gamma_7 = 0$. Note that if $\beta_4 \neq 0$ and $\beta_1 = 0$(resp. $\beta_1 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \beta_4 e_1 - \beta_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_4 = 0$. So let $\beta_4 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_5, [e_1,e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2,e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_2,e_2] = \alpha_5 e_5, [e_1,e_3] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_3,e_1] = -\beta_1 e_4 + \beta_3 e_5, [e_2,e_3] = \beta_5 e_5, [e_3,e_2] = \beta_6 e_5, [e_3,e_3] = \beta_7 e_5, [e_1,e_4] = \gamma_1 e_5, [e_4,e_1] = \gamma_2 e_5,$$
$$[e_2,e_4] = \gamma_3 e_5, [e_4,e_2] = \gamma_4 e_5, [e_3,e_4] = \gamma_5 e_5, [e_4,e_3] = \gamma_6 e_5.$$

If $\beta_1 \neq 0$ and $\alpha_2 = 0$(resp. $\alpha_2 \neq 0$) then with the base change $x_1 = e_1, x_2 = e_3, x_3 = e_2, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = e_2, x_3 = \beta_1 e_2 - \alpha_2 e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then by (4.28) we have $\gamma_2 = -\gamma_1, \gamma_4 = -\gamma_3$ and $\gamma_6 = 0 = \gamma_5$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_2e_5,$$
$$[e_3,e_1]=\beta_3e_5,[e_2,e_3]=\beta_5e_5,[e_3,e_2]=\beta_6e_5,[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1],$$
$$[e_2,e_4]=\gamma_3e_5=-[e_4,e_2].$$

If $\gamma_3\neq 0$ and $\gamma_1=0$(resp. $\gamma_1\neq 0$) then with the base change $x_1=e_2,x_2=e_1,x_3=e_3,x_4=e_4,x_5=e_5$(resp. $x_1=e_1,x_2=\gamma_3e_1-\gamma_1e_2,x_3=e_3,x_4=e_4,x_5=e_5$) we can make $\gamma_3=0$. So we can assume $\gamma_3=0$. Then $\gamma_1\neq 0$ since $\dim(Z(A))=1$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_2e_5,$$
$$[e_3,e_1]=\beta_3e_5,[e_2,e_3]=\beta_5e_5,[e_3,e_2]=\beta_6e_5,[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1].$$

Case 1: Let $\beta_6=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_2e_5,$$
$$[e_3,e_1]=\beta_3e_5,[e_2,e_3]=\beta_5e_5,[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1]. \quad (4.29)$$

Without loss of generality we can assume $\beta_3=0$, because if $\beta_3\neq 0$ then with the base change $x_1=e_1,x_2=e_2,x_3=\gamma_1e_3+\beta_3e_4,x_4=e_4,x_5=e_5$ we can make $\beta_3=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_2e_5,$$
$$[e_2,e_3]=\beta_5e_5,[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1].$$

Case 1.1: Let $\beta_5=0$.

Case 1.1.1: Let $\beta_2=0$. Then $\beta_7\neq 0$ since $\dim(Z(A))=1$. Hence we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,$$
$$[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1]. \quad (4.30)$$

Case 1.1.1.1: Let $\alpha_1=0$. Then we have the following products in A:

$$[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,$$
$$[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1]. \quad (4.31)$$

- If $\alpha_5=0=\alpha_3+\alpha_4$ then the base change $x_1=e_1,x_2=\frac{\beta_7}{\alpha_2\gamma_1}e_2,x_3=e_3,x_4=\frac{\beta_7}{\alpha_2\gamma_1}(\alpha_2e_4+\alpha_3e_5),x_5=\beta_7e_5$ shows that A is isomorphic to $\mathcal{A}_1$.
- If $\alpha_5=0$ and $\alpha_3+\alpha_4\neq0$ then the base change $x_1=\frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1}e_1,x_2=\frac{\alpha_2\beta_7\gamma_1}{(\alpha_3+\alpha_4)^2}e_2,x_3=e_3,x_4=\frac{\beta_7}{\alpha_3+\alpha_4}(\alpha_2e_4+\alpha_3e_5),x_5=\beta_7e_5$ shows that A is isomorphic to $\mathcal{A}_2$.
- If $\alpha_5\neq0$ and $\alpha_3+\alpha_4=0$ then the base change $x_1=e_1,x_2=\frac{\alpha_2\gamma_1}{\alpha_5}e_2,x_3=\frac{\alpha_2\gamma_1}{\sqrt{\alpha_5\beta_7}}e_3,x_4=\frac{\alpha_2\gamma_1}{\alpha_5}(\alpha_2e_4+\alpha_3e_5),x_5=\frac{(\alpha_2\gamma_1)^2}{\alpha_5}e_5$ shows that A is isomorphic to $\mathcal{A}_3$.
- If $\alpha_5\neq0$ and $\alpha_3+\alpha_4\neq0$ then the base change $x_1=\frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1}e_1,x_2=\frac{(\alpha_3+\alpha_4)^2}{\alpha_2\alpha_5\gamma_1}e_2,x_3=\frac{(\alpha_3+\alpha_4)^2}{\alpha_2\gamma_1\sqrt{\alpha_5\beta_7}}e_3,x_4=\frac{(\alpha_3+\alpha_4)^3}{(\alpha_2\gamma_1)^2\alpha_5}(\alpha_2e_4+\alpha_3e_5),x_5=\frac{(\alpha_3+\alpha_4)^4}{(\alpha_2\gamma_1)^2\alpha_5}e_5$ shows that A is isomorphic to $\mathcal{A}_4$.

Case 1.1.1.2: Let $\alpha_1\neq0$. If $(\alpha_3+\alpha_4,\alpha_5)\neq(0,0)$ then the base change $x_1=xe_1+e_2,x_2=e_2,x_3=e_3,x_4=e_4,x_5=e_5$ (where $\alpha_1x^2+(\alpha_3+\alpha_4)x+\alpha_5=0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.31). Hence A is isomorphic to $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ or $\mathcal{A}_4$. So let $\alpha_3+\alpha_4=0=\alpha_5$. Then the base change $x_1=e_1,x_2=\frac{\alpha_1}{\alpha_2\gamma_1}e_2,x_3=\sqrt{\frac{\alpha_1}{\beta_7}}e_3,x_4=\frac{\alpha_1}{\alpha_2\gamma_1}(\alpha_2e_4+\alpha_3e_5),x_5=\alpha_1e_5$ shows that A is isomorphic to $\mathcal{A}_5$.

Case 1.1.2: Let $\beta_2\neq0$. If $\beta_7\neq0$ then the base change $x_1=2\beta_7e_1-\beta_2e_3,x_2=e_2,x_3=-\frac{2\gamma_1}{\beta_2}e_3+e_4,x_4=e_4,x_5=e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.30). Hence A is isomorphic to $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4$ or $\mathcal{A}_5$. So let $\beta_7=0$. Note that if $\alpha_1\neq0$ then with the base change $x_1=\beta_2e_1-\alpha_1e_3,x_2=e_2,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\alpha_1=0$. So we can assume $\alpha_1=0$. Then we have the following products in

A:

$$[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_2e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1].$$

Take $\theta=\frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1}$. The base change $y_1=e_1,y_2=e_2,y_3=\frac{\alpha_2\gamma_1}{\beta_2}e_3,y_4=\alpha_2e_4-\alpha_4e_5,y_5=\alpha_2\gamma_1e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_2]=y_4+\theta y_5,[y_2,y_1]=-y_4,[y_2,y_2]=\frac{\alpha_5}{\alpha_2\gamma_1}y_5,[y_1,y_3]=y_5,[y_1,y_4]=y_5=-[y_4,y_1].$$

Without loss of generality we can assume $\theta=0$ because if $\theta\neq0$ then with the base change $x_1=y_1,x_2=y_2=\theta y_3,x_3=y_3,x_4=y_4,x_5=y_5$ we can make $\theta=0$. Then we have the following products in A:

$$[y_1,y_2]=y_4,[y_2,y_1]=-y_4,[y_2,y_2]=\frac{\alpha_5}{\alpha_2\gamma_1}y_5,[y_1,y_3]=y_5,[y_1,y_4]=y_5=-[y_4,y_1].$$

- If $\alpha_5=0$ then the base change $x_1=y_1,x_2=y_2,x_3=y_3,x_4=y_4,x_5=y_5$ shows that A is isomorphic to $\mathcal{A}_6$.
- If $\alpha_5\neq0$ then the base change $x_1=y_1,x_2=\frac{\alpha_2\gamma_1}{\alpha_5}y_2,x_3=\frac{\alpha_2\gamma_1}{\alpha_5}y_3,x_4=\frac{\alpha_2\gamma_1}{\alpha_5}y_4,x_5=\frac{\alpha_2\gamma_1}{\alpha_5}y_5$ shows that A is isomorphic to $\mathcal{A}_7$.

Case 1.2: Let $\beta_5\neq0$. Without loss of generality we can assume $\beta_2=0$. This is because if $\beta_2\neq0$ then with the base change $x_1=\beta_5e_1-\beta_2e_2,x_2=e_2,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\beta_2=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,$$
$$[e_2,e_3]=\beta_5e_5,[e_3,e_3]=\beta_7e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1].$$

Case 1.2.1: Let $\beta_7=0$. If $\alpha_5\neq0$ then with the base change $x_1=e_1,x_2=\beta_5e_2-\alpha_5e_3,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\alpha_5=0$. So assume $\alpha_5=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_3]=\beta_5e_5,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1].$$

- If $\alpha_1=0$ then the base change $x_1=e_1-\frac{\alpha_3+\alpha_4}{\beta_5}e_3,x_2=e_2,x_3=\frac{\alpha_2\gamma_1}{\beta_5}e_3,x_4=\alpha_2e_4+$

$\alpha_3e_5, x_5 = \alpha_2\gamma_1e_5$ shows that A is isomorphic to $\mathcal{A}_8$.

- If $\alpha_1 \neq 0$ then the base change $x_1 = -e_1 + \frac{\alpha_3+\alpha_4}{\beta_5}e_3, x_2 = \frac{\alpha_1}{\alpha_2\gamma_1}e_2, x_3 = \frac{\alpha_2\gamma_1}{\beta_5}e_3, x_4 = -\frac{\alpha_1}{\alpha_2\gamma_1}(\alpha_2e_4 + \alpha_3e_5), x_5 = \alpha_1e_5$ shows that A is isomorphic to $\mathcal{A}_9$.

Case 1.2.2: Let $\beta_7 \neq 0$.

Case 1.2.2.1: Let $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2e_4 + \alpha_3e_5, [e_2, e_1] = -\alpha_2e_4 + \alpha_4e_5, [e_2, e_2] = \alpha_5e_5,$$
$$[e_2, e_3] = \beta_5e_5, [e_3, e_3] = \beta_7e_5, [e_1, e_4] = \gamma_1e_5 = -[e_4, e_1]. \quad (4.32)$$

- If $\alpha_3+\alpha_4 = 0$ then the base change $x_1 = \sqrt{\frac{\beta_5}{\alpha_2\gamma_1}}e_1, x_2 = \frac{\beta_7}{\beta_5}e_2, x_3 = e_3, x_4 = \sqrt{\frac{\beta_5}{\alpha_2\gamma_1}}\frac{\beta_7}{\beta_5}(\alpha_2e_4+\alpha_3e_5), x_5 = \beta_7e_5$ shows that A is isomorphic to $\mathcal{A}_{10}(\alpha)$.

- If $\alpha_3+\alpha_4 \neq 0$ then the base change $x_1 = \frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1}e_1, x_2 = \frac{(\alpha_3+\alpha_4)^2\beta_7}{\alpha_2\gamma_1\beta_5^2}e_2, x_3 = \frac{(\alpha_3+\alpha_4)^2}{\alpha_2\gamma_1\beta_5}e_3, x_4 = \frac{(\alpha_3+\alpha_4)^3\beta_7}{(\alpha_2\gamma_1\beta_5)^2}(\alpha_2e_4 + \alpha_3e_5), x_5 = \frac{(\alpha_3+\alpha_4)^4\beta_7}{(\alpha_2\gamma_1\beta_5)^2}$ shows that A is isomorphic to $\mathcal{A}_{11}(\alpha)$.

Case 1.2.2.2: Let $\alpha_1 \neq 0$. If $(\alpha_3 + \alpha_4, 4\alpha_5\beta_7 - \beta_5^2) \neq (0,0)$ then the base change $x_1 = \frac{x\beta_7}{\gamma_1}e_1 - \frac{2\beta_7}{\beta_5}e_2 + e_3, x_2 = e_2, x_3 = xe_3 + e_4, x_4 = e_4, x_5 = e_5$ (where $\frac{\alpha_1\beta_7^2}{\gamma_1^2}x^2 - \frac{2(\alpha_3+\alpha_4)\beta_7^2}{\beta_5\gamma_1}x + \frac{\beta_7(4\alpha_5\beta_7-\beta_5^2)}{\beta_5^2} = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.32). Hence A is isomorphic to $\mathcal{A}_{10}(\alpha)$ or $\mathcal{A}_{11}(\alpha)$. So let $\alpha_3 + \alpha_4 = 0 = 4\alpha_5\beta_7 - \beta_5^2$. Then the base change $x_1 = \sqrt{\frac{\alpha_1}{\beta_7}}\frac{\beta_5}{\alpha_2\gamma_1}e_1, x_2 = \frac{\alpha_1}{\alpha_2\gamma_1}e_2, x_3 = \frac{\alpha_1\beta_5}{\alpha_2\gamma_1\beta_7}e_3, x_4 = \sqrt{\frac{\alpha_1}{\beta_7}}\frac{\beta_5\alpha_1}{(\alpha_2\gamma_1)^2}(\alpha_2e_4 + \alpha_3e_5), x_5 = \frac{(\alpha_1\beta_5)^2}{(\alpha_2\gamma_1)^2\beta_7}e_5$ shows that A is isomorphic to $\mathcal{A}_{12}$.

Case 2: Let $\beta_6 \neq 0$. Without loss of generality we can assume $\beta_3 = 0$, because if $\beta_3 \neq 0$ then with the base change $x_1 = \beta_6e_1 - \beta_3e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_3 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1e_5, [e_1, e_2] = \alpha_2e_4 + \alpha_3e_5, [e_2, e_1] = -\alpha_2e_4 + \alpha_4e_5, [e_2, e_2] = \alpha_5e_5, [e_1, e_3] = \beta_2e_5,$$
$$[e_2, e_3] = \beta_5e_5, [e_3, e_2] = \beta_6e_5, [e_3, e_3] = \beta_7e_5, [e_1, e_4] = \gamma_1e_5 = -[e_4, e_1].$$

Note that if $\beta_7 \neq 0$ then the base change $x_1 = e_1, x_2 = \beta_7e_2 - \beta_6e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.29). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}(\alpha), \mathcal{A}_{11}(\alpha)$ or $\mathcal{A}_{12}$. So let $\beta_7 = 0$.

Case 2.1: Let $\alpha_5 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_5, [e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_1, e_3] = \beta_2 e_5,$$
$$[e_2, e_3] = \beta_5 e_5, [e_3, e_2] = \beta_6 e_5, [e_1, e_4] = \gamma_1 e_5 = -[e_4, e_1]. \quad (4.33)$$

Case 2.1.1: Let $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_1, e_3] = \beta_2 e_5,$$
$$[e_2, e_3] = \beta_5 e_5, [e_3, e_2] = \beta_6 e_5, [e_1, e_4] = \gamma_1 e_5 = -[e_4, e_1]. \quad (4.34)$$

- If $\beta_2 = 0$ and $\beta_5 + \beta_6 = 0$ then $\alpha_3 + \alpha_4 \neq 0$ since $Leib(A) \neq 0$. Then the base change $x_1 = \frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1} e_1, x_2 = (\frac{\alpha_2\gamma_1}{\alpha_3+\alpha_4})^2 e_2, x_3 = \frac{(\alpha_3+\alpha_4)^2}{\alpha_2\gamma_1\beta_6} e_3, x_4 = \frac{\alpha_2\gamma_1}{\alpha_3+\alpha_4}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \alpha_2\gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{13}$.

- If $\beta_2 = 0$ and $\beta_5 + \beta_6 \neq 0$ then the base change $x_1 = e_1 - \frac{\alpha_3+\alpha_4}{\beta_5+\beta_6} e_3, x_2 = e_2, x_3 = \frac{\alpha_2\gamma_1}{\beta_6} e_3, x_4 = \alpha_2 e_4 + (\alpha_3 - \frac{(\alpha_3+\alpha_4)\beta_6}{\beta_5+\beta_6}) e_5, x_5 = \alpha_2\gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{14}(\alpha)$.

- If $\beta_2 \neq 0, \beta_5 + \beta_6 = 0$ and $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = \beta_6 e_1, x_2 = \beta_2 e_2 - \alpha_2\beta_6\gamma_1 e_3, x_3 = \alpha_2\beta_6\gamma_1 e_3, x_4 = \beta_2\beta_6(\alpha_2 e_4 + (\alpha_3 - \alpha_2\beta_6\gamma_1) e_5), x_5 = \alpha_2\beta_2\beta_6^2\gamma_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{15}(-1)$.

- If $\beta_2 \neq 0, \beta_5 + \beta_6 = 0$ and $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = \frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1} e_1, x_2 = \frac{(\alpha_3+\alpha_4)\beta_2}{\alpha_2\beta_6\gamma_1} e_2, x_3 = \frac{(\alpha_3+\alpha_4)^2}{\alpha_2\beta_6\gamma_1} e_3, x_4 = \frac{(\alpha_3+\alpha_4)^2\beta_2}{\alpha_2^2\beta_6\gamma_1^2}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{(\alpha_3+\alpha_4)^3\beta_2}{\alpha_2^2\beta_6\gamma_2^2} e_5$ shows that A is isomorphic to $\mathcal{A}_{15}(-1)$.

- If $\beta_2 \neq 0, \beta_5 + \beta_6 \neq 0$ and $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = \frac{\beta_6}{\beta_2} e_1 - \frac{\beta_6}{\beta_5+\beta_6} e_2, x_2 = e_2, x_3 = \frac{\alpha_2\beta_6\gamma_1}{\beta_2^2} e_3 - \frac{\alpha_2\beta_6^2}{\beta_2(\beta_5+\beta_6)} e_4, x_4 = \frac{\beta_6}{\beta_2}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{\alpha_2\beta_6^2\gamma_1}{\beta_2^2} e_5$ shows that A is isomorphic to $\mathcal{A}_{14}(\alpha)$.

- If $\beta_2 \neq 0, \beta_5 + \beta_6 \neq 0$ and $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = \frac{\alpha_3+\alpha_4}{\alpha_2\gamma_1} e_1, x_2 = \frac{(\alpha_3+\alpha_4)\beta_2}{\alpha_2\beta_6\gamma_1} e_2, x_3 = \frac{(\alpha_3+\alpha_4)^2}{\alpha_2\beta_6\gamma_1} e_3, x_4 = \frac{(\alpha_3+\alpha_4)^2\beta_2}{\alpha_2^2\beta_6\gamma_1^2}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{(\alpha_3+\alpha_4)^3\beta_2}{\alpha_2^2\beta_6\gamma_2^2} e_5$ shows that A is isomorphic to $\mathcal{A}_{15}(\alpha)(\alpha \in \mathbb{C}\backslash\{-1\})$.

Case 2.1.2: Let $\alpha_1 \neq 0$.

- If $\beta_2 \neq 0$ then the base change $x_1 = \beta_2 e_1 - \alpha_1 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.34). Hence A is isomorphic to $\mathcal{A}_{13}, \mathcal{A}_{14}(\alpha)$ or $\mathcal{A}_{15}(\alpha)$.

- If $\beta_2 = 0$ and $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = (\alpha_3 + \alpha_4)e_1 - \alpha_1 e_2, x_2 = e_2, x_3 = (\alpha_3 + \alpha_4)\gamma_1 e_3 - \alpha_1 \beta_6 e_4, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.34). Hence A is isomorphic to $\mathcal{A}_{13}, \mathcal{A}_{14}(\alpha)$ or $\mathcal{A}_{15}(\alpha)$.

- If $\beta_2 = 0, \alpha_3 + \alpha_4 = 0$ and $\beta_5 + \beta_6 \neq 0$ then the base change $x_1 = e_1 + e_2 - \frac{\alpha_1}{\beta_5 + \beta_6} e_3, x_2 = e_2, x_3 = \gamma_1 e_3 + \beta_6 e_4, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.34). Hence A is isomorphic to $\mathcal{A}_{13}, \mathcal{A}_{14}(\alpha)$ or $\mathcal{A}_{15}(\alpha)$.

- If $\beta_2 = 0, \alpha_3 + \alpha_4 = 0$ and $\beta_5 + \beta_6 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_1}{\alpha_2 \gamma_1} e_2, x_3 = \frac{\alpha_2 \gamma_1}{\beta_6} e_3, x_4 = \frac{\alpha_1}{\alpha_2 \gamma_1}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \alpha_1 e_5$ shows that A is isomorphic to $\mathcal{A}_{16}$.

Case 2.2: Let $\alpha_5 \neq 0$. If $\beta_5 + \beta_6 \neq 0$ then the base change $x_1 = e_1, x_2 = (\beta_5 + \beta_6)e_2 - \alpha_5 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.33). Hence A is isomorphic to $\mathcal{A}_{13}, \mathcal{A}_{14}(\alpha), \mathcal{A}_{15}(\alpha)$ or $\mathcal{A}_{16}$. So let $\beta_5 + \beta_6 = 0$. Note that if $\alpha_1 \neq 0$ and $\beta_2 = 0$[resp. $\beta_2 \neq 0$] then with the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3 + \frac{\beta_6}{x\gamma_1} e_4, x_4 = e_4, x_5 = e_5$ (where $\alpha_1 x^2 + (\alpha_3 + \alpha_4)x + \alpha_5 = 0$)[resp. $x_1 = \beta_2 e_1 - \alpha_1 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$] we can make $\alpha_1 = 0$. So assume $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5,$$
$$[e_3, e_2] = \beta_6 e_5 = -[e_2, e_3], [e_1, e_4] = \gamma_1 e_5 = -[e_4, e_1].$$

- If $\beta_2 = 0$ and $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_2 \gamma_1}{\alpha_5} e_2, x_3 = \frac{\alpha_2 \gamma_1}{\beta_6} e_3, x_4 = \frac{\alpha_2 \gamma_1}{\alpha_5}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{(\alpha_2 \gamma_1)^2}{\alpha_5} e_5$ shows that A is isomorphic to $\mathcal{A}_{17}$.

- If $\beta_2 = 0$ and $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = \frac{\alpha_3 + \alpha_4}{\alpha_2 \gamma_1} e_1, x_2 = \frac{(\alpha_3 + \alpha_4)^2}{\alpha_2 \alpha_5 \gamma_1} e_2, x_3 = \frac{(\alpha_3 + \alpha_4)^2}{\alpha_2 \beta_6 \gamma_1} e_3, x_4 = \frac{(\alpha_3 + \alpha_4)^3}{(\alpha_2 \gamma_1)^2 \alpha_5}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{(\alpha_3 + \alpha_4)^4}{(\alpha_2 \gamma_1)^2 \alpha_5} e_5$ shows that A is isomorphic to $\mathcal{A}_{18}$.

- If $\beta_2 \neq 0$ then the base change $x_1 = \frac{\alpha_5 \beta_2}{\alpha_2 \beta_6 \gamma_1} e_1, x_2 = \frac{\alpha_5 \beta_2^2}{\alpha_2 \beta_6^2 \gamma_1} e_2 - \frac{(\alpha_3 + \alpha_4)\alpha_5 \beta_2}{\alpha_2 \beta_6^2 \gamma_1} e_3, x_3 = \frac{(\alpha_5 \beta_2)^2}{\alpha_2 \beta_6^3 \gamma_1} e_3, x_4 = \frac{\alpha_5^2 \beta_2^3}{\alpha_2 \beta_6^3 \gamma_1^2} e_4 - \frac{\alpha_5^2 \alpha_4 \beta_2^3}{(\alpha_2 \gamma_1)^2 \beta_6^3} e_5, x_5 = \frac{\alpha_5^3 \beta_2^4}{(\alpha_2 \gamma_1)^2 \beta_6^4} e_5$ shows that A is isomorphic to $\mathcal{A}_{19}$.

$\square$

Remark. 1. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{10}(\alpha_1)$ and $\mathcal{A}_{10}(\alpha_2)$ are not isomorphic.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{11}(\alpha_1)$ and $\mathcal{A}_{11}(\alpha_2)$ are not isomorphic.

3. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{14}(\alpha_1)$ and $\mathcal{A}_{14}(\alpha_2)$ are not isomorphic.

4. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{15}(\alpha_1)$ and $\mathcal{A}_{15}(\alpha_2)$ are not isomorphic.

Theorem 4.3.2. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $\dim(A^2) = 2 = \dim(Leib(A))$ *and* $\dim(A^3) = 1$*. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_2] = x_4, [x_3, x_1] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_2$**:** $[x_1, x_2] = x_4, [x_2, x_2] = x_5, [x_3, x_1] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_3$**:** $[x_1, x_2] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_4$**:** $[x_1, x_2] = x_4, [x_2, x_3] = x_5, [x_3, x_1] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_5$**:** $[x_1, x_1] = x_4, [x_2, x_2] = x_5, [x_3, x_1] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_6$**:** $[x_1, x_1] = x_4, [x_2, x_3] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_7$**:** $[x_1, x_1] = x_4, [x_2, x_3] = x_5, [x_3, x_1] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_8(\alpha)$**:** $[x_1, x_1] = x_4, [x_2, x_3] = \alpha x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}\backslash\{0\}.$

$\mathcal{A}_9$**:** $[x_1, x_2] = x_4, [x_2, x_1] = x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_{10}(\alpha)$**:** $[x_1, x_2] = x_4, [x_2, x_3] = \alpha x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}.$

$\mathcal{A}_{11}(\alpha)$**:** $[x_1, x_2] = x_4, [x_3, x_1] = x_5, [x_2, x_3] = \alpha x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5, \quad \alpha \in \mathbb{C}.$

$\mathcal{A}_{12}$**:** $[x_1, x_2] = x_4, [x_2, x_1] = x_5, [x_3, x_1] = x_5, [x_3, x_2] = x_5, [x_1, x_4] = x_5.$

$\mathcal{A}_{13}$**:** $[x_1, x_1] = x_4, [x_2, x_2] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5.$

$\mathcal{A}_{14}$**:** $[x_1, x_2] = x_4, [x_2, x_2] = x_5, [x_2, x_3] = x_5 = -[x_3, x_2], [x_1, x_4] = x_5.$

$\mathcal{A}_{15}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],[x_1,x_4]=x_5.$

$\mathcal{A}_{16}$: $[x_1,x_2]=x_4,[x_3,x_3]=x_5,[x_1,x_4]=x_5.$

$\mathcal{A}_{17}$: $[x_1,x_2]=x_4,[x_2,x_3]=x_5,[x_3,x_3]=x_5,[x_1,x_4]=x_5.$

$\mathcal{A}_{18}(\alpha)$: $[x_1,x_2]=x_4,[x_2,x_2]=x_5,[x_2,x_3]=\alpha x_5,[x_3,x_3]=x_5,[x_1,x_4]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{19}$: $[x_1,x_2]=x_4,[x_2,x_1]=x_5,[x_3,x_3]=x_5,[x_1,x_4]=x_5.$

$\mathcal{A}_{20}$: $[x_1,x_2]=x_4,[x_2,x_1]=x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,[x_1,x_4]=x_5.$

$\mathcal{A}_{21}(\alpha)$: $[x_1,x_2]=x_4,[x_2,x_1]=x_5,[x_2,x_2]=x_5,[x_2,x_3]=\alpha x_5,[x_3,x_3]=x_5,[x_1,x_4]=x_5,\quad \alpha\in\mathbb{C}.$

Proof. Let $A^3=Z(A)=\mathrm{span}\{\mathrm{e}_5\}$. Extend this to a basis $\{e_4,e_5\}$ of $Leib(A)=A^2$. Then the nonzero products in $A=\mathrm{span}\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4,\mathrm{e}_5\}$ are given by:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5,[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_5e_4+\alpha_6e_5,[e_2,e_2]=\alpha_7e_4+\alpha_8e_5,$$
$$[e_1,e_3]=\beta_1e_4+\beta_2e_5,[e_3,e_1]=\beta_3e_4+\beta_4e_5,[e_2,e_3]=\beta_5e_4+\beta_6e_5,[e_3,e_2]=\beta_7e_4+\beta_8e_5,$$
$$[e_3,e_3]=\gamma_1e_4+\gamma_2e_5,[e_1,e_4]=\gamma_3e_5,[e_2,e_4]=\gamma_4e_5,[e_3,e_4]=\gamma_5e_5.$$

Leibniz identities give the following equations:

$$\begin{cases}\alpha_5\gamma_3=\alpha_1\gamma_4\\ \alpha_7\gamma_3=\alpha_3\gamma_4\\ \beta_5\gamma_3=\beta_1\gamma_4\\ \beta_3\gamma_3=\alpha_1\gamma_5\\ \beta_7\gamma_3=\alpha_3\gamma_5\\ \gamma_1\gamma_3=\beta_1\gamma_5\\ \beta_3\gamma_4=\alpha_5\gamma_5\\ \beta_7\gamma_4=\alpha_7\gamma_5\\ \gamma_1\gamma_4=\beta_5\gamma_5\end{cases}\tag{4.35}$$

Note that if $\gamma_5 \neq 0$ and $\gamma_4 = 0$(resp. $\gamma_4 \neq 0$) then with the base change $x_1 = e_1, x_2 = e_3, x_3 = e_2, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = e_2, x_3 = \gamma_5 e_2 - \gamma_4 e_3, x_4 = e_4, x_5 = e_5$) we can make $\gamma_5 = 0$. So let $\gamma_5 = 0$. Then we have the following products:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \alpha_7 e_4 + \alpha_8 e_5,$$
$$[e_1, e_3] = \beta_1 e_4 + \beta_2 e_5, [e_3, e_1] = \beta_3 e_4 + \beta_4 e_5, [e_2, e_3] = \beta_5 e_4 + \beta_6 e_5, [e_3, e_2] = \beta_7 e_4 + \beta_8 e_5,$$
$$[e_3, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5, [e_2, e_4] = \gamma_4 e_5.$$

Note that if $\gamma_4 \neq 0$ and $\gamma_3 = 0$(resp. $\gamma_3 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = \gamma_4 e_1 - \gamma_3 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\gamma_4 = 0$. So let $\gamma_4 = 0$. Then $\gamma_3 \neq 0$ since $\dim(A^3) = 1$. So by (4.35) we have $\alpha_5 = 0 = \alpha_7 = \beta_5 = \beta_3 = \beta_7 = \gamma_1$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5, [e_1, e_3] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_3, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

If $\beta_1 \neq 0$ and $\alpha_3 = 0$(resp. $\alpha_3 \neq 0$) then with the base change $x_1 = e_1, x_2 = e_3, x_3 = e_2, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = e_2, x_3 = \beta_1 e_2 - \alpha_3 e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5, [e_1, e_3] = \beta_2 e_5,$$
$$[e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_3, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Without loss of generality we can assume $\beta_2 = 0$, because otherwise with the base change $x_1 = e_1, x_2 = e_2, x_3 = \gamma_3 e_3 - \beta_2 e_4, x_4 = e_4, x_5 = e_5$ we can make $\beta_2 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5,$$
$$[e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_3, e_3] = \gamma_2 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1: Let $\gamma_2 = 0$.

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5,$$
$$[e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5. \quad (4.36)$$

Case 1.1: Let $\beta_8 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5,$$
$$[e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5. \quad (4.37)$$

Case 1.1.1: Let $\alpha_1 = 0$. Then $\alpha_3 \neq 0$ since $\dim(A^2) = 2$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5,$$
$$[e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5. \quad (4.38)$$

We can assume $\alpha_2 = 0$, because if $\alpha_2 \neq 0$ then with the base change $x_1 = \gamma_3 e_1 - \alpha_2 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1.1.1.1: Let $\beta_6 = 0$. Then $\beta_4 \neq 0$ since $\dim(Z(A)) = 1$. Without loss of generality assume $\alpha_6 = 0$, because if $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_4 e_2 - \alpha_6 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A.

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_8 e_5, [e_3, e_1] = \beta_4 e_5, [e_1, e_4] = \gamma_3 e_5.$$

- If $\alpha_8 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_3\gamma_3}{\beta_4} e_3, x_4 = \alpha_3 e_4 + \alpha_4 e_5, x_5 = \alpha_3\gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.
- If $\alpha_8 \neq 0$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_3\gamma_3}{\alpha_8} e_2, x_3 = \frac{(\alpha_3\gamma_3)^2}{\alpha_8\beta_4} e_3, x_4 = \frac{\alpha_3\gamma_3}{\alpha_8}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \frac{(\alpha_3\gamma_3)^2}{\alpha_8}$ shows that A is isomorphic to $\mathcal{A}_2$.

Case 1.1.1.2: Let $\beta_6 \neq 0$. If $\alpha_8 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_6 e_2 - \alpha_8 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_8 = 0$. So let $\alpha_8 = 0$. Then the products in A are the following:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Furthermore if $\alpha_6 \neq 0$ then with the base change $x_1 = \beta_6 e_1 - \alpha_6 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. So we can assume $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5.$$

- If $\beta_4 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_3 \gamma_3}{\beta_6} e_3, x_4 = \alpha_3 e_4 + \alpha_4 e_5, x_5 = \alpha_3 \gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_3$.
- If $\beta_4 \neq 0$ then the base change $x_1 = \beta_6 e_1, x_2 = \beta_4 e_2, x_3 = \alpha_3 \beta_6 \gamma_3 e_3, x_4 = \beta_4 \beta_6 (\alpha_3 e_4 + \alpha_4 e_5), x_5 = \alpha_3 \beta_4 \beta_6^2 \gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_4$.

Case 1.1.2: Let $\alpha_1 \neq 0$. If $\alpha_3 \neq 0$ then the base change $x_1 = \alpha_3 e_1 - \alpha_1 e_2, x_2 = e_2, x_3 = \alpha_3 \gamma_3 e_3 + \alpha_1 \beta_6 e_4, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.38). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ or $\mathcal{A}_4$. So let $\alpha_3 = 0$. Without loss of generality we can assume $\alpha_4 = 0$ because if $\alpha_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \alpha_4 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \alpha_8 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1.1.2.1: Let $\beta_6 = 0$. Then $\beta_4 \neq 0$ since $\dim(Z(A)) = 1$. Without loss of generality we can assume $\alpha_6 = 0$ because if $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_4 e_2 - \alpha_6 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Furthermore $\alpha_8 \neq 0$ since A is non-split. Therefore the products in A are the following:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_2, e_2] = \alpha_8 e_5, [e_3, e_1] = \beta_4 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_1 \gamma_3}{\alpha_8}} e_2, x_3 = \frac{\alpha_1 \gamma_3}{\beta_4} e_3, x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1 \gamma_3 e_5$

shows that A is isomorphic to $\mathcal{A}_5$.

Case 1.1.2.2: Let $\beta_6 \neq 0$. Without loss of generality we can assume $\alpha_8 = 0$ because if $\alpha_8 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_6 e_2 - \alpha_8 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_8 = 0$.

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_2, e_1] = \alpha_6 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Furthermore if $\alpha_6 \neq 0$ then with the base change $x_1 = \beta_6 e_1 - \alpha_6 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. So let $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_1, e_4] = \gamma_3 e_5.$$

- If $\beta_4 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_1\gamma_3}{\beta_6} e_3, x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1\gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_6$.
- If $\beta_4 \neq 0$ then the base change $x_1 = e_1, x_2 = \frac{\beta_4}{\beta_6} e_2, x_3 = \frac{\alpha_1\gamma_3}{\beta_4} e_3, x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1\gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_7$.

Case 1.2: Let $\beta_8 \neq 0$.

Case 1.2.1: Let $\alpha_8 = 0$. Then we have the following products in A:

$$\begin{aligned}[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5,&\\ [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5.&\end{aligned} \tag{4.39}$$

Case 1.2.1.1: Let $\alpha_3 = 0$. Then $\alpha_1 \neq 0$ since $\dim(A^2) = 2$. If $\beta_4 \neq 0$ then with the base change $x_1 = \beta_8 e_1 - \beta_4 e_2, x_2 = e_2, x_3 = \beta_8\gamma_3 e_3 + \beta_4\beta_6 e_4, x_4 = e_4, x_5 = e_5$ we can make $\beta_4 = 0$. So we can assume $\beta_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1.2.1.1.1: Let $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5. \tag{4.40}$$

Note that if $\alpha_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \alpha_4 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So let $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_2, e_3] = \beta_6 e_5, [e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Note that if $\beta_6 = 0$ then the base change $x_1 = e_1, x_2 = e_3, x_3 = e_2, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.37). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ or $\mathcal{A}_7$. So let $\beta_6 \neq 0$. The base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_1 \gamma_3}{\beta_8} e_3, x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1 \gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_8(\alpha)$.

Case 1.2.1.1.2: Let $\alpha_6 \neq 0$. If $\beta_6 \neq 0$ then the base change $x_1 = \beta_6 e_1 - \alpha_6 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.40). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7$ or $\mathcal{A}_8(\alpha)$. Now let $\beta_6 = 0$. Then the base change $x_1 = e_1, x_2 = e_3, x_3 = \gamma_3 e_2 - \alpha_4 e_4, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.37). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6$ or $\mathcal{A}_7$.

Case 1.2.1.2: Let $\alpha_3 \neq 0$. If $\alpha_1 \neq 0$ then with the base change $x_1 = \alpha_3 e_1 - \alpha_1 e_2, x_2 = e_2, x_3 = \alpha_3 \gamma_3 e_3 + \alpha_1 \beta_6 e_4, x_4 = e_4, x_5 = e_5$ we can make $\alpha_1 = 0$. So let $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_3, e_1] = \beta_4 e_5, [e_2, e_3] = \beta_6 e_5,$$
$$[e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5.$$

Case 1.2.1.2.1: Let $\beta_6 = 0$. Note that if $\alpha_2 \neq 0$ then with the base change $x_1 = \gamma_3 e_1 - \alpha_2 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. So we can assume $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_3, e_1] = \beta_4 e_5, [e_3, e_2] = \beta_8 e_5, [e_1, e_4] = \gamma_3 e_5.$$

- If $\beta_4 = 0$ and $\alpha_6 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_3 \gamma_3}{\beta_8} e_3, x_4 = \alpha_3 e_4 + \alpha_4 e_5, x_5 = \alpha_3 \gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_{10}(0)$.
- If $\beta_4 = 0$ and $\alpha_6 \neq 0$ then the base change $x_1 = \frac{\alpha_6}{\alpha_3 \gamma_3} e_1, x_2 = e_2, x_3 = \frac{\alpha_6^2}{\alpha_3 \beta_8 \gamma_3} e_3, x_4 = \frac{\alpha_6}{\alpha_3 \gamma_3}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \frac{\alpha_6^2}{\alpha_3 \gamma_3} e_5$ shows that A is isomorphic to $\mathcal{A}_9$.
- If $\beta_4 \neq 0$ and $\alpha_6 = 0$ then the base change $x_1 = \beta_8 e_1, x_2 = \beta_4 e_2, x_3 = \alpha_3 \beta_8 \gamma_3 e_3, x_4 =$

$\beta_4\beta_8(\alpha_3e_4 + \alpha_4e_5), x_5 = \alpha_3\beta_4\beta_8^2\gamma_3e_5$ shows that A is isomorphic to $\mathcal{A}_{11}(0)$.

- If $\beta_4 \neq 0$ and $\alpha_6 \neq 0$ then the base change $x_1 = \frac{\alpha_6}{\alpha_3\gamma_3}e_1, x_2 = \frac{\alpha_6\beta_4}{\alpha_3\beta_8\gamma_3}e_2, x_3 = \frac{\alpha_6^2}{\alpha_3\beta_8\gamma_3}e_3, x_4 = \frac{\alpha_6^2\beta_4}{\alpha_3^2\beta_8\gamma_3^2}(\alpha_3e_4 + \alpha_4e_5), x_5 = \frac{\alpha_6^3\beta_4}{\alpha_3^2\beta_8\gamma_3^2}e_5$ shows that A is isomorphic to $\mathcal{A}_{12}$.

Case 1.2.1.2.2: Let $\beta_6 \neq 0$. Without loss of generality we can assume $\alpha_6 = 0$ because if $\alpha_6 \neq 0$ then with the base change $x_1 = \beta_6e_1 - \alpha_6e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2e_5, [e_1, e_2] = \alpha_3e_4 + \alpha_4e_5, [e_3, e_1] = \beta_4e_5, [e_2, e_3] = \beta_6e_5, [e_3, e_2] = \beta_8e_5, [e_1, e_4] = \gamma_3e_5.$$

Note that if $\alpha_2 \neq 0$ then with the base change $x_1 = \gamma_3e_1 - \alpha_2e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. So we can assume $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3e_4 + \alpha_4e_5, [e_3, e_1] = \beta_4e_5, [e_2, e_3] = \beta_6e_5, [e_3, e_2] = \beta_8e_5, [e_1, e_4] = \gamma_3e_5.$$

- If $\beta_4 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_3\gamma_3}{\beta_8}e_3, x_4 = \alpha_3e_4 + \alpha_4e_5, x_5 = \alpha_3\gamma_3e_5$ shows that A is isomorphic to $\mathcal{A}_{10}(\alpha)(\alpha \in \mathbb{C}\backslash\{0\})$.
- If $\beta_4 \neq 0$ then the base change $x_1 = \beta_8e_1, x_2 = \beta_4e_2, x_3 = \alpha_3\beta_8\gamma_3e_3, x_4 = \beta_4\beta_8(\alpha_3e_4 + \alpha_4e_5), x_5 = \alpha_3\beta_4\beta_8^2\gamma_3e_5$ shows that A is isomorphic to $\mathcal{A}_{11}(\alpha)(\alpha \in \mathbb{C}\backslash\{0\})$.

Case 1.2.2: Let $\alpha_8 \neq 0$. If $\beta_6 + \beta_8 \neq 0$ then the base change $x_1 = e_1, x_2 = (\beta_6 + \beta_8)e_2 - \alpha_8e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with nonzero products given by (4.39). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8(\alpha), \mathcal{A}_9, \mathcal{A}_{10}(\alpha), \mathcal{A}_{11}(\alpha)$ or $\mathcal{A}_{12}$. So let $\beta_6 + \beta_8 = 0$. Note that if $\beta_4 \neq 0$ then with the base change $x_1 = -\beta_6e_1 - \beta_4e_2, x_2 = e_2, x_3 = \gamma_3e_3 - \beta_4e_4, x_4 = e_4, x_5 = e_5$ we can make $\beta_4 = 0$. So let $\beta_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1e_4 + \alpha_2e_5, [e_1, e_2] = \alpha_3e_4 + \alpha_4e_5, [e_2, e_1] = \alpha_6e_5, [e_2, e_2] = \alpha_8e_5,$$
$$[e_2, e_3] = \beta_6e_5 = -[e_3, e_2], [e_1, e_4] = \gamma_3e_5.$$

We can assume $\alpha_6 = 0$, because if $\alpha_6 \neq 0$ then with the base change $x_1 = \beta_6e_1 - \alpha_6e_3, x_2 =$

$e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_8 e_5, [e_2, e_3] = \beta_6 e_5 = -[e_3, e_2], [e_1, e_4] = \gamma_3 e_5.$$

Case 1.2.2.1: Let $\alpha_3 = 0$. If $\alpha_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \alpha_4 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So let $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_2, e_2] = \alpha_8 e_5, [e_2, e_3] = \beta_6 e_5 = -[e_3, e_2], [e_1, e_4] = \gamma_3 e_5.$$

The base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_1\gamma_3}{\alpha_8}} e_2, x_3 = \frac{\sqrt{\alpha_1\alpha_8\gamma_3}}{\beta_6} e_3, x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \alpha_1\gamma_3 e_5$ shows that A is isomorphic to $\mathcal{A}_{13}$.

Case 1.2.2.2: Let $\alpha_3 \neq 0$. Without loss of generality we can assume $\alpha_4 = 0$ because if $\alpha_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \alpha_4 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4, [e_2, e_2] = \alpha_8 e_5, [e_2, e_3] = \beta_6 e_5 = -[e_3, e_2], [e_1, e_4] = \gamma_3 e_5.$$

Similarly if $\alpha_2 \neq 0$ then with the base change $x_1 = \gamma_3 e_1 - \alpha_2 e_4, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. So we can assume $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4, [e_1, e_2] = \alpha_3 e_4, [e_2, e_2] = \alpha_8 e_5, [e_2, e_3] = \beta_6 e_5 = -[e_3, e_2], [e_1, e_4] = \gamma_3 e_5.$$

- If $\alpha_1 = 0$ then the base change $x_1 = \sqrt{\frac{\alpha_8\beta_6}{\alpha_3\gamma_3}} e_1, x_2 = \beta_6 e_2, x_3 = \alpha_8 e_3, x_4 = \sqrt{\frac{\alpha_8\beta_6}{\alpha_3\gamma_3}} \alpha_3\beta_6 e_4, x_5 = \alpha_8\beta_6^2 e_5$ shows that A is isomorphic to $\mathcal{A}_{14}$.
- If $\alpha_1 \neq 0$ then the base change $x_1 = \frac{\alpha_1\alpha_8}{\alpha_3^2\gamma_3} e_1, x_2 = \frac{\alpha_1^2\alpha_8}{\alpha_3^3\gamma_3} e_2, x_3 = \frac{\alpha_1^2\alpha_8^2}{\alpha_3^3\beta_6\gamma_3} e_3, x_4 = \frac{\alpha_1^3\alpha_8^2}{\alpha_3^4\gamma_3^2} e_4, x_5 = \frac{\alpha_1^4\alpha_8^3}{\alpha_3^6\gamma_3^2} e_5$ shows that A is isomorphic to $\mathcal{A}_{15}$.

Case 2: Let $\gamma_2 \neq 0$.

Case 2.1: Let $\alpha_3 = 0$. If $\alpha_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \alpha_4 e_4, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So let $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5,[e_2,e_1]=\alpha_6e_5,[e_2,e_2]=\alpha_8e_5,[e_3,e_1]=\beta_4e_5,[e_2,e_3]=\beta_6e_5,[e_3,e_2]=\beta_8e_5,$$
$$[e_3,e_3]=\gamma_2e_5,[e_1,e_4]=\gamma_3e_5.$$

If $\alpha_8=0$[resp. $\alpha_8\neq 0$] then the base change $x_1=e_1,x_2=e_3,x_3=e_2,x_4=e_4,x_5=e_5$[resp. $x_1=e_1,x_2=e_2,x_3=e_2+xe_3,x_4=e_4,x_5=e_5$ (where $x^2\gamma_2+x(\beta_6+\beta_8)+\alpha_8=0$)] shows that A is isomorphic to an algebra with nonzero products given by (4.36). Hence A is isomorphic to $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4,\mathcal{A}_5,\mathcal{A}_6,\mathcal{A}_7$, $\mathcal{A}_8(\alpha),\mathcal{A}_9,\mathcal{A}_{10}(\alpha),\mathcal{A}_{11}(\alpha)\mathcal{A}_{12},\mathcal{A}_{13},\mathcal{A}_{14}$ or $\mathcal{A}_{15}$.

Case 2.2: Let $\alpha_3\neq 0$. If $\alpha_1\neq 0$ then with the base change $x_1=\alpha_3e_1-\alpha_1e_2,x_2=e_2,x_3=\gamma_3e_3+\beta_6e_4,x_4=e_4,x_5=e_5$ we can make $\alpha_1=0$. So let $\alpha_1=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_2e_5,[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_6e_5,[e_2,e_2]=\alpha_8e_5,[e_3,e_1]=\beta_4e_5,$$
$$[e_2,e_3]=\beta_6e_5,[e_3,e_3]=\gamma_2e_5,[e_1,e_4]=\gamma_3e_5.$$

Note that if $\alpha_2\neq 0$ then with the base change $x_1=\gamma_3e_1-\alpha_2e_4,x_2=e_2,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\alpha_2=0$. So let $\alpha_2=0$. Then we have the following products in A:

$$[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_6e_5,[e_2,e_2]=\alpha_8e_5,[e_3,e_1]=\beta_4e_5,$$
$$[e_2,e_3]=\beta_6e_5,[e_3,e_3]=\gamma_2e_5,[e_1,e_4]=\gamma_3e_5.$$

If $\beta_4\neq 0$ then with the base change $x_1=\gamma_2e_1-\beta_4e_3,x_2=e_2,x_3=\gamma_3e_3+\beta_4e_4,x_4=e_4,x_5=e_5$ we can make $\beta_4=0$. Then assume $\beta_4=0$. Hence we have the following products in A:

$$[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_6e_5,[e_2,e_2]=\alpha_8e_5,[e_2,e_3]=\beta_6e_5,[e_3,e_3]=\gamma_2e_5,[e_1,e_4]=\gamma_3e_5.$$

- If $\alpha_6=0,\alpha_8=0$ and $\beta_6=0$ then the base change $x_1=e_1,x_2=\frac{\gamma_2}{\alpha_3\gamma_3}e_2,x_3=e_3,x_4=\frac{\gamma_2}{\alpha_3\gamma_3}(\alpha_3e_4+\alpha_4e_5),x_5=\gamma_2e_5$ shows that A is isomorphic to $\mathcal{A}_{16}$.
- If $\alpha_6=0,\alpha_8=0$ and $\beta_6\neq 0$ then the base change $x_1=\frac{\beta_6}{\sqrt{\alpha_3\gamma_3}}e_1,x_2=\gamma_2e_2,x_3=\beta_6e_3,x_4=\frac{\beta_6\gamma_2}{\sqrt{\alpha_3\gamma_3}}(\alpha_3e_4+\alpha_4e_5),x_5=\beta_6^2\gamma_2e_5$ shows that A is isomorphic to $\mathcal{A}_{17}$.
- If $\alpha_6=0$ and $\alpha_8\neq 0$ then the base change $x_1=e_1,x_2=\frac{\alpha_3\gamma_3}{\alpha_8}e_2,x_3=\frac{\alpha_3\gamma_3}{\sqrt{\alpha_8\gamma_2}}e_3,x_4=\frac{\alpha_3\gamma_3}{\alpha_8}(\alpha_3e_4+\alpha_4e_5),x_5=\frac{\alpha_3^2\gamma_3^2}{\alpha_8}e_5$ shows that A is isomorphic to $\mathcal{A}_{18}(\alpha)$.

- If $\alpha_6 \neq 0, \alpha_8 = 0$ and $\beta_6 = 0$ then the base change $x_1 = \frac{\alpha_6}{\alpha_3\gamma_3}e_1, x_2 = \frac{\alpha_3\gamma_2\gamma_3}{\alpha_6^2}e_2, x_3 = e_3, x_4 = \frac{\gamma_2}{\alpha_6}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \gamma_2 e_5$ shows that A is isomorphic to $\mathcal{A}_{19}$.

- If $\alpha_6 \neq 0, \alpha_8 = 0$ and $\beta_6 \neq 0$ then the base change $x_1 = \frac{\alpha_6}{\alpha_3\gamma_3}e_1, x_2 = \frac{\alpha_6^2\gamma_2}{\alpha_3\beta_6^2\gamma_3}e_2, x_3 = \frac{\alpha_6^2}{\alpha_3\beta_6\gamma_3}e_3, x_4 = \frac{\alpha_6^3\gamma_2}{(\alpha_3\beta_6\gamma_3)^2}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \frac{\alpha_6^4\gamma_2}{(\alpha_3\beta_6\gamma_3)^2}e_5$ shows that A is isomorphic to $\mathcal{A}_{20}$.

- If $\alpha_6 \neq 0$ and $\alpha_8 \neq 0$ then the base change $x_1 = \frac{\alpha_6}{\alpha_3\gamma_3}e_1, x_2 = \frac{\alpha_6^2}{\alpha_3\alpha_8\gamma_3}e_2, x_3 = \frac{\alpha_6^2}{\alpha_3\gamma_3\sqrt{\alpha_8\gamma_2}}e_3, x_4 = \frac{\alpha_6^3}{\alpha_3^2\alpha_8\gamma_3^2}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \frac{\alpha_6^4}{\alpha_3^2\alpha_8\gamma_3^2}e_5$ shows that A is isomorphic to $\mathcal{A}_{21}(\alpha)$.

□

Remark. 1. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_8(\alpha_1)$ and $\mathcal{A}_8(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.

2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{10}(\alpha_1)$ and $\mathcal{A}_{10}(\alpha_2)$ are not isomorphic.

3. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{11}(\alpha_1)$ and $\mathcal{A}_{11}(\alpha_2)$ are not isomorphic.

4. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{18}(\alpha_1)$ and $\mathcal{A}_{18}(\alpha_2)$ are isomorphic if and only if $\alpha_2 = -\alpha_1$.

5. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{21}(\alpha_1)$ and $\mathcal{A}_{21}(\alpha_2)$ are isomorphic if and only if $\alpha_2 = -\alpha_1$.

4.3.2 $\dim(A^3) = 0$

Let $\dim(A^2) = 2$ and $A^3 = 0$. Then by Lemma 4.0.2 and Lemma 4.0.3 we have $A^2 = Z(A)$. Also since $Leib(A) \subseteq A^2$ we have $\dim(Leib(A)) = 1$ or 2.

Theorem 4.3.3. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with* $\dim(A^2) = 2, A^3 = 0$ *and* $\dim(Leib(A)) = 1$. *Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$**:** $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_3, x_3] = x_5.$

$\mathcal{A}_2$**:** $[x_1, x_2] = x_4 = -[x_2, x_1], [x_3, x_1] = x_5.$

$\mathcal{A}_3$**:** $[x_1, x_2] = x_4, [x_2, x_1] = -x_4 + x_5, [x_3, x_1] = x_5, [x_3, x_3] = x_5.$

$\mathcal{A}_4(\alpha)$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_1,x_3]=x_5,[x_3,x_1]=\alpha x_5, \quad \alpha\in\mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_5$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+x_5,[x_1,x_3]=x_5=-[x_3,x_1]$.

$\mathcal{A}_6(\alpha)$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_1,x_3]=x_5,[x_3,x_1]=\alpha x_5,[x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}$.

$\mathcal{A}_7(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+\alpha x_5,[x_3,x_1]=x_5,[x_3,x_2]=x_5,[x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0\}$.

$\mathcal{A}_8$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+x_5,[x_3,x_1]=x_5,[x_3,x_2]=x_5$.

$\mathcal{A}_9$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_1,x_3]=x_5,[x_3,x_2]=x_5$.

$\mathcal{A}_{10}$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+x_5,[x_1,x_3]=x_5,[x_3,x_2]=x_5$.

$\mathcal{A}_{11}$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+x_5,[x_1,x_3]=x_5,[x_3,x_1]=x_5,[x_3,x_2]=x_5,[x_3,x_3]=x_5$.

$\mathcal{A}_{12}(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+x_5,[x_1,x_3]=x_5,[x_3,x_1]=\alpha x_5,[x_2,x_3]=x_5,[x_3,x_2]=\alpha x_5, \quad \alpha\in\mathbb{C}$.

$\mathcal{A}_{13}$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2]$.

$\mathcal{A}_{14}$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_5,[x_1,x_3]=x_5=-[x_3,x_1]$.

Proof. Let $Leib(A)=\mathrm{span}\{e_5\}$. Extend this to a basis of $\{e_4,e_5\}$ of $A^2=Z(A)$. Then the nonzero products in $A=\mathrm{span}\{e_1,e_2,e_3,e_4,e_5\}$ are given by:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_1e_4+\beta_2e_5,$$
$$[e_3,e_1]=-\beta_1e_4+\beta_3e_5,[e_2,e_3]=\beta_4e_4+\beta_5e_5,[e_3,e_2]=-\beta_4e_4+\beta_6e_5,[e_3,e_3]=\beta_7e_5.$$

Without loss of generality we can assume $\beta_4=0$, because if $\beta_4\neq0$ and $\beta_1=0$(resp. $\beta_1\neq0$) then with the base change $x_1=e_2,x_2=e_1,x_3=e_3,x_4=e_4,x_5=e_5$(resp. $x_1=e_1,x_2=\beta_4e_1-\beta_1e_2,x_3=e_3,x_4=e_4,x_5=e_5$) we can make $\beta_4=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_5,[e_1,e_2]=\alpha_2e_4+\alpha_3e_5,[e_2,e_1]=-\alpha_2e_4+\alpha_4e_5,[e_2,e_2]=\alpha_5e_5,[e_1,e_3]=\beta_1e_4+\beta_2e_5,$$
$$[e_3,e_1]=-\beta_1e_4+\beta_3e_5,[e_2,e_3]=\beta_5e_5,[e_3,e_2]=\beta_6e_5,[e_3,e_3]=\beta_7e_5.$$

Note that if $\beta_1\neq0$ and $\alpha_2=0$(resp. $\alpha_2\neq0$) then with the base change $x_1=e_1,x_2=e_3,x_3=e_2,x_4=e_4,x_5=e_5$(resp. $x_1=e_1,x_2=e_2,x_3=\beta_1e_2-\alpha_2e_3,x_4=e_4,x_5=e_5$) we can make $\beta_1 = 0$. So let $\beta_1 = 0$. Then $\alpha_2 \neq 0$ since $\dim(A^2) = 2$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_5, [e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5,$$
$$[e_3, e_1] = \beta_3 e_5, [e_2, e_3] = \beta_5 e_5, [e_3, e_2] = \beta_6 e_5, [e_3, e_3] = \beta_7 e_5.$$

If $\alpha_1 \neq 0$ and $(\alpha_3 + \alpha_4, \alpha_5) \neq (0,0)$[resp. $\alpha_3 + \alpha_4 = 0 = \alpha_5$] then with the base change $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ (where $\alpha_1 x^2 + (\alpha_3 + \alpha_4)x + \alpha_5 = 0$)[resp. $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$] we can make $\alpha_1 = 0$. So we can assume $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_1] = \beta_3 e_5,$$
$$[e_2, e_3] = \beta_5 e_5, [e_3, e_2] = \beta_6 e_5, [e_3, e_3] = \beta_7 e_5.$$

Case 1: Let $\alpha_5 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_1] = \beta_3 e_5,$$
$$[e_2, e_3] = \beta_5 e_5, [e_3, e_2] = \beta_6 e_5, [e_3, e_3] = \beta_7 e_5. \quad (4.41)$$

Case 1.1: Let $\beta_5 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_1] = \beta_3 e_5,$$
$$[e_3, e_2] = \beta_6 e_5, [e_3, e_3] = \beta_7 e_5. \quad (4.42)$$

Case 1.1.1: Let $\beta_6 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_1] = \beta_3 e_5,$$
$$[e_3, e_3] = \beta_7 e_5. \quad (4.43)$$

Case 1.1.1.1: Let $\beta_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_3, e_1] = \beta_3 e_5, [e_3, e_3] = \beta_7 e_5. \quad (4.44)$$

Case 1.1.1.1.1: Let $\beta_3 = 0$. Then $\beta_7 \neq 0$ since $\dim(Z(A)) = 2$. Note that $\alpha_3 + \alpha_4 \neq 0$ since A is non-split. Then the base change $x_1 = \frac{\beta_7}{\alpha_3+\alpha_4}e_1, x_2 = e_2, x_3 = e_3, x_4 = \frac{\beta_7}{\alpha_3+\alpha_4}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_7 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

Case 1.1.1.1.2: Let $\beta_3 \neq 0$.

- If $\beta_7 = 0$ then the base change $x_1 = e_1, x_2 = \beta_3 e_2 - (\alpha_3 + \alpha_4)e_3, x_3 = e_3, x_4 = \beta_3(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_3 e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\beta_7 \neq 0$ and $\alpha_3+\alpha_4 = 0$ then the base change $x_1 = \beta_7 e_1 - \beta_3 e_3, x_2 = e_2, x_3 = -\beta_3 e_3, x_4 = \beta_7(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_3^2\beta_7 e_5$ shows that A is isomorphic to $\mathcal{A}_6(0)$.

- If $\beta_7 \neq 0$ and $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = \beta_7 e_1, x_2 = \frac{\beta_3^2}{\alpha_3+\alpha_4}e_2, x_3 = \beta_3 e_3, x_4 = \frac{\beta_3^2\beta_7}{\alpha_3+\alpha_4}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_3^2\beta_7 e_5$ shows that A is isomorphic to $\mathcal{A}_3$.

Case 1.1.1.2: Let $\beta_2 \neq 0$.

- If $\beta_7 = 0$ and $\beta_2+\beta_3 \neq 0$ then the base change $x_1 = e_1, x_2 = (\beta_2+\beta_3)e_2-(\alpha_3+\alpha_4)e_3, x_3 = e_3, x_4 = \alpha_2(\beta_2+\beta_3)e_4+(\alpha_3\beta_3-\alpha_4\beta_2)e_5, x_5 = \beta_2 e_5$ shows that A is isomorphic to $\mathcal{A}_4(\alpha)$.

- If $\beta_7 = 0$ and $\beta_2 + \beta_3 = 0$ then $\alpha_3 + \alpha_4 \neq 0$ since $Leib(A) \neq 0$. Then the base change $x_1 = e_1, x_2 = \beta_2 e_2, x_3 = (\alpha_3 + \alpha_4)e_3, x_4 = \beta_2(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_2(\alpha_3 + \alpha_4)e_5$ shows that A is isomorphic to $\mathcal{A}_5$.

- If $\beta_7 \neq 0$ and $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = \beta_7 e_1, x_2 = e_2, x_3 = \beta_2 e_3, x_4 = \beta_7(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_2^2\beta_7 e_5$ shows that A is isomorphic to $\mathcal{A}_6(\alpha)$.

- If $\beta_7 \neq 0$ and $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = -\beta_7 e_1 - \frac{\beta_2\beta_3}{\alpha_3+\alpha_4}e_2 + \beta_2 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.44). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ or $\mathcal{A}_6(0)$.

Case 1.1.2: Let $\beta_6 \neq 0$.

Case 1.1.2.1: Let $\beta_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_3, e_1] = \beta_3 e_5,$$
$$[e_3, e_2] = \beta_6 e_5, [e_3, e_3] = \beta_7 e_5. \quad (4.45)$$

- If $\beta_3 = 0$ then the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.43). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5$ or $\mathcal{A}_6(\alpha)$.

- If $\beta_3 \neq 0$ and $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = e_1, x_2 = \beta_6 e_1 - \beta_3 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.43). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5$ or $\mathcal{A}_6(\alpha)$.

- If $\beta_3 \neq 0, \alpha_3 + \alpha_4 \neq 0$ and $\beta_7 \neq 0$ then the base change $x_1 = \beta_6 e_1, x_2 = \beta_3 e_2, x_3 = \frac{\beta_3\beta_6}{\beta_7} e_3, x_4 = \beta_3\beta_6(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{(\beta_3\beta_6)^2}{\beta_7} e_5$ shows that A is isomorphic to $\mathcal{A}_7(\alpha)$.

- If $\beta_3 \neq 0, \alpha_3 + \alpha_4 \neq 0$ and $\beta_7 = 0$ then the base change $x_1 = \beta_6 e_1, x_2 = \beta_3 e_2, x_3 = (\alpha_3 + \alpha_4) e_3, x_4 = \beta_3\beta_6(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_3\beta_6(\alpha_3 + \alpha_4) e_5$ shows that A is isomorphic to $\mathcal{A}_8$.

Case 1.1.2.2: Let $\beta_2 \neq 0$.

Case 1.1.2.2.1: Let $\beta_7 = 0$. Note that if $\beta_3 \neq 0$ then with the base change $x_1 = \beta_6 e_1 - \beta_3 e_2 + \frac{\beta_3(\alpha_3+\alpha_4)}{\beta_2} e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_3 = 0$. So we can assume $\beta_3 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_2] = \beta_6 e_5.$$

- If $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = \beta_6 e_1, x_2 = \beta_2 e_2, x_3 = e_3, x_4 = \beta_2\beta_6(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_2\beta_6 e_5$ shows that A is isomorphic to $\mathcal{A}_9$.

- If $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = \beta_6^2 e_1, x_2 = \beta_2\beta_6 e_2, x_3 = \beta_6(\alpha_3 + \alpha_4) e_3, x_4 = \beta_2\beta_6^3(\alpha_2 e_4 + \alpha_3 e_5), x_5 = (\alpha_3 + \alpha_4)\beta_2\beta_6^3 e_5$ shows that A is isomorphic to $\mathcal{A}_{10}$.

Case 1.1.2.2.2: Let $\beta_7 \neq 0$. Take $\theta = \beta_7(\alpha_3 + \alpha_4) - \beta_2\beta_6$.

- If $\theta \neq 0$ then the base change $x_1 = \beta_7 e_1 + \frac{\beta_2\beta_3\beta_7}{\theta} e_2 - \beta_2 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.45). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha)$ or $\mathcal{A}_8$.

- If $\theta = 0$ and $\beta_3 = 0$ then the base change $x_1 = \beta_7 e_1 - \beta_2 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.45). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha)$ or $\mathcal{A}_8$.

- If $\theta = 0$ and $\beta_3 \neq 0$ then the base change $x_1 = \beta_6 e_1 + (\beta_2 - \beta_3) e_2, x_2 = \beta_2 e_2, x_3 = \frac{\beta_2\beta_6}{\beta_7} e_3, x_4 = \beta_2\beta_6(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \frac{(\beta_2\beta_6)^2}{\beta_7} e_5$ shows that A is isomorphic to $\mathcal{A}_{11}$.

Case 1.2: Let $\beta_5 \neq 0$.

- If $\beta_2 = 0$ then the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.42). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha), \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}$ or $\mathcal{A}_{11}$.

- If $\beta_2 \neq 0$ and $\alpha_3 + \alpha_4 = 0$ then the base change $x_1 = e_1, x_2 = \beta_5 e_1 - \beta_2 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.42). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha), \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}$ or $\mathcal{A}_{11}$.

- If $\beta_2 \neq 0, \alpha_3 + \alpha_4 \neq 0$ and $(\beta_7, \beta_5\beta_3 - \beta_2\beta_6) \neq (0,0)$ then the base change $x_1 = e_1, x_2 = xe_1 - \frac{x\beta_2+\beta_7}{\beta_5}e_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $(\alpha_3 + \alpha_4)\beta_2 x^2 + ((\alpha_3 + \alpha_4)\beta_7 - \beta_3\beta_5 + \beta_2\beta_6)x + \beta_6\beta_7 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.42). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha), \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}$ or $\mathcal{A}_{11}$.

- If $\beta_2 \neq 0, \alpha_3 + \alpha_4 \neq 0$ and $(\beta_7, \beta_5\beta_3 - \beta_2\beta_6) = (0,0)$ hen the base change $x_1 = \beta_5 e_1, x_2 = \beta_2 e_2, x_3 = (\alpha_3 + \alpha_4)e_3, x_4 = \beta_2\beta_5(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \beta_2\beta_5(\alpha_3 + \alpha_4)e_5$ shows that A is isomorphic to $\mathcal{A}_{12}(\alpha)$.

Case 2: Let $\alpha_5 \neq 0$. Note that if $(\beta_5 + \beta_6, \beta_7) \neq (0,0)$ then the base change $x_1 = e_1, x_2 = xe_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\alpha_5 x^2 + (\beta_5 + \beta_6)x + \beta_7 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.41). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha), \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}$ or $\mathcal{A}_{12}(\alpha)$. So let $\beta_5 + \beta_6 = 0 = \beta_7$. Then the products in A are the following:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5, [e_2, e_1] = -\alpha_2 e_4 + \alpha_4 e_5, [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_1] = \beta_3 e_5,$$
$$[e_2, e_3] = \beta_5 e_5 = -[e_3, e_2].$$

Furthermore, if $\alpha_3 + \alpha_4 \neq 0$ then the base change $x_1 = e_1, x_2 = \alpha_5 e_1 - (\alpha_3 + \alpha_4)e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.41). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha), \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}$ or $\mathcal{A}_{12}(\alpha)$. So we can assume $\alpha_3 + \alpha_4 = 0$. Then the products in A are the following:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5, [e_3, e_1] = \beta_3 e_5, [e_2, e_3] = \beta_5 e_5 = -[e_3, e_2].$$

If $\beta_2 + \beta_3 \neq 0$ then the base change $x_1 = e_1, x_2 = -\frac{\alpha_5}{\beta_2+\beta_3}e_1 + e_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$

shows that A is isomorphic to an algebra with the nonzero products given by (4.41). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4(\alpha), \mathcal{A}_5, \mathcal{A}_6(\alpha), \mathcal{A}_7(\alpha), \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{A}_{11}$ or $\mathcal{A}_{12}(\alpha)$. So let $\beta_2 + \beta_3 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5 = -[e_3, e_1], [e_2, e_3] = \beta_5 e_5 = -[e_3, e_2].$$

Case 2.1: Let $\beta_2 = 0$. Then $\beta_5 \neq 0$ since $\dim(Z(A)) = 2$. Then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_5}{\beta_5} e_3, x_4 = \alpha_2 e_4 + \alpha_3 e_5, x_5 = \alpha_5 e_5$ shows that A is isomorphic to $\mathcal{A}_{13}$.

Case 2.2: Let $\beta_2 \neq 0$. Without loss of generality we can assume $\beta_5 = 0$ because if $\beta_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_5 e_1 - \beta_2 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_5 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_2, e_2] = \alpha_5 e_5, [e_1, e_3] = \beta_2 e_5 = -[e_3, e_1].$$

Then the base change $x_1 = \frac{\alpha_5}{\beta_2} e_1, x_2 = e_2, x_3 = e_3, x_4 = \frac{\alpha_5}{\beta_2}(\alpha_2 e_4 + \alpha_3 e_5), x_5 = \alpha_5 e_5$ shows that A is isomorphic to $\mathcal{A}_{14}$. □

Remark.

1. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{-1\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_4(\alpha_1)$ and $\mathcal{A}_4(\alpha_2)$ are not isomorphic.
2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_6(\alpha_1)$ and $\mathcal{A}_6(\alpha_2)$ are isomorphic if and only if $\alpha_2 = \frac{1}{\alpha_1}$.
3. If $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\{0\}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_7(\alpha_1)$ and $\mathcal{A}_7(\alpha_2)$ are not isomorphic.
4. If $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\alpha_1 \neq \alpha_2$, then $\mathcal{A}_{12}(\alpha_1)$ and $\mathcal{A}_{12}(\alpha_2)$ are not isomorphic.

Theorem 4.3.4. *Let A be a 5−dimensional non-split non-Lie nilpotent Leibniz algebra with $\dim(A^2) = 2 = \dim(Leib(A))$ and $A^3 = 0$. Then A is isomorphic to a Leibniz algebra spanned by $\{x_1, x_2, x_3, x_4, x_5\}$ with the nonzero products given by one of the following:*

$\mathcal{A}_1$: $[x_1, x_1] = x_5, [x_1, x_2] = x_4, [x_3, x_3] = x_5.$

$\mathcal{A}_2$: $[x_1, x_1] = x_5, [x_1, x_2] = x_4, [x_2, x_1] = x_5, [x_3, x_3] = x_5.$

$\mathcal{A}_3(\alpha)$: $[x_1, x_1] = x_5, [x_1, x_2] = x_4, [x_2, x_1] = \alpha x_5, [x_2, x_2] = x_5, [x_3, x_3] = x_5, \quad \alpha \in \mathbb{C}.$

$\mathcal{A}_4(\alpha)$: $[x_1, x_2] = x_4, [x_2, x_1] = \alpha x_4 + x_5, [x_3, x_3] = x_5, \quad \alpha \in \mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_5(\alpha)$: $[x_1,x_2]=x_4, [x_2,x_1]=\alpha x_4, [x_2,x_2]=x_5, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_6(\alpha)$: $[x_1,x_2]=x_4, [x_2,x_1]=\alpha x_4+x_5, [x_2,x_2]=x_5, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_7(\alpha,\beta)$: $[x_1,x_1]=x_5, [x_1,x_2]=x_4, [x_2,x_1]=\alpha x_4+\beta x_5, [x_2,x_2]=x_5, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{-1,0\}, \beta\in\mathbb{C}$.

$\mathcal{A}_8$: $[x_1,x_2]=x_5, [x_2,x_2]=x_4, [x_3,x_3]=x_5$.

$\mathcal{A}_9(\alpha)$: $[x_1,x_2]=\alpha x_5, [x_2,x_1]=x_5, [x_2,x_2]=x_4, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}$.

$\mathcal{A}_{10}$: $[x_1,x_1]=x_5, [x_1,x_2]=x_5, [x_2,x_2]=x_4, [x_3,x_3]=x_5$.

$\mathcal{A}_{11}$: $[x_1,x_2]=x_4+x_5=-[x_2,x_1], [x_2,x_2]=x_4, [x_3,x_3]=x_5$.

$\mathcal{A}_{12}$: $[x_1,x_2]=x_4+x_5, [x_2,x_1]=-x_4, [x_2,x_2]=x_4, [x_3,x_3]=x_5$.

$\mathcal{A}_{13}(\alpha,\beta)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=-x_4+\beta x_5, [x_2,x_2]=x_4, [x_3,x_3]=x_5, \quad \alpha,\beta\in\mathbb{C}$.

$\mathcal{A}_{14}$: $[x_1,x_2]=x_4, [x_2,x_3]=x_5$.

$\mathcal{A}_{15}$: $[x_1,x_1]=x_5, [x_1,x_2]=x_4, [x_2,x_3]=x_5$.

$\mathcal{A}_{16}(\alpha)$: $[x_1,x_2]=\alpha x_4, [x_2,x_1]=x_4, [x_2,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_{17}(\alpha)$: $[x_1,x_1]=x_5, [x_1,x_2]=\alpha x_4, [x_2,x_1]=x_4, [x_2,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{-1\}$.

$\mathcal{A}_{18}$: $[x_1,x_1]=x_4, [x_1,x_2]=x_5, [x_2,x_3]=x_5$.

$\mathcal{A}_{19}$: $[x_1,x_1]=x_4, [x_1,x_2]=x_4, [x_2,x_3]=x_5$.

$\mathcal{A}_{20}$: $[x_1,x_1]=x_4, [x_1,x_2]=x_4+x_5, [x_2,x_3]=x_5$.

$\mathcal{A}_{21}$: $[x_1,x_1]=x_4, [x_1,x_2]=x_4+x_5, [x_2,x_1]=x_4, [x_2,x_3]=x_5$.

$\mathcal{A}_{22}(\alpha)$: $[x_1,x_1]=x_4, [x_1,x_2]=\alpha x_4, [x_2,x_1]=x_4, [x_2,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0,1\}$.

$\mathcal{A}_{23}(\alpha)$: $[x_1,x_1]=x_4, [x_1,x_2]=\alpha x_4+x_5, [x_2,x_1]=x_4, [x_2,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0,1\}$.

$\mathcal{A}_{24}(\alpha)$: $[x_1,x_2]=x_4, [x_2,x_2]=\alpha x_5, [x_2,x_3]=x_5, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}$.

$\mathcal{A}_{25}(\alpha)$: $[x_1,x_2]=x_4, [x_2,x_1]=x_5, [x_2,x_2]=\alpha x_5, [x_2,x_3]=x_5, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}$.

$\mathcal{A}_{26}(\alpha,\beta)$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_5,[x_2,x_2]=\beta x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha,\beta\in\mathbb{C}.$

$\mathcal{A}_{27}(\alpha,\beta)$**:** $[x_1,x_2]=\alpha x_4,[x_2,x_1]=x_4,[x_2,x_2]=\beta x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1,0\},\beta\in\mathbb{C}.$

$\mathcal{A}_{28}(\alpha,\beta)$**:** $[x_1,x_2]=\alpha x_4+x_5,[x_2,x_1]=x_4,[x_2,x_2]=\beta x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1\},\beta\in\mathbb{C}.$

$\mathcal{A}_{29}(\alpha,\beta,\gamma)$**:** $[x_1,x_1]=x_5,[x_1,x_2]=\alpha x_4+\beta x_5,[x_2,x_1]=x_4,[x_2,x_2]=\gamma x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1\},\beta,\gamma\in\mathbb{C}.$

$\mathcal{A}_{30}(\alpha)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=x_5,[x_2,x_2]=\alpha x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0\}.$

$\mathcal{A}_{31}$**:** $[x_1,x_1]=x_4,[x_2,x_1]=x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5.$

$\mathcal{A}_{32}(\alpha,\beta,\gamma)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=x_4+\alpha x_5,[x_2,x_1]=\beta x_5,[x_2,x_2]=\gamma x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha,\beta,\gamma\in\mathbb{C}.$

$\mathcal{A}_{33}(\alpha,\beta,\gamma)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=x_4+\alpha x_5,[x_2,x_1]=x_4+\beta x_5,[x_2,x_2]=\gamma x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha,\beta,\gamma\in\mathbb{C},(\alpha,\beta)\neq(0,0).$

$\mathcal{A}_{34}(\alpha,\beta,\gamma,\theta)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=\alpha x_4+\beta x_5,[x_2,x_1]=x_4+\gamma x_5,[x_2,x_2]=\theta x_5,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0,1\},\beta,\gamma,\theta\in\mathbb{C},(\beta,\gamma,\theta)\neq(0,0,0).$

$\mathcal{A}_{35}$**:** $[x_1,x_2]=x_5,[x_2,x_2]=x_4,[x_2,x_3]=x_5.$

$\mathcal{A}_{36}$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_4,[x_2,x_3]=x_5.$

$\mathcal{A}_{37}$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_4,[x_2,x_3]=x_5.$

$\mathcal{A}_{38}$**:** $[x_1,x_1]=x_5,[x_2,x_1]=ix_5,[x_2,x_2]=x_4,[x_2,x_3]=x_5,[x_3,x_3]=x_5.$

$\mathcal{A}_{39}(\alpha,\beta)$**:** $[x_1,x_1]=\alpha x_5,[x_1,x_2]=x_4,[x_2,x_1]=-x_4+\beta x_5,[x_2,x_2]=x_4,[x_2,x_3]=x_5,[x_3,x_3]=x_5,\quad \alpha,\beta\in\mathbb{C}.$

$\mathcal{A}_{40}(\alpha)$**:** $[x_1,x_1]=x_5,[x_2,x_1]=x_4,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{41}$**:** $[x_1,x_1]=x_5,[x_2,x_1]=x_4,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2].$

$\mathcal{A}_{42}(\alpha)$**:** $[x_2,x_1]=x_4,[x_1,x_3]=x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{43}(\alpha)$**:** $[x_2,x_1]=x_4,[x_2,x_2]=x_5,[x_1,x_3]=x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{44}(\alpha,\beta)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{45}(\alpha,\beta)$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1\},\beta\in\mathbb{C}.$

$\mathcal{A}_{46}(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4+x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{47}(\alpha)$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{48}(\alpha,\beta)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_1,x_3]=x_5,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1\},\beta\in\mathbb{C}.$

$\mathcal{A}_{49}(\alpha,\beta)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4+x_5,[x_1,x_3]=x_5,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{50}(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{51}(\alpha)$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{52}(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_2,x_2]=x_5,[x_1,x_3]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{53}(\alpha)$**:** $[x_1,x_1]=x_4,[x_1,x_3]=x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0,1\}.$

$\mathcal{A}_{54}$**:** $[x_1,x_1]=x_4,[x_2,x_1]=x_5,[x_1,x_3]=x_5,[x_3,x_2]=x_5.$

$\mathcal{A}_{55}(\alpha,\beta)$**:** $[x_1,x_1]=x_4,[x_2,x_1]=x_4,[x_1,x_3]=\alpha x_5,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0\},\beta\in\mathbb{C}.$

$\mathcal{A}_{56}$**:** $[x_1,x_1]=x_4,[x_2,x_1]=x_4+x_5,[x_2,x_3]=x_5=-[x_3,x_2].$

$\mathcal{A}_{57}(\alpha,\beta)$**:** $[x_1,x_1]=x_4,[x_2,x_1]=x_4+x_5,[x_1,x_3]=\alpha x_5,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C},\beta\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{58}(\alpha,\beta,\gamma)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4,[x_1,x_3]=\beta x_5,[x_2,x_3]=\gamma x_5,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C}\backslash\{0\},\gamma\in\mathbb{C}.$

$\mathcal{A}_{59}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4+x_5,[x_1,x_3]=x_5,[x_3,x_2]=x_5.$

$\mathcal{A}_{60}(\alpha,\beta)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4+x_5,[x_2,x_1]=\alpha x_4,[x_1,x_3]=(1+\beta)x_5,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C},\alpha(1+\beta)\neq\beta.$

$\mathcal{A}_{61}(\alpha,\beta)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4+x_5,[x_2,x_1]=x_4,[x_1,x_3]=\alpha x_5,[x_2,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C},\alpha\neq 1+\beta,\alpha\neq\beta.$

$\mathcal{A}_{62}(\alpha,\beta)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4+x_5,[x_2,x_1]=\alpha x_4,[x_1,x_3]=\beta x_5,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C}\backslash\{0,1\}.$

$\mathcal{A}_{63}(\alpha,\beta,\gamma)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4+x_5,[x_2,x_1]=\alpha x_4,[x_1,x_3]=\beta x_5,[x_2,x_3]=\gamma x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0,1\},\beta\in\mathbb{C},\gamma\in\mathbb{C}\backslash\{0\},\beta\neq 1+\gamma,\alpha\beta\neq\gamma,(\alpha+1)\beta\neq\gamma.$

$\mathcal{A}_{64}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4+\frac{1}{\alpha}x_5,[x_2,x_1]=x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1,0\}.$

$\mathcal{A}_{65}$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=x_4+x_5,[x_3,x_2]=x_5.$

$\mathcal{A}_{66}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=x_4+x_5,[x_1,x_3]=\alpha x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0\}.$

$\mathcal{A}_{67}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4+x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0,1\}.$

$\mathcal{A}_{68}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=\frac{\alpha}{1+\alpha}x_4+x_5,[x_1,x_3]=(1+\alpha)x_5,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{-1,0,1\}.$

$\mathcal{A}_{69}$: $[x_1,x_1]=x_4,[x_2,x_1]=x_5,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2].$

$\mathcal{A}_{70}(\alpha)$: $[x_1,x_1]=x_4,[x_2,x_1]=x_4+\alpha x_5,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{0,1\}.$

$\mathcal{A}_{71}(\alpha)$: $[x_1,x_1]=x_4,[x_2,x_1]=x_4+\alpha x_5,[x_2,x_2]=x_5,[x_1,x_3]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{0\}.$

$\mathcal{A}_{72}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=x_4+\alpha x_5,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{0,\frac{1}{2}\}.$

$\mathcal{A}_{73}(\alpha)$: $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=x_4+\alpha x_5,[x_2,x_2]=x_5,[x_1,x_3]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{\frac{1}{2}\}.$

$\mathcal{A}_{74}(\alpha,\beta)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4+\beta x_5,[x_2,x_2]=x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{0,1\},\beta\in\mathbb{C}.$

$\mathcal{A}_{75}(\alpha)$**:** $[x_1,x_1]=x_4,[x_1,x_2]=x_4,[x_2,x_1]=\frac{\alpha-1}{\alpha}x_4,[x_2,x_2]=x_5,[x_1,x_3]=\alpha x_5,[x_2,x_3]=x_5=-[x_3,x_2],\quad \alpha\in\mathbb{C}\backslash\{0,1\}.$

$\mathcal{A}_{76}(\alpha)$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_4,[x_2,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{77}(\alpha)$**:** $[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_4,[x_1,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0\}.$

$\mathcal{A}_{78}(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=-x_4+x_5,[x_2,x_2]=x_4,[x_1,x_3]=\alpha x_5,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}\backslash\{0,1\}.$

$\mathcal{A}_{79}$**:** $[x_1,x_1]=x_5,[x_1,x_2]=x_4=-[x_2,x_1],[x_2,x_2]=x_4,[x_1,x_3]=x_5,[x_3,x_2]=x_5.$

$\mathcal{A}_{80}$**:** $[x_1,x_2]=x_4,[x_2,x_1]=x_5,[x_2,x_3]=x_4.$

$\mathcal{A}_{81}(\alpha,\beta)$**:** $[x_1,x_2]=x_4+\alpha x_5,[x_2,x_1]=\beta x_4,[x_2,x_2]=x_5,[x_1,x_3]=x_5,[x_2,x_3]=x_4,\quad \alpha\in\mathbb{C},\beta\in\mathbb{C}\backslash\{0\}.$

$\mathcal{A}_{82}(\alpha)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_4+(\alpha+1)x_5,[x_2,x_3]=x_4,[x_3,x_2]=x_5,\quad \alpha\in\mathbb{C}.$

$\mathcal{A}_{83}$**:** $[x_1,x_2]=x_4+2x_5,[x_2,x_1]=-x_5,[x_2,x_2]=x_5,[x_2,x_3]=x_4,[x_3,x_2]=x_5.$

$\mathcal{A}_{84}(\alpha,\beta)$**:** $[x_1,x_2]=x_4,[x_2,x_1]=\alpha x_5,[x_2,x_2]=\beta x_5,[x_1,x_3]=x_5,[x_2,x_3]=x_4,[x_3,x_2]=x_5,\quad \alpha,\beta\in\mathbb{C}.$

$\mathcal{A}_{85}(\alpha,\beta,\gamma)$**:** $[x_1,x_2]=x_4+\alpha x_5,[x_2,x_1]=\beta x_4,[x_2,x_2]=\gamma x_5,[x_1,x_3]=x_5,[x_2,x_3]=x_4,[x_3,x_2]=x_5,\quad \alpha,\gamma\in\mathbb{C},\beta\in\mathbb{C}\backslash\{0\}.$

$\mathcal{A}_{86}$**:** $[x_1,x_2]=x_4,[x_1,x_3]=x_5,[x_3,x_1]=x_5,[x_2,x_3]=x_4.$

$\mathcal{A}_{86}$**:** $[x_1,x_2]=x_4+x_5,[x_1,x_3]=-x_5,[x_3,x_1]=x_5,[x_2,x_3]=x_4.$

$\mathcal{A}_{87}(\alpha,\beta)$**:** $[x_1,x_2]=x_4+x_5,[x_2,x_1]=\alpha x_4,[x_1,x_3]=\beta x_5,[x_3,x_1]=x_5,[x_2,x_3]=x_4,\quad \alpha\in\mathbb{C},\beta\in\mathbb{C}\backslash\{-1\}.$

$\mathcal{A}_{88}(\alpha,\beta)$**:** $[x_1,x_2]=x_4+\alpha x_5,[x_2,x_1]=\beta x_4+x_5,[x_3,x_1]=x_5,[x_2,x_3]=x_4,\quad \alpha\in\mathbb{C}\backslash\{0\},\beta\in\mathbb{C},\alpha\beta\neq 1.$

$\mathcal{A}_{89}(\alpha,\beta,\gamma)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4+x_5, [x_1,x_3]=\gamma x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha,\beta\in\mathbb{C}, \gamma\in\mathbb{C}\backslash\{0\}, \alpha\neq\gamma$.

$\mathcal{A}_{90}(\alpha)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=-x_4+\frac{-1-\alpha^2}{\alpha}x_5, [x_2,x_2]=x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha\in\mathbb{C}\backslash\{0\}$.

$\mathcal{A}_{91}(\alpha,\beta,\gamma)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4+\gamma x_5, [x_2,x_2]=x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha,\beta,\gamma\in\mathbb{C}, \alpha\gamma-\alpha^2\beta+1\neq 0$.

$\mathcal{A}_{92}(\alpha,\beta,\gamma)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4, [x_2,x_2]=x_5, [x_1,x_3]=\gamma x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha,\beta\in\mathbb{C}, \gamma\in\mathbb{C}\backslash\{0\}$.

$\mathcal{A}_{93}(\alpha,\beta)$: $[x_1,x_2]=x_4-\frac{2}{\alpha}x_5, [x_2,x_1]=\beta x_4+\alpha x_5, [x_2,x_2]=x_5, [x_1,x_3]=-\frac{1}{\alpha^2}x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha\in\mathbb{C}\backslash\{-1,0,1\}, \beta\in\mathbb{C}\backslash\{0\}$.

$\mathcal{A}_{94}(\alpha,\beta,\gamma)$: $[x_1,x_2]=x_4+2\beta\gamma x_5, [x_2,x_1]=\alpha x_4+\beta x_5, [x_2,x_2]=x_5, [x_1,x_3]=\gamma x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha\in\mathbb{C}, \beta,\gamma\in\mathbb{C}\backslash\{0\}, \beta^2\gamma\neq -1$.

$\mathcal{A}_{95}(\alpha,\beta)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_5, [x_2,x_2]=x_5, [x_1,x_3]=-x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha\in\mathbb{C}, \beta\in\mathbb{C}\backslash\{0\}, \alpha\neq -2\beta, \beta^2+\alpha\beta+1\neq 0$.

$\mathcal{A}_{96}(\alpha,\beta,\gamma,\theta)$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4+\gamma x_5, [x_2,x_2]=x_5, [x_1,x_3]=\theta x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4, \quad \alpha,\beta\in\mathbb{C}, \gamma\in\mathbb{C}\backslash\{0\}, \theta\in\mathbb{C}\backslash\{-1,0\}, \alpha\neq 2\gamma\theta$.

$\mathcal{A}_{97}(\alpha)$: $[x_1,x_2]=x_4, [x_1,x_3]=\alpha x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0\}$.

$\mathcal{A}_{98}(\alpha)$: $[x_1,x_2]=x_4+x_5, [x_2,x_1]=\alpha x_4, [x_1,x_3]=-x_5, [x_3,x_1]=\alpha x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0\}$.

$\mathcal{A}_{99}(\alpha,\beta)$: $[x_1,x_2]=x_4+x_5, [x_2,x_1]=\alpha x_4, [x_1,x_3]=\beta x_5, [x_3,x_1]=\alpha x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0,1\}, \beta\in\mathbb{C}\backslash\{-1,0\}, \alpha\neq\beta$.

$\mathcal{A}_{100}(\alpha,\beta)$: $[x_1,x_2]=x_4+x_5, [x_2,x_1]=\alpha x_4, [x_1,x_3]=\alpha x_5, [x_3,x_1]=\beta x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha\in\mathbb{C}\backslash\{0\}, \alpha\neq\beta$.

$\mathcal{A}_{101}(\alpha,\beta,\gamma)$: $[x_1,x_2]=x_4+x_5, [x_2,x_1]=\alpha x_4, [x_1,x_3]=\beta x_5, [x_3,x_1]=\gamma x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha,\beta,\gamma\in\mathbb{C}, \alpha\neq\beta, \alpha\neq\gamma$.

$\mathcal{R}_1$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4+x_5, [x_1,x_3]=\gamma x_5, [x_3,x_1]=\theta x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha,\beta,\gamma,\theta\in\mathbb{C}$.

$\mathcal{R}_2$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4+\gamma x_5, [x_1,x_3]=\theta x_5, [x_3,x_1]=\delta x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha,\beta,\gamma,\theta,\delta\in\mathbb{C}.$

$\mathcal{R}_3$: $[x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_4+\gamma x_5, [x_2,x_2]=\theta x_5, [x_1,x_3]=\delta x_5, [x_3,x_1]=\lambda x_5, [x_2,x_3]=x_4, [x_3,x_3]=x_5, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda\in\mathbb{C}.$

$\mathcal{R}_4$: $[x_1,x_1]=x_5, [x_1,x_2]=x_4+\alpha x_5, [x_2,x_1]=\beta x_5, [x_2,x_2]=\gamma x_5, [x_1,x_3]=\theta x_5=-[x_3,x_1], [x_2,x_3]=x_4, [x_3,x_2]=\delta x_5, \quad \alpha,\beta,\gamma,\theta,\delta\in\mathbb{C}.$

$\mathcal{R}_5$: $[x_1,x_1]=x_4, [x_1,x_2]=\alpha x_5, [x_2,x_1]=\beta x_5, [x_2,x_2]=\gamma x_5, [x_1,x_3]=\theta x_5, [x_3,x_1]=\delta x_5, [x_2,x_3]=x_4, [x_3,x_2]=\lambda x_5, [x_3,x_3]=\mu x_5, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda,\mu\in\mathbb{C}.$

$\mathcal{R}_6$: $[x_1,x_1]=x_4+x_5, [x_1,x_2]=\alpha x_5, [x_2,x_1]=\beta x_5, [x_2,x_2]=\gamma x_5, [x_1,x_3]=\theta x_5=-[x_3,x_1], [x_2,x_3]=x_4, [x_3,x_2]=\delta x_5, \quad \alpha,\beta,\gamma,\theta,\delta\in\mathbb{C}.$

$\mathcal{R}_7$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_1]=\theta x_5, [x_2,x_2]=\delta x_5, [x_1,x_3]=\lambda x_5, [x_2,x_3]=x_4=-[x_3,x_2], \quad \alpha,\beta,\gamma,\theta,\delta,\lambda\in\mathbb{C}.$

$\mathcal{R}_8$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_1]=\theta x_5, [x_1,x_3]=\delta x_5, [x_2,x_3]=x_4+x_5, [x_3,x_2]=-x_4, \quad \alpha,\beta,\gamma,\theta,\delta\in\mathbb{C}.$

$\mathcal{R}_9$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_2]=\theta x_5, [x_1,x_3]=\delta x_5, [x_3,x_1]=x_5, [x_2,x_3]=x_4+\lambda x_5, [x_3,x_2]=-x_4, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda\in\mathbb{C}.$

$\mathcal{R}_{10}$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_1]=\theta x_5, [x_1,x_3]=\delta x_5, [x_3,x_1]=\lambda x_5, [x_2,x_3]=x_4+\mu x_5, [x_3,x_2]=-x_4, [x_3,x_3]=x_5, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda,\mu\in\mathbb{C}.$

$\mathcal{R}_{11}$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_1]=\theta x_5, [x_2,x_2]=\delta x_5, [x_2,x_3]=\lambda x_4+\mu x_5, [x_3,x_2]=x_4, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda,\mu\in\mathbb{C}.$

$\mathcal{R}_{12}$: $[x_1,x_1]=x_4, [x_1,x_2]=\alpha x_4+\beta x_5, [x_2,x_1]=\gamma x_5, [x_2,x_2]=\theta x_5, [x_1,x_3]=x_5, [x_2,x_3]=\delta x_4+\lambda x_5, [x_3,x_2]=x_4, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda\in\mathbb{C}.$

$\mathcal{R}_{13}$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_1]=\theta x_5, [x_2,x_2]=\delta x_5, [x_1,x_3]=\lambda x_5, [x_3,x_1]=x_5, [x_2,x_3]=\mu x_4+\omega x_5, [x_3,x_2]=x_4, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda,\mu,\omega\in\mathbb{C}.$

$\mathcal{R}_{14}$: $[x_1,x_1]=x_4+\alpha x_5, [x_1,x_2]=\beta x_4+\gamma x_5, [x_2,x_1]=\theta x_5, [x_2,x_2]=\delta x_5, [x_1,x_3]=\lambda x_5, [x_2,x_3]=\mu x_4+\omega x_5, [x_3,x_2]=x_4, [x_3,x_3]=x_5, \quad \alpha,\beta,\gamma,\theta,\delta,\lambda,\mu,\omega\in\mathbb{C}.$

$\mathcal{R}_{15}$**:** $[x_1, x_1] = \alpha x_5, [x_1, x_2] = \beta x_4 + \gamma x_5, [x_2, x_1] = x_4 + \theta x_5, [x_2, x_2] = \delta x_5, [x_1, x_3] = x_4 + \lambda x_5, [x_3, x_1] = \mu x_5, [x_2, x_3] = \omega x_5, [x_3, x_2] = x_4, [x_3, x_3] = \varphi x_5, \quad \alpha, \beta, \gamma, \theta, \delta, \lambda, \mu, \omega, \varphi \in \mathbb{C}$.

Proof. Let $Leib(A) = A^2 = Z(A) = \text{span}\{\mathrm{e}_4, \mathrm{e}_5\}$. Then the nonzero products in $A = \text{span}\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \mathrm{e}_4, \mathrm{e}_5\}$ are given by:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_3 e_4 + \beta_4 e_5, [e_3, e_1] = \beta_5 e_4 + \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_3 e_4 + \gamma_4 e_5,$$
$$[e_3, e_3] = \gamma_5 e_4 + \gamma_6 e_5.$$

Without loss of generality we can assume $\gamma_5 = 0$ because if $\gamma_5 \neq 0$ and $\beta_1 = 0$(resp. $\beta_1 \neq 0$) then with the base change $x_1 = e_1, x_2 = e_3, x_3 = e_2, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = e_2, x_3 = e_2 + x e_3, x_4 = e_{4,5} = e_5$ where $\gamma_5 x^2 + (\gamma_1 + \gamma_3)x + \beta_1 = 0$) we can make $\gamma_5 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_3 e_4 + \beta_4 e_5, [e_3, e_1] = \beta_5 e_4 + \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_3 e_4 + \gamma_4 e_5,$$
$$[e_3, e_3] = \gamma_6 e_5.$$

Note that if $\beta_5 \neq 0$ and $\gamma_3 = 0$(resp. $\gamma_3 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = \gamma_3 e_1 - \beta_5 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_{4,5} = e_5$) we can make $\beta_5 = 0$. So we can assume $\beta_5 = 0$. Then the products in A are the following:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_3 e_4 + \beta_4 e_5, [e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_3 e_4 + \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1: Let $\gamma_3 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_3 e_4 + \beta_4 e_5, [e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5,$$
$$[e_3, e_3] = \gamma_6 e_5. \quad (4.46)$$

If $\beta_3 \neq 0$ and $\gamma_1 = 0$(resp. $\gamma_1 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = \gamma_1 e_1 - \beta_3 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_3 = 0$. So let $\beta_3 = 0$. Then the products in A are the following:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1.1: Let $\gamma_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5. \tag{4.47}$$

Note that if $\beta_6 \neq 0$ and $\gamma_4 = 0$(resp. $\gamma_4 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = \gamma_4 e_1 - \beta_6 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_6 = 0$. So we can assume $\beta_6 = 0$. Then the products in A are the following:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1.1.1: Let $\gamma_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_3] = \gamma_6 e_5. \tag{4.48}$$

Without loss of generality we can assume $\beta_4 = 0$ because if $\beta_4 \neq 0$ and $\gamma_2 = 0$(resp. $\gamma_2 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = \gamma_2 e_1 - \beta_4 e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$) we can make $\beta_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_2, e_3] = \gamma_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1.1.1.1: Let $\gamma_2 = 0$. Then $\gamma_6 \neq 0$ since $\dim(Z(A)) = 2$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_3, e_3] = \gamma_6 e_5. \quad (4.49)$$

Note that if $\alpha_1 \neq 0$ and $\beta_1 = 0$(resp. $\beta_1 \neq 0$) then with the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$(resp. $x_1 = xe_1 + e_2, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ where $\alpha_1 x^2 + (\alpha_3 + \alpha_5)x + \beta_1 = 0$) we can make $\alpha_1 = 0$. So we can assume $\alpha_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1.1.1.1.1: Let $\beta_1 = 0$. Then $\alpha_3 + \alpha_5 \neq 0$ since $\dim(Leib(A)) = 2$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5,$$
$$[e_3, e_3] = \gamma_6 e_5. \quad (4.50)$$

Case 1.1.1.1.1.1: Let $\alpha_5 = 0$. Then $\alpha_3 \neq 0$ since $\dim(A^2) = 2$. Hence we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_3, e_3] = \gamma_6 e_5. \quad (4.51)$$

- If $\alpha_2 = 0, \beta_2 = 0$ then $\alpha_6 \neq 0$ since A is non-split. Then the base change $x_1 = e_1, x_2 = \frac{\gamma_6}{\alpha_6} e_2, x_3 = e_3, x_4 = \frac{\gamma_6}{\alpha_6}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_4(0)$.
- If $\alpha_2 = 0, \beta_2 \neq 0$ and $\alpha_6 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \sqrt{\frac{\beta_2}{\gamma_6}} e_3, x_4 = \alpha_3 e_4 + \alpha_4 e_5, x_5 = \beta_2 e_5$ shows that A is isomorphic to $\mathcal{A}_5(0)$.
- If $\alpha_2 = 0, \beta_2 \neq 0$ and $\alpha_6 \neq 0$ then the base change $x_1 = \beta_2 e_1, x_2 = \alpha_6 e_2, x_3 = \sqrt{\frac{\alpha_6^2 \beta_2}{\gamma_6}} e_3, x_4 = \alpha_6 \beta_2 (\alpha_3 e_4 + \alpha_4 e_5), x_5 = \alpha_6^2 \beta_2 e_5$ shows that A is isomorphic to $\mathcal{A}_6(0)$.

- If $\alpha_2 \neq 0, \beta_2 = 0$ and $\alpha_6 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \sqrt{\frac{\alpha_2}{\gamma_6}}e_3, x_4 = \alpha_3 e_4 + \alpha_4 e_5, x_5 = \alpha_2 e_5$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\alpha_2 \neq 0, \beta_2 = 0$ and $\alpha_6 \neq 0$ then the base change $x_1 = \alpha_6 e_1, x_2 = \alpha_2 e_2, x_3 = \sqrt{\frac{\alpha_2\alpha_6^2}{\gamma_6}}e_3, x_4 = \alpha_2\alpha_6(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \alpha_2\alpha_6^2 e_5$ shows that A is isomorphic to $\mathcal{A}_2$.

- If $\alpha_2 \neq 0$ and $\beta_2 \neq 0$ then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_2}{\beta_2}}e_2, x_3 = \sqrt{\frac{\alpha_2}{\gamma_6}}e_3, x_4 = \sqrt{\frac{\alpha_2}{\beta_2}}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \alpha_2 e_5$ shows that A is isomorphic to $\mathcal{A}_3(\alpha)$.

Case 1.1.1.1.1.2: Let $\alpha_5 \neq 0$. If $\alpha_3 = 0$ then the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.51). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3(\alpha), \mathcal{A}_4(\alpha), \mathcal{A}_5(\alpha)$ or $\mathcal{A}_6(\alpha)$. Then suppose $\alpha_3 \neq 0$.

Case 1.1.1.1.1.2.1: Let $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_3, e_3] = \gamma_6 e_5. \tag{4.52}$$

- If $\beta_2 = 0$ then $\alpha_3\alpha_6 - \alpha_4\alpha_5 \neq 0$ since A is non-split. Then the base change $x_1 = e_1, x_2 = \frac{\alpha_3\gamma_6}{\alpha_3\alpha_6 - \alpha_4\alpha_5}e_2, x_3 = e_3, x_4 = \frac{\alpha_3\gamma_6}{\alpha_3\alpha_6 - \alpha_4\alpha_5}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_4(\alpha)(\alpha \in \mathbb{C}\backslash\{-1, 0\})$.

- If $\beta_2 \neq 0$ and $\alpha_3\alpha_6 - \alpha_4\alpha_5 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \sqrt{\frac{\beta_2}{\gamma_6}}e_3, x_4 = \alpha_3 e_4 + \alpha_4 e_5, x_5 = \beta_2 e_5$ shows that A is isomorphic to $\mathcal{A}_5(\alpha)(\alpha \in \mathbb{C}\backslash\{-1, 0\})$.

- If $\beta_2 \neq 0$ and $\alpha_3\alpha_6 - \alpha_4\alpha_5 \neq 0$ then the base change $x_1 = \beta_2 e_1, x_2 = \frac{\alpha_3\alpha_6 - \alpha_4\alpha_5}{\alpha_3}e_2, x_3 = \frac{\alpha_3\alpha_6 - \alpha_4\alpha_5}{\alpha_3}\sqrt{\frac{\beta_2}{\gamma_6}}e_3, x_4 = \frac{(\alpha_3\alpha_6 - \alpha_4\alpha_5)\beta_2}{\alpha_3}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \beta_2(\frac{\alpha_3\alpha_6 - \alpha_4\alpha_5}{\alpha_3})^2 e_5$ shows that A is isomorphic to $\mathcal{A}_6(\alpha)(\alpha \in \mathbb{C}\backslash\{-1, 0\})$.

Case 1.1.1.1.1.2.2: Let $\alpha_2 \neq 0$. If $\beta_2 = 0$ then the base change $x_1 = e_2, x_2 = e_1, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.52). Hence A is isomorphic to $\mathcal{A}_4(\alpha), \mathcal{A}_5(\alpha)$ or $\mathcal{A}_6(\alpha)$. So let $\beta_2 \neq 0$. Then the base change $x_1 = e_1, x_2 = \sqrt{\frac{\alpha_2}{\beta_2}}e_2, x_3 = \sqrt{\frac{\alpha_2}{\gamma_6}}e_3, x_4 = \sqrt{\frac{\alpha_2}{\beta_2}}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \alpha_2 e_5$ shows that A is isomorphic to $\mathcal{A}_7(\alpha, \beta)$.

Case 1.1.1.1.2: Let $\beta_1 \neq 0$. If $\alpha_3 + \alpha_5 \neq 0$ then the base change $x_1 = e_1, x_2 = \beta_1 e_1 - (\alpha_3 + \alpha_5)e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.50). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3(\alpha), \mathcal{A}_4(\alpha), \mathcal{A}_5(\alpha)$, $\mathcal{A}_6(\alpha)$ or $\mathcal{A}_7(\alpha, \beta)$. So let $\alpha_3 + \alpha_5 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_3 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1.1.1.1.2.1: Let $\alpha_3 = 0$.

Case 1.1.1.1.2.1.1: Let $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

- If $\alpha_6 = 0$ then $\alpha_4 \neq 0$ since A is non-split. Then the base change $x_1 = \frac{\gamma_6}{\alpha_4} e_1, x_2 = e_2, x_3 = e_3, x_4 = \beta_1 e_4 + \beta_2 e_5, x_5 = \gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_8$.
- If $\alpha_6 \neq 0$ then the base change $x_1 = \frac{\gamma_6}{\alpha_6} e_1, x_2 = e_2, x_3 = e_3, x_4 = \beta_1 e_4 + \beta_2 e_5, x_5 = \gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_9(\alpha)$.

Case 1.1.1.1.2.1.2: Let $\alpha_2 \neq 0$. Without loss of generality, we can assume $\alpha_6 = 0$, because if $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_6 e_1 - \alpha_2 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Note that $\alpha_4 \neq 0$ since A is non-split. Then the base change $x_1 = \alpha_4 e_1, x_2 = \alpha_2 e_2, x_3 = \sqrt{\frac{\alpha_2 \alpha_4^2}{\gamma_6}} e_3, x_4 = \alpha_2^2(\beta_1 e_4 + \beta_2 e_5), x_5 = \alpha_2 \alpha_4^2 e_5$ shows that A is isomorphic to $\mathcal{A}_{10}$.

Case 1.1.1.1.2.2: Let $\alpha_3 \neq 0$. Take $\theta_1 = \frac{\alpha_4 \beta_1 - \beta_2 \alpha_3}{\alpha_3 \gamma_6}$ and $\theta_2 = \frac{\alpha_6 \beta_1 + \beta_2 \alpha_3}{\alpha_3 \gamma_6}$. The base change $y_1 = \frac{\beta_1}{\alpha_3} e_1, y_2 = e_2, y_3 = e_3, y_4 = \beta_1 e_4 + \beta_2 e_5, y_5 = \gamma_6 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \frac{\alpha_2 \beta_1^2}{\alpha_3^2 \gamma_6} y_5, [y_1, y_2] = y_4 + \theta_1 y_5, [y_2, y_1] = -y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_3, y_3] = y_5.$$

- If $\alpha_2 = 0$ and $\theta_2 = -\theta_1$ then $\theta_1, \theta_2 \neq 0$ since A is non-split. Then the base change $x_1 = y_1, x_2 = y_2, x_3 = \sqrt{\theta_1} y_3, x_4 = y_4, x_5 = \theta_1 y_5$ shows that A is isomorphic to $\mathcal{A}_{11}$.
- If $\alpha_2 = 0$ and $\theta_2 \neq -\theta_1$ then the base change $x_1 = y_1, x_2 = \frac{-\theta_2}{\theta_1 + \theta_2} y_1 + y_2, x_3 = \sqrt{\theta_1 + \theta_2} y_3, x_4 = y_4 - \theta_2 y_5, x_5 = (\theta_1 + \theta_2) y_5$ shows that A is isomorphic to $\mathcal{A}_{12}$.

- If $\alpha_2 \neq 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = \sqrt{\frac{\alpha_2\beta_1^2}{\alpha_3^2\gamma_6}}y_3, x_4 = y_4, x_5 = \frac{\alpha_2\beta_1^2}{\alpha_3^2\gamma_6}y_5$ shows that A is isomorphic to $\mathcal{A}_{13}(\alpha,\beta)$.

Case 1.1.1.2: Let $\gamma_2 \neq 0$.

Case 1.1.1.2.1: Let $\beta_1 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1e_4 + \alpha_2e_5, [e_1,e_2] = \alpha_3e_4 + \alpha_4e_5, [e_2,e_1] = \alpha_5e_4 + \alpha_6e_5, [e_2,e_2] = \beta_2e_5,$$
$$[e_2,e_3] = \gamma_2e_5, [e_3,e_3] = \gamma_6e_5. \quad (4.53)$$

Case 1.1.1.2.1.1: Let $\gamma_6 = 0$. Note that if $\beta_2 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_2e_2 - \beta_2e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_2 = 0$. So we can assume $\beta_2 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1e_4 + \alpha_2e_5, [e_1,e_2] = \alpha_3e_4 + \alpha_4e_5, [e_2,e_1] = \alpha_5e_4 + \alpha_6e_5, [e_2,e_3] = \gamma_2e_5.$$

Case 1.1.1.2.1.1.1: Let $\alpha_1 = 0$. Then $\alpha_3 + \alpha_5 \neq 0$ since $\dim(Leib(A)) = 2$.

Case 1.1.1.2.1.1.1.1: Let $\alpha_5 = 0$. Without loss of generality we can assume $\alpha_6 = 0$ because if $\alpha_6 \neq 0$ then with the base change $x_1 = \gamma_2e_1 - \alpha_6e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2e_5, [e_1,e_2] = \alpha_3e_4 + \alpha_4e_5, [e_2,e_3] = \gamma_2e_5.$$

- If $\alpha_2 = 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = e_3, x_4 = \alpha_3e_4 + \alpha_4e_5, x_5 = \gamma_2e_5$ shows that A is isomorphic to $\mathcal{A}_{14}$.

- If $\alpha_2 \neq 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_2}{\gamma_2}e_3, x_4 = \alpha_3e_4 + \alpha_4e_5, x_5 = \alpha_2e_5$ shows that A is isomorphic to $\mathcal{A}_{15}$.

Case 1.1.1.2.1.1.1.2: Let $\alpha_5 \neq 0$. Take $\theta = \alpha_4\alpha_5 - \alpha_3\alpha_6$.

- If $\alpha_2 = 0$ and $\theta = 0$(resp. $\theta \neq 0$) then the base change $x_1 = e_1, x_2 = e_2, x_3 = e_3, x_4 = \alpha_5e_4 + \alpha_6e_5, x_5 = \gamma_2e_5$(resp. $x_1 = \frac{\alpha_3\gamma_2}{\theta}e_1 + e_3, x_2 = e_2, x_3 = e_3, x_4 = \frac{\alpha_3\alpha_5\gamma_2}{\theta}e_4 + (\frac{\alpha_3\alpha_6\gamma_2}{\theta} + \gamma_2)e_5, x_5 = \gamma_2e_5$) shows that A is isomorphic to $\mathcal{A}_{16}(\alpha)$.

- If $\alpha_2 \neq 0$ and $\theta = 0$(resp. $\theta \neq 0$) then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_2}{\gamma_2}e_3, x_4 = \alpha_5 e_4 + \alpha_6 e_5, x_5 = \alpha_2 e_5$(resp. $x_1 = \frac{\alpha_3\gamma_2}{\theta}e_1 + e_3, x_2 = e_2, x_3 = \frac{\alpha_2}{\gamma_2}(\frac{\alpha_3\gamma_2}{\theta})^2 e_3, x_4 = \frac{\alpha_3\alpha_5\gamma_2}{\theta}e_4 + (\frac{\alpha_3\alpha_6\gamma_2}{\theta} + \gamma_2)e_5, x_5 = \alpha_2(\frac{\alpha_3\gamma_2}{\theta})^2 e_5$) shows that A is isomorphic to $\mathcal{A}_{17}(\alpha)$.

Case 1.1.1.2.1.1.2: Let $\alpha_1 \neq 0$.

Case 1.1.1.2.1.1.2.1: Let $\alpha_5 = 0$. Note that if $\alpha_6 \neq 0$ then with the base change $x_1 = \gamma_2 e_1 - \alpha_6 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. So let $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_3] = \gamma_2 e_5. \tag{4.54}$$

- If $\alpha_3 = 0$ then $\alpha_4 \neq 0$ since A is non-split. Then the base change $x_1 = \gamma_2 e_1, x_2 = e_2, x_3 = \alpha_4 e_3, x_4 = \gamma_2^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \alpha_4\gamma_2 e_5$ shows that A is isomorphic to $\mathcal{A}_{18}$.

- If $\alpha_3 \neq 0$ and $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 0$ then the base change $x_1 = \alpha_3 e_1, x_2 = \alpha_1 e_2, x_3 = e_3, x_4 = \alpha_3^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \alpha_1\gamma_2 e_5$ shows that A is isomorphic to $\mathcal{A}_{19}$.

- If $\alpha_3 \neq 0$ and $\alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$ then the base change $x_1 = \alpha_3 e_1, x_2 = \alpha_1 e_2, x_3 = \frac{\alpha_3(\alpha_1\alpha_4 - \alpha_2\alpha_3)}{\alpha_1\gamma_2}e_3, x_4 = \alpha_3^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \alpha_3(\alpha_1\alpha_4 - \alpha_2\alpha_3)e_5$ shows that A is isomorphic to $\mathcal{A}_{20}$.

Case 1.1.1.2.1.1.2.2: Let $\alpha_5 \neq 0$. Take $\theta_1 = \alpha_5(\alpha_1\alpha_4 - \alpha_2\alpha_3)$ and $\theta_2 = \alpha_5(\alpha_1\alpha_6 - \alpha_2\alpha_5)$. Then the base change $y_1 = \alpha_5 e_1, y_2 = \alpha_1 e_2, y_3 = e_3, y_4 = \alpha_5^2(\alpha_1 e_4 + \alpha_2 e_5), y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = \frac{\alpha_3}{\alpha_5}y_4 + \theta_1 y_5, [y_2, y_1] = y_4 + \theta_2 y_5, [y_2, y_3] = \alpha_1\gamma_2 y_5.$$

Note that if $\theta_2 \neq 0$ then with the base change $x_1 = \alpha_1\gamma_2 y_1 - \theta_2 y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\theta_2 = 0$. So we can assume $\theta_2 = 0$. Then we have the following products in A:

$$[y_1, y_1] = y_4, [y_1, y_2] = \frac{\alpha_3}{\alpha_5}y_4 + \theta_1 y_5, [y_2, y_1] = y_4, [y_2, y_3] = \alpha_1\gamma_2 y_5.$$

- If $\alpha_3 = 0$ then the base change $x_1 = y_1, x_2 = y_1 - y_2 - \frac{\theta_1}{\alpha_1\gamma_2}y_3, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.54). Hence A is isomorphic to $\mathcal{A}_{18}, \mathcal{A}_{19}$ or $\mathcal{A}_{20}$.

- If $\alpha_3 \neq 0, \alpha_3 = \alpha_5$ and $\theta_1 = 0$ then the base change $x_1 = -iy_1 + iy_2, x_2 = -iy_3, x_3 = -y_1, x_4 = \alpha_1\gamma_2 y_5, x_5 = y_4$ shows that A is isomorphic to $\mathcal{A}_1$.

- If $\alpha_3 \neq 0, \alpha_3 = \alpha_5$ and $\theta_1 \neq 0$ then the base change $x_1 = \alpha_1\gamma_2 y_1, x_2 = \alpha_1\gamma_2 y_2, x_3 = \theta_1 y_3, x_4 = (\alpha_1\gamma_2)^2 y_4, x_5 = \theta_1(\alpha_1\gamma_2)^2 y_5$ shows that A is isomorphic to $\mathcal{A}_{21}$.

- If $\alpha_3 \neq 0, \alpha_3 \neq \alpha_5$ and $\theta_1 = 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = \alpha_1\gamma_2 y_5$ shows that A is isomorphic to $\mathcal{A}_{22}(\alpha)$.

- If $\alpha_3 \neq 0, \alpha_3 \neq \alpha_5$ and $\theta_1 \neq 0$ then the base change $x_1 = \alpha_1\gamma_2 y_1, x_2 = \alpha_1\gamma_2 y_2, x_3 = \theta_1 y_3, x_4 = (\alpha_1\gamma_2)^2 y_4, x_5 = \theta_1(\alpha_1\gamma_2)^2 y_5$ shows that A is isomorphic to $\mathcal{A}_{23}(\alpha)$.

Case 1.1.1.2.1.2: Let $\gamma_6 \neq 0$.

Case 1.1.1.2.1.2.1: Let $\alpha_1 = 0$. Then $\alpha_3 + \alpha_5 \neq 0$ since $\dim(Leib(A)) = 2$.

Case 1.1.1.2.1.2.1.1: Let $\alpha_5 = 0$. Then $\alpha_3 \neq 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

- If $\alpha_2 = 0$ and $\alpha_6 = 0$ then the base change $x_1 = e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = \gamma_6(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{24}(\alpha)$.

- If $\alpha_2 = 0$ and $\alpha_6 \neq 0$ then the base change $x_1 = \frac{\gamma_2^2}{\alpha_6} e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = \frac{\gamma_2^2\gamma_6}{\alpha_6}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{25}(\alpha)$.

- If $\alpha_2 \neq 0$ then the base change $x_1 = \sqrt{\frac{\gamma_2^2\gamma_6}{\alpha_2}} e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = \gamma_6\sqrt{\frac{\gamma_2^2\gamma_6}{\alpha_2}}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{26}(\alpha, \beta)$.

Case 1.1.1.2.1.2.1.2: Let $\alpha_5 \neq 0$. Take $\theta = \alpha_4\alpha_5 - \alpha_3\alpha_6$.

- If $\alpha_2 = 0, \theta = 0, \alpha_3 = 0$ and $\beta_2 = 0$ then the base change $x_1 = \gamma_2 e_3, x_2 = -\gamma_6 e_2 + \gamma_2 e_3, x_3 = -e_1, x_4 = \gamma_2^2\gamma_6 e_5, x_5 = \gamma_6(\alpha_5 e_4 + \alpha_6 e_5)$ shows that A is isomorphic to $\mathcal{A}_{19}$.

- If $\alpha_2 = 0, \theta = 0, \alpha_3 = 0$ and $\beta_2 \neq 0$ then the base change $x_1 = \frac{x\gamma_2+\gamma_6}{\gamma_6} e_3, x_2 = xe_2 + e_3, x_3 = e_1, x_4 = \frac{(x\gamma_2+\gamma_6)^2}{\gamma_6} e_5, x_5 = x(\alpha_5 e_4 + \alpha_6 e_5)$ (where $\beta_2 x^2 + \gamma_2 x + \gamma_6 = 0$) shows that A is isomorphic to $\mathcal{A}_{22}(\alpha)$.

- If $\alpha_2 = 0, \theta = 0$ and $\alpha_3 \neq 0$ then the base change $x_1 = e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = \gamma_6(\alpha_5 e_4 + \alpha_6 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{27}(\alpha, \beta)$.

- If $\alpha_2 = 0$ and $\theta \neq 0$ then the base change $x_1 = \frac{\alpha_5\gamma_2^2}{\theta}e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = \frac{\alpha_5\gamma_2^2\gamma_6}{\theta}(\alpha_5 e_4 + \alpha_6 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{28}(\alpha, \beta)$.

- If $\alpha_2 \neq 0$ then the base change $x_1 = \sqrt{\frac{\gamma_2^2\gamma_6}{\alpha_2}}e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = \gamma_6\sqrt{\frac{\gamma_2^2\gamma_6}{\alpha_2}}(\alpha_5 e_4 + \alpha_6 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{29}(\alpha, \beta, \gamma)$.

Case 1.1.1.2.1.2.2: Let $\alpha_1 \neq 0$.
Case 1.1.1.2.1.2.2.1: Let $\alpha_5 = 0$.

- If $\alpha_3 = 0, \alpha_6 = 0$ and $\beta_2 = 0$ then $\alpha_4 \neq 0$ since A is non-split. Then the base change $x_1 = \frac{-\gamma_2^2}{\alpha_4}e_1 - \gamma_6 e_2, x_2 = e_2, x_3 = \gamma_6 e_2 - \gamma_2 e_3, x_4 = \frac{\alpha_1\gamma_2^4}{\alpha_4^2}e_4 + (\frac{\alpha_2\gamma_2^4}{\alpha_4^2} + \gamma_2^2\gamma_6)e_5, x_5 = -\gamma_2^2 e_5$ shows that A is isomorphic to $\mathcal{A}_{18}$.

- If $\alpha_3 = 0, \alpha_6 = 0$ and $\beta_2 \neq 0$ then $\alpha_4 \neq 0$ since A is non-split. Then the base change $x_1 = \frac{\gamma_2^2}{\alpha_4}e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = (\frac{\gamma_2^2}{\alpha_4})^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{30}(\alpha)$.

- If $\alpha_3 = 0, \alpha_6 \neq 0$ and $(\alpha_4, \beta_2) = (0,0)$ then the base change $x_1 = e_1, x_2 = \frac{\alpha_6\gamma_6}{\gamma_2^2}e_2, x_3 = \frac{\alpha_6}{\gamma_2}e_3, x_4 = \alpha_1 e_4 + \alpha_2 e_5, x_5 = \frac{\alpha_6^2\gamma_6}{\gamma_2^2}e_5$ shows that A is isomorphic to $\mathcal{A}_{31}$.

- If $\alpha_3 = 0$ and $\alpha_6 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{30}(\alpha)$.

- If $\alpha_3 \neq 0$ then the base change $x_1 = \frac{\alpha_3\gamma_6}{\alpha_1}e_1, x_2 = \gamma_6 e_2, x_3 = \gamma_2 e_3, x_4 = (\frac{\alpha_3\gamma_6}{\alpha_1})^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \gamma_2^2\gamma_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{32}(\alpha, \beta, \gamma)$.

Case 1.1.1.2.1.2.2.2: Let $\alpha_5 \neq 0$. Take $\theta_1 = \frac{\alpha_5\gamma_6(\alpha_1\alpha_6 - \alpha_2\alpha_5)}{\alpha_1^2\gamma_2^2}$ and $\theta_2 = \frac{\alpha_5\gamma_6(\alpha_1\alpha_4 - \alpha_2\alpha_3)}{\alpha_1^2\gamma_2^2}$. The base change $y_1 = \frac{\alpha_5}{\alpha_1}e_1, y_2 = e_2, y_3 = \frac{\gamma_2}{\gamma_6}e_3, x_4 = (\frac{\alpha_5}{\alpha_1})^2(\alpha_1 e_4 + \alpha_2 e_5), x_5 = \frac{\gamma_2^2}{\gamma_6}e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = \frac{\alpha_3}{\alpha_5}y_4 + \theta_1 y_5, [y_2, y_1] = y_4 + \theta_2 y_5, [y_2, y_2] = \frac{\beta_2\gamma_6}{\gamma_2^2}y_5, [y_2, y_3] = y_5, [y_3, y_3] = y_5.$$

- If $\alpha_3 = 0$ then the base change $x_1 = y_1, x_2 = y_1 - y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{32}(\alpha, \beta, \gamma)$.

- If $\alpha_3 \neq 0, \alpha_3 = \alpha_5, \theta_1 = 0, \theta_2 = 0$ and $\frac{\beta_2\gamma_6}{\gamma_2^2} = 0$ then the base change $x_1 = -iy_1 + iy_2, x_2 = -iy_1 + iy_2 - iy_3, x_3 = -y_1, x_4 = y_4 + y_5, x_5 = y_4$ shows that A is isomorphic to $\mathcal{A}_3(1)$.

- If $\alpha_3 \neq 0, \alpha_3 = \alpha_5, \theta_1 = 0, \theta_2 = 0$ and $\frac{\beta_2\gamma_6}{\gamma_2^2} = \frac{1}{4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{13}(\alpha, \beta)((\alpha+\beta)^2 = 2(\alpha-\beta))$.

- If $\alpha_3 \neq 0, \alpha_3 = \alpha_5, \theta_1 = 0, \theta_2 = 0$ and $\frac{\beta_2\gamma_6}{\gamma_2^2} \in \mathbb{C}\backslash\{0, \frac{1}{4}\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_7(\alpha, \sqrt{(\alpha-1)^2})$.

- If $\alpha_3 \neq 0, \alpha_3 = \alpha_5$ and $(\theta_1, \theta_2) \neq (0,0)$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{33}(\alpha, \beta, \gamma)$.

- If $\alpha_3 \neq 0, \alpha_3 \neq \alpha_5$ and $\theta_1 = 0 = \theta_2 = \beta_2$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(0,0,\gamma)(\gamma \in \mathbb{C}\backslash\{0\})$.

- If $\alpha_3 \neq 0, \alpha_3 \neq \alpha_5$ and $(\theta_1, \theta_2, \frac{\beta_2\gamma_6}{\gamma_2^2}) \neq (0,0,0)$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{34}(\alpha, \beta, \gamma, \theta)$.

Case 1.1.1.2.2: Let $\beta_1 \neq 0$. If $(\alpha_1, \alpha_3 + \alpha_5) \neq (0,0)$ then the base change $x_1 = e_1, x_2 = e_1 + xe_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\beta_1 x^2 + (\alpha_3+\alpha_5)x + \alpha_1 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.53). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_3(\alpha), \mathcal{A}_7(\alpha,\beta), \mathcal{A}_{13}(\alpha,\beta), \mathcal{A}_{14}, \mathcal{A}_{15}, \mathcal{A}_{16}(\alpha), \mathcal{A}_{17}(\alpha), \mathcal{A}_{18}, \mathcal{A}_{19}, \mathcal{A}_{20}, \mathcal{A}_{21}, \mathcal{A}_{22}(\alpha), \mathcal{A}_{23}(\alpha), \mathcal{A}_{24}(\alpha), \mathcal{A}_{25}(\alpha), \mathcal{A}_{26}(\alpha,\beta), \mathcal{A}_{27}(\alpha,\beta), \mathcal{A}_{28}(\alpha,\beta), \mathcal{A}_{29}(\alpha,\beta,\gamma), \mathcal{A}_{30}(\alpha), \mathcal{A}_{31}, \mathcal{A}_{32}(\alpha,\beta,\gamma), \mathcal{A}_{33}(\alpha,\beta,\gamma)$ or $\mathcal{A}_{34}(\alpha,\beta,\gamma,\theta)$. So let $\alpha_1 = 0 = \alpha_3 + \alpha_5$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_3 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_2, e_3] = \gamma_2 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 1.1.1.2.2.1: Let $\gamma_6 = 0$. Then $\gamma_2 \neq 0$ since $\dim(Z(A)) = 2$. Take $\theta_1 = \frac{\alpha_4\beta_1 - \alpha_3\beta_2}{\beta_1\gamma_2}$ and $\theta_2 = \frac{\alpha_6\beta_1 + \alpha_3\beta_2}{\beta_1\gamma_2}$. Then the base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 = \beta_1 e_4 + \beta_2 e_5, y_5 = \gamma_2 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \frac{\alpha_2}{\gamma_2} y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4 + \theta_1 y_5, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_2, y_3] = y_5.$$

Note that if $\theta_2 \neq 0$ then with the base change $x_1 = \gamma_2 y_1 - \theta_2 y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 =$

y_5 we can make $\theta_2 = 0$. So let $\theta_2 = 0$. Then we have the following products in A:

$$[y_1, y_1] = \frac{\alpha_2}{\gamma_2} y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4 + \theta_1 y_5, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4, [y_2, y_2] = y_4, [y_2, y_3] = y_5.$$

- If $\alpha_3 = 0$ and $\alpha_2 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_9(0)$.
- If $\alpha_3 = 0$ and $\alpha_2 = 0$ then $\theta_1 \neq 0$ since A is non-split. Then the base change $x_1 = y_1, x_2 = y_2, x_3 = \theta_1 y_3, x_4 = y_4, x_5 = \theta_1 y_5$ shows that A is isomorphic to $\mathcal{A}_{35}$.
- If $\alpha_3 \neq 0$ and $\alpha_2 = 0$ then the base change $x_1 = \frac{1}{(\frac{\alpha_3}{\beta_1})^{2/3}} y_1 - \frac{\theta_1}{(\frac{\alpha_3}{\beta_1})^{2/3}} y_3, x_2 = -y_1 + (\frac{\alpha_3}{\beta_1})^{1/3} y_2 + \theta_1 y_3, x_3 = \frac{1}{(\frac{\alpha_3}{\beta_1})^{1/3}} y_3, x_4 = (\frac{\alpha_3}{\beta_1})^{2/3} y_4 + \frac{\theta_1}{(\frac{\alpha_3}{\beta_1})^{1/3}} y_5, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{36}$.
- If $\alpha_3 \neq 0$ and $\alpha_2 \neq 0$ then the base change $x_1 = y_1 + \theta_1 y_3, x_2 = -\frac{\alpha_3 \gamma_2 \theta_1}{\alpha_2 \beta_1} y_1 + \frac{\alpha_3}{\beta_1} y_2, x_3 = \frac{\alpha_2 \beta_1}{\alpha_3 \gamma_2} y_3, x_4 = (\frac{\alpha_3}{\beta_1})^2 y_4, x_5 = \frac{\alpha_2}{\gamma_2} y_5$ shows that A is isomorphic to $\mathcal{A}_{37}$.

Case 1.1.1.2.2.2: Let $\gamma_6 \neq 0$. Take $\theta_1 = \frac{\alpha_4 \beta_1 - \alpha_3 \beta_2}{\beta_1 \gamma_6}$ and $\theta_2 = \frac{\alpha_6 \beta_1 + \alpha_3 \beta_2}{\beta_1 \gamma_6}$. Then the base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 = \beta_1 e_4 + \beta_2 e_5, y_5 = \gamma_6 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \frac{\alpha_2}{\gamma_6} y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4 + \theta_1 y_5, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_2, y_3] = \frac{\gamma_2}{\gamma_6} y_5, [y_3, y_3] = y_5.$$

Without loss of generality we can assume $\theta_1 = 0$ because if $\theta_1 \neq 0$ then with the base change $x_1 = y_1, x_2 = x y_1 + y_2, x_3 = y_3, x_4 = y_4 + (\frac{\alpha_2}{\gamma_6} x^2 + (\theta_1 + \theta_2) x) y_5, x_5 = y_5$(where $\frac{\alpha_2 \alpha_3}{\beta_1 \gamma_6} x^2 + (\frac{\alpha_3(\theta_1 + \theta_2)}{\beta_1} - \frac{\alpha_2}{\gamma_6}) x - \theta_1 = 0$) we can make $\theta_1 = 0$. Then the products in A are the following:

$$[y_1, y_1] = \frac{\alpha_2}{\gamma_6} y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_2, y_3] = \frac{\gamma_2}{\gamma_6} y_5, [y_3, y_3] = y_5.$$

- If $\frac{\alpha_3}{\beta_1} = 0$ and $\frac{\alpha_2}{\gamma_6} = 0$ then $\theta_2 \neq 0$ since A is non-split. Then w.s.c.o.b. A is isomorphic to $\mathcal{A}_9(0)$.
- If $\frac{\alpha_3}{\beta_1} = 0, \frac{\alpha_2}{\gamma_6} \neq 0$ and $(\frac{\gamma_6 \theta_2}{\alpha_2 \gamma_2})^2 + 1 = 0$ then the base change $x_1 = \sqrt{\frac{\gamma_2^2}{\alpha_2 \gamma_6}} y_1, x_2 = y_2, x_3 = \frac{\gamma_2}{\gamma_6} y_3, x_4 = y_4, x_5 = (\frac{\gamma_2}{\gamma_6})^2 y_5$ shows that A is isomorphic to $\mathcal{A}_{38}$.
- If $\frac{\alpha_3}{\beta_1} = 0, \frac{\alpha_2}{\gamma_6} \neq 0$ and $(\frac{\gamma_6 \theta_2}{\alpha_2 \gamma_2})^2 + 1 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{10}$.

- If $\frac{\alpha_3}{\beta_1} \neq 0$ then the base change $x_1 = \frac{\beta_1}{\alpha_3} y_1, x_2 = y_2, x_3 = \frac{\gamma_2}{\gamma_6} y_3, x_4 = y_4, x_5 = (\frac{\gamma_2}{\gamma_6})^2 y_5$ shows that A is isomorphic to $\mathcal{A}_{39}(\alpha, \beta)$.

Case 1.1.2: Let $\gamma_4 \neq 0$. If $\gamma_6 \neq 0$ then the base change $x_1 = e_1, x_2 = \gamma_6 e_2 - \gamma_4 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.48). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{38}$ or $\mathcal{A}_{39}(\alpha, \beta)$.

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5.$$

Case 1.1.2.1: Let $\beta_1 = 0$. Then the products in A are the following:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5. \quad (4.55)$$

Case 1.1.2.1.1: Let $\alpha_1 = 0$. Then $\alpha_3 + \alpha_5 \neq$ since $\dim(Leib(A)) = 2$. Note that if $\alpha_4 \neq 0$ then with the base change $x_1 = \gamma_4 e_1 - \alpha_4 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So we can assume $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5.$$

Case 1.1.2.1.1.1: Let $\alpha_3 = 0$. Then $\alpha_5 \neq 0$.

Case 1.1.2.1.1.1.1: Let $\beta_4 = 0$.

Case 1.1.2.1.1.1.1.1: Let $\beta_2 = 0$. Note that if $\alpha_2 = 0$ then $\gamma_2 + \gamma_4 \neq 0$ since $\dim(Leib(A)) = 2$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5. \quad (4.56)$$

- If $\alpha_2 = 0$ and $\gamma_2 = 0$ then the base change $x_1 = e_3, x_2 = \frac{\alpha_5}{\gamma_4} e_2, x_3 = \frac{\gamma_4}{\alpha_5^2} e_1, x_4 = \alpha_5 e_5, x_5 = e_4 + \frac{\alpha_6}{\alpha_5} e_5$ shows that A is isomorphic to $\mathcal{A}_{14}$.

- If $\alpha_2 = 0$ and $\gamma_2 \neq 0$ then the base change $x_1 = e_3, x_2 = \frac{\gamma_4}{\gamma_2} e_2, x_3 = \frac{\gamma_2}{\gamma_4} e_1, x_4 = \gamma_4 e_5, x_5 = \alpha_5 e_4 + \alpha_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{16}(\alpha)$.

- If $\alpha_2 \neq 0$ then the base change $x_1 = e_1, x_2 = e_2, x_3 = \frac{\alpha_2}{\gamma_4} e_3, x_4 = \alpha_5 e_4 + \alpha_6 e_5, x_5 = \alpha_2 e_5$ shows that A is isomorphic to $\mathcal{A}_{40}(\alpha)$.

Case 1.1.2.1.1.1.1.2: Let $\beta_2 \neq 0$. If $\gamma_2 + \gamma_4 \neq 0$ then the base change $x_1 = e_1, x_2 = -\frac{\gamma_2+\gamma_4}{\beta_2} e_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.56). Hence A is isomorphic to $\mathcal{A}_{14}, \mathcal{A}_{16}(\alpha)$ or $\mathcal{A}_{40}(\alpha)$. So let $\gamma_2 + \gamma_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_2, e_3] = \gamma_2 e_5 = -[e_3, e_2].$$

- If $\alpha_2 = 0$ then the base change $x_1 = -\frac{\beta_2}{\gamma_2} e_3, x_2 = e_2, x_3 = e_1, x_4 = \beta_2 e_5, x_5 = \alpha_5 e_4 + \alpha_6 e_5$ shows that A is isomorphic to $\mathcal{A}_{36}$.
- If $\alpha_2 \neq 0$ then the base change $x_1 = \sqrt{\frac{\beta_2}{\alpha_2}} e_1, x_2 = e_2, x_3 = \frac{\beta_2}{\gamma_2} e_3, x_4 = \sqrt{\frac{\beta_2}{\alpha_2}} (\alpha_5 e_4 + \alpha_6 e_5), x_5 = \beta_2 e_5$ shows that A is isomorphic to $\mathcal{A}_{41}$.

Case 1.1.2.1.1.1.2: Let $\beta_4 \neq 0$.

- If $\alpha_2 = 0$ and $\beta_2 = 0$ then the base change $x_1 = \gamma_4 e_1, x_2 = \beta_4 e_2, x_3 = e_3, x_4 = \beta_4 \gamma_4 (\alpha_5 e_4 + \alpha_6 e_5), x_5 = \beta_4 \gamma_4 e_5$ shows that A is isomorphic to $\mathcal{A}_{42}(\alpha)$.
- If $\alpha_2 = 0$ and $\beta_2 \neq 0$ then the base change $x_1 = \gamma_4 e_1, x_2 = \beta_4 e_2, x_3 = \frac{\beta_2 \beta_4}{\gamma_4} e_3, x_4 = \beta_4 \gamma_4 (\alpha_5 e_4 + \alpha_6 e_5), x_5 = \beta_2 \beta_4^2 e_5$ shows that A is isomorphic to $\mathcal{A}_{43}(\alpha)$.
- If $\alpha_2 \neq 0$ and $\frac{\beta_2 \beta_4^2}{\alpha_2 \gamma_4^2} + \frac{\gamma_2}{\gamma_4} = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{42}(\alpha)$.
- If $\alpha_2 \neq 0$ and $\frac{\beta_2 \beta_4^2}{\alpha_2 \gamma_4^2} + \frac{\gamma_2}{\gamma_4} \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{43}(\alpha)$.

Case 1.1.2.1.1.2: Let $\alpha_3 \neq 0$.

Case 1.1.2.1.1.2.1: Let $\beta_2 = 0$. Then the products in A are the following:

$$\begin{aligned} &[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, \\ &\qquad [e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5. \end{aligned} \tag{4.57}$$

Case 1.1.2.1.1.2.1.1: Let $\beta_4 = 0$. Take $\theta_1 = \frac{\alpha_3 \alpha_6 - \alpha_4 \alpha_5}{\alpha_3}$ and $\theta_2 = \frac{\alpha_5 \gamma_4 - \alpha_3 \gamma_2}{\alpha_3}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 = \alpha_3 e_4 + \alpha_4 e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1]=\alpha_2y_5,[y_1,y_2]=y_4,[y_2,y_1]=\frac{\alpha_5}{\alpha_3}y_4+\theta_1y_5,[y_2,y_3]=\gamma_2y_5,[y_3,y_2]=\gamma_4y_5.$$

Note that if $\theta_2\neq 0$ then we can assume $\theta_1=0$, because if $\theta_1\neq 0$ then with the base change $x_1=\theta_2y_1+\theta_1y_3,x_2=y_2,x_3=y_3,x_4=\theta_2y_4+\gamma_4\theta_1y_5,x_5=y_5$ we can make $\theta_1=0$. Then we have the following products in A:

$$[y_1,y_1]=\alpha_2y_5,[y_1,y_2]=y_4,[y_2,y_1]=\frac{\alpha_5}{\alpha_3}y_4,[y_2,y_3]=\gamma_2y_5,[y_3,y_2]=\gamma_4y_5.$$

- If $\alpha_2=0$ then the base change $x_1=y_1,x_2=y_2,x_3=y_3,x_4=y_4,x_5=\gamma_4y_5$ shows that A is isomorphic to $\mathcal{A}_{44}(\alpha,\beta)(\alpha\neq\beta)$.
- If $\alpha_2\neq 0$ then the base change $x_1=y_1,x_2=y_2,x_3=\frac{\alpha_2}{\gamma_4}y_3,x_4=y_4,x_5=\alpha_2y_5$ shows that A is isomorphic to $\mathcal{A}_{45}(\alpha,\beta)(\alpha\neq\beta)$.

Now suppose $\theta_2=0$.

- If $\theta_1=0$ and $\alpha_2=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{44}(\alpha,\alpha)$.
- If $\theta_1=0$ and $\alpha_2\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{45}(\alpha,\alpha)$.
- If $\theta_1\neq 0$ and $\alpha_2=0$ then $x_1=y_1,x_2=y_2,x_3=\frac{\theta_1}{\gamma_4}y_3,x_4=y_4,x_5=\theta_1y_5$ shows that A is isomorphic to $\mathcal{A}_{46}(\alpha)$.
- If $\theta_1\neq 0$ and $\alpha_2\neq 0$ then $x_1=y_1,x_2=\frac{\alpha_2}{\theta_1}y_2,x_3=\frac{\theta_1}{\gamma_4}y_3,x_4=\frac{\alpha_2}{\theta_1}y_4,x_5=\alpha_2y_5$ shows that A is isomorphic to $\mathcal{A}_{47}(\alpha)$.

Case 1.1.2.1.1.2.1.2: Let $\beta_4\neq 0$. Note that if $\alpha_2\neq 0$ then with the base change $x_1=\beta_4e_1-\alpha_2e_3,x_2=e_2,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\alpha_2=0$. So we can assume $\alpha_2=0$. Then we have the following products in A:

$$[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_5e_4+\alpha_6e_5,[e_1,e_3]=\beta_4e_5,[e_2,e_3]=\gamma_2e_5,[e_3,e_2]=\gamma_4e_5.$$

- If $\alpha_3\alpha_6-\alpha_4\alpha_5=0$ then the base change $x_1=e_1,x_2=\frac{\beta_4}{\gamma_4}e_2,x_3=e_3,x_4=\frac{\beta_4}{\gamma_4}(\alpha_3e_4+\alpha_4e_5),x_5=\beta_4e_5$ shows that A is isomorphic to $\mathcal{A}_{48}(\alpha,\beta)$.
- If $\alpha_3\alpha_6-\alpha_4\alpha_5\neq 0$ and $\gamma_4=-\gamma_2$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(\alpha,-1)$.

- If $\alpha_3\alpha_6 - \alpha_4\alpha_5 \neq 0$ and $\gamma_4 \neq -\gamma_2$ then the base change $x_1 = e_1, x_2 = \frac{\beta_4}{\gamma_4}e_2, x_3 = \frac{\alpha_3\alpha_6-\alpha_4\alpha_5}{\alpha_5\gamma_4}e_3, x_4 = \frac{\beta_4}{\gamma_4}(\alpha_3 e_4 + \alpha_4 e_5), x_5 = \frac{(\alpha_3\alpha_6-\alpha_4\alpha_5)\beta_4}{\alpha_5\gamma_4}e_5$ shows that A is isomorphic to $\mathcal{A}_{49}(\alpha,\beta)$.

Case 1.1.2.1.1.2.2: Let $\beta_2 \neq 0$. If $\gamma_2 + \gamma_4 \neq 0$ then the base change $x_1 = e_1, x_2 = -\frac{\gamma_2+\gamma_4}{\beta_2}e_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.57). Hence A is isomorphic to $\mathcal{A}_{44}(\alpha,\beta), \mathcal{A}_{45}(\alpha,\beta), \mathcal{A}_{46}(\alpha)$, $\mathcal{A}_{47}(\alpha), \mathcal{A}_{48}(\alpha,\beta)$ or $\mathcal{A}_{49}(\alpha,\beta)$. So let $\gamma_2 + \gamma_4 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2,e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_2] = \beta_2 e_5, [e_1,e_3] = \beta_4 e_5,$$
$$[e_2,e_3] = \gamma_2 e_5 = -[e_3,e_2].$$

Case 1.1.2.1.1.2.2.1: Let $\beta_4 = 0$. Take $\theta = \frac{\alpha_3\alpha_6-\alpha_4\alpha_5}{\alpha_3}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 = \alpha_3 e_4 + \alpha_4 e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1] = \alpha_2 y_5, [y_1,y_2] = y_4, [y_2,y_1] = \frac{\alpha_5}{\alpha_3}y_4 + \theta y_5, [y_2,y_2] = \beta_2 y_5, [y_2,y_3] = \gamma_2 y_5 = -[y_3,y_2].$$

Note that if $\theta \neq 0$ then with the base change $x_1 = y_1 - \frac{\alpha_3\theta}{(\alpha_3+\alpha_5)\gamma_2}y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\theta = 0$. So let $\theta = 0$. Then we have the following products in A:

$$[y_1,y_1] = \alpha_2 y_5, [y_1,y_2] = y_4, [y_2,y_1] = \frac{\alpha_5}{\alpha_3}y_4, [y_2,y_2] = \beta_2 y_5, [y_2,y_3] = \gamma_2 y_5 = -[y_3,y_2].$$

- If $\alpha_2 = 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = \frac{\beta_2}{\gamma_2}y_3, x_4 = y_4, x_5 = \beta_2 y_5$ shows that A is isomorphic to $\mathcal{A}_{50}(\alpha)$.
- If $\alpha_2 \neq 0$ then the base change $x_1 = \sqrt{\frac{\beta_2}{\alpha_2}}y_1, x_2 = y_2, x_3 = \frac{\beta_2}{\gamma_2}y_3, x_4 = \sqrt{\frac{\beta_2}{\alpha_2}}y_4, x_5 = \beta_2 y_5$ shows that A is isomorphic to $\mathcal{A}_{51}(\alpha)$.

Case 1.1.2.1.1.2.2.2: Let $\beta_4 \neq 0$. Without loss of generality we can assume $\alpha_2 = 0$, because if $\alpha_2 \neq 0$ then with the base change $x_1 = \beta_4 e_1 - \alpha_2 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1,e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2,e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_2] = \beta_2 e_5, [e_1,e_3] = \beta_4 e_5, [e_2,e_3] = \gamma_2 e_5 = -[e_3,e_2].$$

Take $\theta = \frac{(\alpha_4\alpha_5-\alpha_3\alpha_6)\beta_4\gamma_2}{\alpha_5\beta_2}$. The base change $y_1 = \frac{\gamma_2}{\beta_4}e_1, y_2 = e_2, y_3 = \frac{\beta_2}{\gamma_2}e_3, y_4 = \frac{\gamma_2}{\beta_4}(\alpha_3 e_4 + \frac{\alpha_3\alpha_6}{\alpha_5}e_5), y_5 = \beta_2 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = y_4 + \theta y_5, [y_2, y_1] = \frac{\alpha_5}{\alpha_3}y_4, [y_2, y_2] = y_5, [y_1, y_3] = y_5, [y_2, y_3] = y_5 = -[y_3, y_2].$$

Note that if $\theta \neq 0$ then with the base change $x_1 = y_1, x_2 = y_2 - \theta y_3, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\theta = 0$. So we can assume $\theta = 0$. Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{52}(\alpha)$.

Case 1.1.2.1.2: Let $\alpha_1 \neq 0$.

Case 1.1.2.1.2.1: Let $\beta_2 = 0$.

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5,$$
$$[e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5. \quad (4.58)$$

Case 1.1.2.1.2.1.1: Let $\alpha_3 = 0$. If $\alpha_4 \neq 0$ then with the base change $x_1 = \gamma_4 e_1 - \alpha_4 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So we can assume $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_1, e_3] = \beta_4 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5.$$

Take $\theta = \frac{\alpha_1\alpha_6-\alpha_2\alpha_5}{\alpha_1\gamma_4}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 = \alpha_1 e_4 + \alpha_2 e_5, y_5 = \gamma_4 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_2, y_1] = \frac{\alpha_5}{\alpha_1}y_4 + \theta y_5, [y_1, y_3] = \frac{\beta_4}{\gamma_4}y_5, [y_2, y_3] = \frac{\gamma_2}{\gamma_4}y_5, [y_3, y_2] = y_5.$$

Notice that if $\alpha_5 = 0$ and $\theta = 0$ then $\beta_4 \neq 0$ since A is non-split.

- If $\alpha_5 = 0, \theta = 0$ and $\frac{\gamma_2}{\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{18}$.
- If $\alpha_5 = 0, \theta = 0$ and $\frac{\gamma_2}{\gamma_4} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{38}$.
- If $\alpha_5 = 0, \theta = 0$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C}\backslash\{0,1\}$ then the base change $x_1 = y_1, x_2 = \frac{\beta_4}{\gamma_4}y_2, x_3 = y_3, x_4 = y_4, x_5 = \frac{\beta_4}{\gamma_4}y_5$ shows that A is isomorphic to $\mathcal{A}_{53}(\alpha)$.
- If $\alpha_5 = 0, \theta \neq 0, \beta_4 = 0$ and $\frac{\gamma_2}{\gamma_4} = 0$ then the base change $x_1 = -y_1 + \frac{1}{\theta}y_2, x_2 = \theta y_3, x_3 = \frac{1}{\theta}y_2 + \theta y_3, x_4 = y_4 - y_5, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{31}$.

- If $\alpha_5 = 0, \theta \neq 0, \beta_4 = 0$ and $\frac{\gamma_2}{\gamma_4} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{38}$.
- If $\alpha_5 = 0, \theta \neq 0, \beta_4 = 0$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C} \backslash \{0, 1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{53}(\alpha)$.
- If $\alpha_5 = 0, \theta \neq 0, \beta_4 \neq 0$ and $\frac{\gamma_2}{\gamma_4} = 0$ then the base change $x_1 = y_1, x_2 = \frac{\beta_4}{\gamma_4} y_2, x_3 = \theta y_3, x_4 = y_4, x_5 = \frac{\beta_4 \theta}{\gamma_4} y_5$ shows that A is isomorphic to $\mathcal{A}_{54}$.
- If $\alpha_5 = 0, \theta \neq 0, \beta_4 \neq 0$ and $\frac{\gamma_2}{\gamma_4} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{10}$.
- If $\alpha_5 = 0, \theta \neq 0, \beta_4 \neq 0$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C} \backslash \{0, 1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{30}(\alpha)$.
- If $\alpha_5 \neq 0, \theta = 0, \beta_4 = 0$ and $\frac{\gamma_2}{\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{24}(0)$.
- If $\alpha_5 \neq 0, \theta = 0, \beta_4 = 0$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C} \backslash \{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{27}(\alpha, 0)$. Note that $\frac{\gamma_2}{\gamma_4} \neq -1$ since $\dim(Leib(A)) = 2$.
- If $\alpha_5 \neq 0, \theta = 0$ and $\beta_4 \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{\alpha_1}{\alpha_5} y_2, x_3 = y_3, x_4 = y_4, x_5 = \frac{\alpha_1}{\alpha_5} y_5$ shows that A is isomorphic to $\mathcal{A}_{55}(\alpha, \beta)$.
- If $\alpha_5 \neq 0, \theta \neq 0, \frac{\gamma_2}{\gamma_4} = -1$ and $\beta_4 = 0$ then the base change $x_1 = y_1, x_2 = \frac{\alpha_1}{\alpha_5} y_2, x_3 = -\theta y_3, x_4 = y_4, x_5 == \frac{\alpha_1 \theta}{\alpha_5} y_5$ shows that A is isomorphic to $\mathcal{A}_{56}$.
- If $\alpha_5 \neq 0, \theta \neq 0, \frac{\gamma_2}{\gamma_4} = -1$ and $\beta_4 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, -1)$.
- If $\alpha_5 \neq 0, \theta \neq 0$ and $\frac{\gamma_2}{\gamma_4} \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{57}(\alpha, \beta)$.

Case 1.1.2.1.2.1.2: Let $\alpha_3 \neq 0$. Take $\theta_1 = \frac{\alpha_1 \alpha_4 - \alpha_2 \alpha_3}{\alpha_1 \gamma_4}$ and $\theta_2 = \frac{\alpha_1 \alpha_6 - \alpha_2 \alpha_5}{\alpha_1 \gamma_4}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_1}{\alpha_3} e_2, y_3 = e_3, y_4 = \alpha_1 e_4 + \alpha_2 e_5, y_5 = \frac{\alpha_1 \gamma_4}{\alpha_3} e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = y_4 + \theta_1 y_5, [y_2, y_1] = \frac{\alpha_5}{\alpha_3} y_4 + \theta_2 y_5, [y_1, y_3] = \frac{\alpha_3 \beta_4}{\alpha_1 \gamma_4} y_5, [y_2, y_3] = \frac{\gamma_2}{\gamma_4} y_5, [y_3, y_2] = y_5.$$

Note that if $\theta_1 = 0, \theta_2 = 0$ and $\frac{\alpha_3 \beta_4}{\alpha_1 \gamma_4} = 0$ then $\frac{\gamma_2}{\gamma_4} \neq -1$ since $\dim(Leib(A)) = 2$.

- If $\theta_2 = 0, \theta_1 = 0, \frac{\alpha_3 \beta_4}{\alpha_1 \gamma_4} = 0$ and $\frac{\alpha_5}{\alpha_3} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_5(\alpha)$.
- If $\theta_2 = 0, \theta_1 = 0, \frac{\alpha_3 \beta_4}{\alpha_1 \gamma_4} = 0, \frac{\alpha_5}{\alpha_3} \neq 1$ and $\frac{\gamma_2}{\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{24}(\alpha)$.
- If $\theta_2 = 0, \theta_1 = 0, \frac{\alpha_3 \beta_4}{\alpha_1 \gamma_4} = 0, \frac{\alpha_5}{\alpha_3} \neq 1$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C} \backslash \{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{27}(\alpha, \beta)$.

- If $\theta_2 = 0, \theta_1 = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0$ and $\frac{\alpha_5}{\alpha_3} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, \beta)$.
- If $\theta_2 = 0, \theta_1 = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0$ and $\frac{\alpha_5}{\alpha_3} \neq 0$ then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{58}(\alpha, \beta, \gamma)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3}(1 + \frac{\gamma_2}{\gamma_4}) = \frac{\gamma_2}{\gamma_4}$ and $\frac{\gamma_2}{\gamma_4} = 0$ then the base change $x_1 = \frac{1}{\theta_1}y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{59}$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3}(1 + \frac{\gamma_2}{\gamma_4}) = \frac{\gamma_2}{\gamma_4}$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C}\backslash\{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(\frac{\alpha-1}{\alpha}, \alpha, \alpha - 1)(\alpha \in \mathbb{C}\backslash\{0, 1\})$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\frac{\alpha_5}{\alpha_3}(1 + \frac{\gamma_2}{\gamma_4}) \neq \frac{\gamma_2}{\gamma_4}$ then the base change $x_1 = \frac{1}{\theta_1}y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{60}(\alpha, \beta)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 0, \frac{\gamma_2}{\gamma_4} = 0$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{24}(0)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 0, \frac{\gamma_2}{\gamma_4} = 0$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \in \mathbb{C}\backslash\{0, 1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, \alpha)(\alpha \in \mathbb{C}\backslash\{-1, 0\})$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\gamma_2}{\gamma_4} = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, \alpha - 1)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1+\frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C}\backslash\{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{57}(\alpha, \beta)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4}$ and $\frac{\gamma_2}{\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_5(0)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4}$ and $\frac{\gamma_2}{\gamma_4} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(-1, \alpha, \alpha)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1+\frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} = 1$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4}$ then the base change $x_1 = \frac{1}{\theta_1}y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{61}(\alpha, \beta)$.
- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0, 1\}, \frac{\gamma_2}{\gamma_4} = 0$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{24}(\alpha)(\alpha \in \mathbb{C}\backslash\{0\})$.

- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\gamma_2}{\gamma_4} = 0$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \in \mathbb{C}\backslash\{0,1\}$ then the base change $x_1 = \frac{1}{\theta_1}y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{62}(\alpha,\beta)$.

- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\gamma_2}{\gamma_4} \neq 0$ and $\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(\alpha,\beta,\alpha\beta)(\alpha \in \mathbb{C}\backslash\{0,1\}, \beta \in \mathbb{C}\{0\})$.

- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\gamma_2}{\gamma_4} \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4}$ and $(\frac{\alpha_5}{\alpha_3} + 1)\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{62}(\alpha,\beta)$.

- If $\theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\gamma_2}{\gamma_4} \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4}$ and $(\frac{\alpha_5}{\alpha_3} + 1)\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4}$ then the base change $x_1 = \frac{1}{\theta_1}y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{63}(\alpha,\beta,\gamma)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\theta_1 = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha,\beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\theta_1 \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(0,\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \theta_1\frac{\gamma_2}{\gamma_4} = 1 - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0$ then the base change $x_1 = \frac{1}{\theta_2}y_1, x_2 = \frac{1}{\theta_2}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_2})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{64}(\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}, \theta_1\frac{\gamma_2}{\gamma_4} = 1 - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha,\beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}$ and $\theta_1\frac{\gamma_2}{\gamma_4} \neq 1 - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{57}(\alpha,\beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0$ and $\frac{\gamma_2\theta_1}{\gamma_4} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_5(\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0, \frac{\gamma_2\theta_1}{\gamma_4} \neq 1$ and $\frac{\gamma_2}{\gamma_4} = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(1,-1)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0, \frac{\gamma_2\theta_1}{\gamma_4} \neq 1$ and $\frac{\gamma_2}{\gamma_4} = 0$ then the base change $x_1 = \frac{1}{\theta_2}y_1, x_2 = \frac{1}{\theta_2}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_2})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{65}$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0, \frac{\gamma_2\theta_1}{\gamma_4} \neq 1$ and $\frac{\gamma_2}{\gamma_4} \in \mathbb{C}\backslash\{-1,0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{61}(0,\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\theta_1 = \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(0, \alpha, \alpha - 1)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\theta_1 \neq \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(1, \alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4}$ then the base change $x_1 = \frac{1}{\theta_2}y_1, x_2 = \frac{1}{\theta_2}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_2})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{66}(\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{61}(\alpha, \beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) = \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{27}(\alpha, \beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) = \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(\alpha, \beta, \gamma)(\alpha \in \mathbb{C}\backslash\{0,1\}, \beta \in \mathbb{C}\backslash\{0\}, \gamma \in \mathbb{C})$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) \neq \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4} - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 0$ then the base change $x_1 = \frac{1}{\theta_2}y_1, x_2 = \frac{1}{\theta_2}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_2})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{67}(\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) \neq \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} = \frac{\gamma_2}{\gamma_4} - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{62}(\alpha, \beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) \neq \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4} - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\frac{\alpha_5}{\alpha_3}(1+\frac{\gamma_2}{\gamma_4}) = \frac{\gamma_2}{\gamma_4}$ then the base change $x_1 = \frac{1}{\theta_2}y_1, x_2 = \frac{1}{\theta_2}y_2, x_3 = y_3, x_4 = (\frac{1}{\theta_2})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{68}(\alpha)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) \neq \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4} - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}, \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} = 1 + \frac{\gamma_2}{\gamma_4}$ and $\frac{\alpha_5}{\alpha_3}(1 + \frac{\gamma_2}{\gamma_4}) \neq \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(\alpha, \beta)$.

- If $\theta_2 \neq 0, \frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \theta_1(\frac{\alpha_5\beta_4}{\alpha_1\gamma_4} - \frac{\gamma_2}{\gamma_4}) \neq \frac{\alpha_3\beta_4}{\alpha_1\gamma_4} - 1, \frac{\alpha_5\beta_4}{\alpha_1\gamma_4} \neq \frac{\gamma_2}{\gamma_4} - \frac{\alpha_3\beta_4}{\alpha_1\gamma_4}$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_4} \neq 1 + \frac{\gamma_2}{\gamma_4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{63}(\alpha, \beta, \gamma)$.

Case 1.1.2.1.2.2: Let $\beta_2 \neq 0$. If $\gamma_2 + \gamma_4 \neq 0$ then the base change $x_1 = e_1, x_2 = -\frac{\gamma_2+\gamma_4}{\beta_2}e_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.58). Hence A is isomorphic to $\mathcal{A}_5(\alpha), \mathcal{A}_{10}, \mathcal{A}_{18}, \mathcal{A}_{24}(\alpha), \mathcal{A}_{27}(\alpha, \beta), \mathcal{A}_{30}(\alpha), \mathcal{A}_{31}$,

$\mathcal{A}_{38}, \mathcal{A}_{45}(\alpha,\beta), \mathcal{A}_{48}(\alpha,\beta), \mathcal{A}_{53}(\alpha), \mathcal{A}_{54}, \mathcal{A}_{55}(\alpha,\beta), \mathcal{A}_{56}, \mathcal{A}_{57}(\alpha,\beta), \mathcal{A}_{58}(\alpha,\beta,\gamma), \mathcal{A}_{59}, \mathcal{A}_{60}(\alpha,\beta)$, $\mathcal{A}_{61}(\alpha,\beta), \mathcal{A}_{62}(\alpha,\beta), \mathcal{A}_{63}(\alpha,\beta,\gamma)$, $\mathcal{A}_{64}(\alpha), \mathcal{A}_{65}, \mathcal{A}_{66}(\alpha), \mathcal{A}_{67}(\alpha)$ or $\mathcal{A}_{68}(\alpha)$. So let $\gamma_2+\gamma_4 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_4+\alpha_2 e_5, [e_1,e_2] = \alpha_3 e_4+\alpha_4 e_5, [e_2,e_1] = \alpha_5 e_4+\alpha_6 e_5, [e_2,e_2] = \beta_2 e_5,$$
$$[e_1,e_3] = \beta_4 e_5, [e_2,e_3] = \gamma_2 e_5 = -[e_3,e_2].$$

Case 1.1.2.1.2.2.1: Let $\alpha_3 = 0$. Note that if $\alpha_4 \neq 0$ then with the base change $x_1 = \gamma_2 e_1+\alpha_4 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So we can assume $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_1 e_4+\alpha_2 e_5, [e_2,e_1] = \alpha_5 e_4+\alpha_6 e_5, [e_2,e_2] = \beta_2 e_5, [e_1,e_3] = \beta_4 e_5, [e_2,e_3] = \gamma_2 e_5 = -[e_3,e_2].$$

Take $\theta = \frac{\alpha_1\alpha_6-\alpha_2\alpha_5}{\alpha_1\beta_2}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \frac{\beta_2}{\gamma_2}e_3, y_4 = \alpha_1 e_4+\alpha_2 e_5, y_5 = \beta_2 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1] = y_4, [y_2,y_1] = \frac{\alpha_5}{\alpha_1}y_4+\theta y_5, [y_2,y_2] = y_5, [y_1,y_3] = \frac{\beta_4}{\gamma_2}y_5, [y_2,y_3] = y_5 = -[y_3,y_2].$$

- If $\alpha_5 = 0$ and $\frac{\beta_4}{\gamma_2} = 0$ then $\theta \neq 0$ since A is non-split. Then the base change $x_1 = y_1, x_2 = \theta y_2, x_3 = \theta y_3, x_4 = y_4, x_5 = \theta^2 y_5$ shows that A is isomorphic to $\mathcal{A}_{69}$.
- If $\alpha_5 = 0$ and $\frac{\beta_4}{\gamma_2} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{30}(\frac{1}{4})$.
- If $\alpha_5 \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_2} = 0$ and $\frac{\alpha_5\theta}{\alpha_1} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{39}(0,0)$.
- If $\alpha_5 \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_2} = 0$ and $\frac{\alpha_5\theta}{\alpha_1} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{59}$.
- If $\alpha_5 \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_2} = 0$ and $\frac{\alpha_5\theta}{\alpha_1} \in \mathbb{C}\backslash\{0,1\}$ then the base change $x_1 = \frac{\alpha_5}{\alpha_1}y_1, x_2 = y_2, x_3 = y_3, x_4 = (\frac{\alpha_5}{\alpha_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{70}(\alpha)$.
- If $\alpha_5 \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_2} = 1$ and $\frac{\alpha_5\theta}{\alpha_1} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{59}$.
- If $\alpha_5 \neq 0, \frac{\alpha_5\beta_4}{\alpha_1\gamma_2} = 1$ and $\frac{\alpha_5\theta}{\alpha_1} \neq 0$ then the base change $x_1 = \frac{\alpha_5}{\alpha_1}y_1, x_2 = y_2, x_3 = y_3, x_4 = (\frac{\alpha_5}{\alpha_1})^2 y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{71}(\alpha)$.
- If $\alpha_5 \neq 0$ and $\frac{\alpha_5\beta_4}{\alpha_1\gamma_2} \in \mathbb{C}\backslash\{0,1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(0,\alpha)$.

Case 1.1.2.1.2.2.2: Let $\alpha_3 \neq 0$. Take $\theta_1 = \frac{(\alpha_1\alpha_4 - \alpha_2\alpha_3)\alpha_3}{\alpha_1^2\beta_2}$ and $\theta_2 = \frac{(\alpha_1\alpha_6 - \alpha_2\alpha_5)\alpha_3}{\alpha_1^2\beta_2}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_1}{\alpha_3}e_2, y_3 = \frac{\alpha_1\beta_2}{\alpha_3\gamma_2}e_3, y_4 = \alpha_1 e_4 + \alpha_2 e_5, y_5 = \frac{\alpha_1^2\beta_2}{\alpha_3^2}e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = y_4 + \theta_1 y_5, [y_2, y_1] = \frac{\alpha_5}{\alpha_3}y_4 + \theta_2 y_5, [y_2, y_2] = y_5, [y_1, y_3] = \frac{\alpha_3\beta_4}{\alpha_1\gamma_2}y_5,$$
$$[y_2, y_3] = y_5 = -[y_3, y_2].$$

- If $\frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\theta_1 + \theta_2 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{39}(0,0)$.
- If $\frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\theta_1 + \theta_2 = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{56}$.
- If $\frac{\alpha_5}{\alpha_3} = 0, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\theta_1 + \theta_2 \in \mathbb{C}\backslash\{0,1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{70}(\alpha)$.
- If $\frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_2} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{57}(\alpha, \alpha - 1)$.
- If $\frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\theta_1 + \theta_2 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{11}$.
- If $\frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\theta_1 + \theta_2 = \frac{1}{2}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(1,-1)$.
- If $\frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\theta_1 + \theta_2 \in \mathbb{C}\backslash\{0, \frac{1}{2}\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{72}(\alpha)$.
- If $\frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 1$ and $\theta_2 = \frac{1}{2}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{66}(1)$.
- If $\frac{\alpha_5}{\alpha_3} = 1, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 1$ and $\theta_2 \neq \frac{1}{2}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{73}(\alpha)$.
- If $\frac{\alpha_5}{\alpha_3} = 1$ and $\frac{\alpha_3\beta_4}{\alpha_1\gamma_2} \in \mathbb{C}\backslash\{0,1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{61}(\alpha, 2\alpha - 1)$.
- If $\frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\frac{\alpha_5}{\alpha_3}(\theta_1+\theta_2) = 1-(\theta_1+\theta_2)$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{60}(\alpha,-1)$.
- If $\frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} = 0$ and $\frac{\alpha_5}{\alpha_3}(\theta_1+\theta_2) \neq 1-(\theta_1+\theta_2)$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{74}(\alpha,\beta)$.
- If $\frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} \neq 0$ and $\frac{\alpha_3\alpha_5\beta_4}{\alpha_1\alpha_3\gamma_2} = \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} - 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{75}(\alpha)$.
- If $\frac{\alpha_5}{\alpha_3} \in \mathbb{C}\backslash\{0,1\}, \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} \neq 0$ and $\frac{\alpha_3\alpha_5\beta_4}{\alpha_1\alpha_3\gamma_2} \neq \frac{\alpha_3\beta_4}{\alpha_1\gamma_2} - 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{63}(\alpha,\beta,1+\beta)$.

Case 1.1.2.2: Let $\beta_1 \neq 0$. If $(\alpha_1, \alpha_3 + \alpha_5) \neq (0,0)$ then the base change $x_1 = e_1, x_2 = e_1 + xe_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\beta_1 x^2 + (\alpha_3 + \alpha_5)x + \alpha_1 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.55). Hence A is isomorphic to $\mathcal{A}_5(\alpha), \mathcal{A}_{10}, \mathcal{A}_{11}, \mathcal{A}_{14}, \mathcal{A}_{15}, \mathcal{A}_{16}(\alpha), \mathcal{A}_{17}(\alpha), \mathcal{A}_{18}, \mathcal{A}_{24}(\alpha), \mathcal{A}_{27}(\alpha,\beta), \mathcal{A}_{30}(\alpha), \mathcal{A}_{31}, \mathcal{A}_{36}, \mathcal{A}_{38}, \mathcal{A}_{39}(\alpha,\beta)$, $\mathcal{A}_{40}(\alpha), \mathcal{A}_{41}, \mathcal{A}_{42}(\alpha), \mathcal{A}_{43}(\alpha), \mathcal{A}_{44}(\alpha,\beta), \mathcal{A}_{45}(\alpha,\beta), \mathcal{A}_{46}(\alpha), \mathcal{A}_{47}(\alpha), \mathcal{A}_{48}(\alpha,\beta), \mathcal{A}_{49}(\alpha,\beta), \mathcal{A}_{50}(\alpha)$, $\mathcal{A}_{51}(\alpha), \mathcal{A}_{52}(\alpha), \mathcal{A}_{53}(\alpha), \mathcal{A}_{54}, \mathcal{A}_{55}(\alpha,\beta), \mathcal{A}_{56}, \mathcal{A}_{57}(\alpha,\beta), \mathcal{A}_{58}(\alpha,\beta,\gamma), \mathcal{A}_{59}, \mathcal{A}_{60}(\alpha,\beta), \mathcal{A}_{61}(\alpha,\beta)$, $\mathcal{A}_{62}(\alpha,\beta), \mathcal{A}_{63}(\alpha,\beta,\gamma), \mathcal{A}_{64}(\alpha), \mathcal{A}_{65}, \mathcal{A}_{66}(\alpha), \mathcal{A}_{67}(\alpha), \mathcal{A}_{68}(\alpha), \mathcal{A}_{69}, \mathcal{A}_{70}(\alpha), \mathcal{A}_{71}(\alpha), \mathcal{A}_{72}(\alpha), \mathcal{A}_{73}(\alpha)$, $\mathcal{A}_{74}(\alpha,\beta)$ or $\mathcal{A}_{75}(\alpha)$. Then let $\alpha_1 = 0 = \alpha_3 + \alpha_5$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2,e_1] = -\alpha_3 e_4 + \alpha_6 e_5, [e_2,e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1,e_3] = \beta_4 e_5, [e_2,e_3] = \gamma_2 e_5, [e_3,e_2] = \gamma_4 e_5.$$

Case 1.1.2.2.1: Let $\alpha_3 = 0$.

Case 1.1.2.2.1.1: Let $\beta_4 = 0$. Note that if $\alpha_4 \neq 0$ then with the base change $x_1 = \gamma_4 e_1 - \alpha_4 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So we can assume $\alpha_4 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_2,e_1] = \alpha_6 e_5, [e_2,e_2] = \beta_1 e_4 + \beta_2 e_5, [e_2,e_3] = \gamma_2 e_5, [e_3,e_2] = \gamma_4 e_5.$$

Then the base change $x_1 = e_3, x_2 = e_2, x_3 = e_1, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.48). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{38}$ or $\mathcal{A}_{39}(\alpha,\beta)$.

Case 1.1.2.2.1.2: Let $\beta_4 \neq 0$. Without loss of generality we can assume $\alpha_2 = 0$, because if $\alpha_2 \neq 0$ then with the base change $x_1 = \beta_4 e_1 - \alpha_2 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1,e_2] = \alpha_4 e_5, [e_2,e_1] = \alpha_6 e_5, [e_2,e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1,e_3] = \beta_4 e_5, [e_2,e_3] = \gamma_2 e_5, [e_3,e_2] = \gamma_4 e_5.$$

If $\gamma_2 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_2 e_1 - \beta_4 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\gamma_2 = 0$. So we can assume $\gamma_2 = 0$. Then we have the following products in A:

$$[e_1,e_2] = \alpha_4 e_5, [e_2,e_1] = \alpha_6 e_5, [e_2,e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1,e_3] = \beta_4 e_5, [e_3,e_2] = \gamma_4 e_5.$$

If $\alpha_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_4 e_2 - \alpha_4 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_4 = 0$. So we can assume $\alpha_4 = 0$. Then we have the following products in A:

$$[e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5, [e_3, e_2] = \gamma_4 e_5.$$

- If $\alpha_6 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{31}$.
- If $\alpha_6 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{54}$.

Case 1.1.2.2.2: Let $\alpha_3 \neq 0$.

Case 1.1.2.2.2.1: Let $\beta_4 = 0$. Take $\theta_1 = \frac{\alpha_4\beta_1 - \alpha_3\beta_2}{\beta_1}$ and $\theta_2 = \frac{\alpha_6\beta_1 + \alpha_3\beta_2}{\beta_1}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 = \beta_1 e_4 + \beta_2 e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \alpha_2 y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4 + \theta_1 y_5, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4,$$
$$[y_2, y_3] = \gamma_2 y_5, [y_3, y_2] = \gamma_4 y_5.$$

Without loss of generality we can assume $\theta_1 = 0$, because if $\theta_1 \neq 0$ then with the base change $x_1 = \gamma_4 y_1 - \theta_1 y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\theta_1 = 0$. Then we have the following products in A:

$$[y_1, y_1] = \alpha_2 y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_2, y_3] = \gamma_2 y_5, [y_3, y_2] = \gamma_4 y_5.$$

- If $\alpha_2 = 0$ then $(\theta_2, \frac{\gamma_2}{\gamma_4}) \neq (0, -1)$ since $\dim(Leib(A)) = 2$. Then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{50}(\alpha)$.
- If $\alpha_2 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{76}(\alpha)$.

Case 1.1.2.2.2.2: Let $\beta_4 \neq 0$. Note that if $\gamma_2 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_2 e_1 - \beta_4 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\gamma_2 = 0$. So we can assume $\gamma_2 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = -\alpha_3 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_2] = \gamma_4 e_5.$$

Take $\theta_1 = \frac{\alpha_4\beta_1 - \alpha_3\beta_2}{\beta_1}$ and $\theta_2 = \frac{\alpha_6\beta_1 + \alpha_3\beta_2}{\beta_1}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = e_3, y_4 =$

$\beta_1 e_4 + \beta_2 e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \alpha_2 y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4 + \theta_1 y_5, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_1, y_3] = \beta_4 y_5,$$
$$[y_3, y_2] = \gamma_4 y_5.$$

Without loss of generality we can assume $\theta_1 = 0$, because if $\theta_1 \neq 0$ then with the base change $x_1 = \gamma_4 y_1 - \theta_1 y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\theta_1 = 0$. Then we have the following products in A:

$$[y_1, y_1] = \alpha_2 y_5, [y_1, y_2] = \frac{\alpha_3}{\beta_1} y_4, [y_2, y_1] = -\frac{\alpha_3}{\beta_1} y_4 + \theta_2 y_5, [y_2, y_2] = y_4, [y_1, y_3] = \beta_4 y_5, [y_3, y_2] = \gamma_4 y_5.$$

- If $\alpha_2 = 0$ and $\theta_2 = 0$ then the base change $x_1 = \frac{\beta_1}{\alpha_3} y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = \gamma_4 y_5$ shows that A is isomorphic to $\mathcal{A}_{77}(\alpha)$.
- If $\alpha_2 = 0, \theta_2 \neq 0$ and $\frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{77}(1)$.
- If $\alpha_2 = 0, \theta_2 \neq 0$ and $\frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} \in \mathbb{C} \backslash \{0, 1\}$ then the base change $x_1 = \frac{\beta_1}{\alpha_3} y_1, x_2 = y_2, x_3 = \frac{\beta_1 \theta_2}{\alpha_3 \gamma_4} y_3, x_4 = y_4, x_5 = \frac{\beta_1 \theta_2}{\alpha_3} y_5$ shows that A is isomorphic to $\mathcal{A}_{78}(\alpha)$.
- If $\alpha_2 \neq 0$ and $\frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} = 1$ then the base change $x_1 = \frac{\beta_1}{\alpha_3} y_1, x_2 = y_2, x_3 = \frac{\alpha_2 \beta_1^2}{\alpha_3^2 \gamma_4} y_3, x_4 = y_4, x_5 = \frac{\alpha_2 \beta_1^2}{\alpha_3^2} y_5$ shows that A is isomorphic to $\mathcal{A}_{79}$.
- If $\alpha_2 \neq 0, \frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} \in \mathbb{C} \backslash \{0, 1\}$ and $\frac{\beta_4 \theta_2}{\alpha_2 \gamma_4} (\frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} - 1) = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{77}(\alpha) (\alpha \in \mathbb{C} \backslash \{0, 1\})$.
- If $\alpha_2 \neq 0, \frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} \in \mathbb{C} \backslash \{0, 1\}$ and $\frac{\beta_4 \theta_2}{\alpha_2 \gamma_4} (\frac{\beta_1 \beta_4}{\alpha_3 \gamma_4} - 1) \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{78}(\alpha)$.

Case 1.2: Let $\gamma_1 \neq 0$.

Case 1.2.1: Let $\alpha_1 = 0$.

Case 1.2.1.1: Let $\alpha_3 = 0$. Then if $\alpha_5 = 0$(resp. $\alpha_5 \neq 0$) then the base change $x_1 = e_3, x_2 = e_2, x_3 = e_1, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = e_2, x_3 = \gamma_1 e_1 - \alpha_5 e_3, x_4 = e_4, x_5 = e_5$) shows that A is isomorphic to an algebra with the nonzero products given by (4.47). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{78}(\alpha)$ or $\mathcal{A}_{79}$.

Case 1.2.1.2: Let $\alpha_3 \neq 0$.

Case 1.2.1.2.1: Let $\alpha_2 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5. \quad (4.59)$$

Case 1.2.1.2.1.1: Let $\gamma_6 = 0$.

Case 1.2.1.2.1.1.1: Let $\beta_6 = 0$. If $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5.$$

Take $\theta_1 = \frac{\alpha_4\gamma_1 - \alpha_3\gamma_2}{\alpha_3}, \theta_2 = \frac{\alpha_6\gamma_1 - \alpha_5\gamma_2}{\alpha_3}$. The base change $y_1 = \frac{\gamma_1}{\alpha_3} e_1, y_2 = e_2, y_3 = e_3, y_4 = \gamma_1 e_4 + \gamma_2 e_5, y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = y_4 + \theta_1 y_5, [y_2, y_1] = \frac{\alpha_5}{\alpha_3} y_4 + \theta_2 y_5, [y_2, y_2] = \beta_2 y_5, [y_1, y_3] = \frac{\beta_4\gamma_1}{\alpha_3} y_5, [y_2, y_3] = y_4, [y_3, y_2] = \gamma_4 y_5.$$

- If $\gamma_4 = 0, \beta_4 = 0, \theta_2 = 0$ and $\theta_1 = 0$ then $\beta_2 \neq 0$ since $\dim(Leib(A)) = 2$. Then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{35}$.

- If $\gamma_4 = 0, \beta_4 = 0, \theta_2 = 0$ and $\theta_1 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{14}$.

- If $\gamma_4 = 0, \beta_4 = 0, \theta_2 \neq 0$ and $\frac{\theta_1}{\theta_2} = -1$ then $\beta_2 \neq 0$ since $\dim(Leib(A)) = 2$. Then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{36}$.

- If $\gamma_4 = 0, \beta_4 = 0, \theta_2 \neq 0$ and $\frac{\theta_1}{\theta_2} = 0$ then the base change $x_1 = y_1 - \frac{\alpha_5}{\alpha_3} y_3, x_2 = -\frac{\beta_2}{\theta_2} y_1 + y_2 + \frac{(\alpha_3+\alpha_5)\beta_2}{\alpha_3\theta_2} y_3, x_3 = y_3, x_4 = y_4, x_5 = \theta_2 y_5$ shows that A is isomorphic to $\mathcal{A}_{80}$.

- If $\gamma_4 = 0, \beta_4 = 0, \theta_2 \neq 0$ and $\frac{\theta_1}{\theta_2} \in \mathbb{C}\setminus\{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{16}(\alpha)(\alpha \in \mathbb{C}\setminus\{-1, 0\})$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 = 0, \theta_1 = 0$ and $\frac{\alpha_5}{\alpha_3} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(0, 0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 = 0, \theta_1 = 0$ and $\frac{\alpha_5}{\alpha_3} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, \alpha)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 = 0, \theta_1 \neq 0$ and $\frac{\alpha_5}{\alpha_3} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{26}(0,0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 = 0, \theta_1 \neq 0$ and $\frac{\alpha_5}{\alpha_3} \neq 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(\alpha, 0, 0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\alpha_3\theta_1}{\beta_4\gamma_1} = -4$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(0,0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 \neq 0, \frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\alpha_3\theta_1}{\beta_4\gamma_1} \neq -4$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(0,0,0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 = 0, \beta_2 \neq 0$ and $\frac{\alpha_5}{\alpha_3} \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{\beta_4\gamma_1}{\alpha_3\theta_2}y_2, x_3 = y_3, x_4 = \frac{\beta_4\gamma_1}{\alpha_3\theta_2}y_4, x_5 = \frac{\beta_4\gamma_1}{\alpha_3}y_5$ shows that A is isomorphic to $\mathcal{A}_{81}(\alpha, \beta)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 \neq 0, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5}{\alpha_3} = -1$ and $\frac{\alpha_5\theta_1}{\alpha_3\theta_2} = \frac{\alpha_5}{\alpha_3} + 2$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, \alpha)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 \neq 0, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5}{\alpha_3} = -1$ and $\frac{\alpha_5\theta_1}{\alpha_3\theta_2} \neq \frac{\alpha_5}{\alpha_3} + 2$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{81}(\alpha, \beta)(\alpha, \beta \in \mathbb{C}\backslash\{0\})$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 \neq 0, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5}{\alpha_3} \neq -1, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5\theta_1}{\alpha_3\theta_2} = 1$ and $\frac{\alpha_5}{\alpha_3} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{26}(0,0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 \neq 0, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5}{\alpha_3} \neq -1, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5\theta_1}{\alpha_3\theta_2} = 1$ and $\frac{\alpha_5}{\alpha_3} \neq 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(\alpha, 0, 0)$.

- If $\gamma_4 = 0, \beta_4 \neq 0, \theta_2 \neq 0, \frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5}{\alpha_3} \neq -1$ and $\frac{\beta_2\beta_4\gamma_1}{\alpha_3\theta_2^2}(\frac{\alpha_5}{\alpha_3})^2 + \frac{\alpha_5\theta_1}{\alpha_3\theta_2} \neq 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{81}(\alpha, \beta)$.

- If $\gamma_4 \neq 0, \beta_4 = 0, \frac{\theta_2}{\gamma_4} = \frac{\alpha_5}{\alpha_3} - \frac{\theta_1}{\gamma_4} + 1$ and $\beta_2 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{82}(\alpha)$.

- If $\gamma_4 \neq 0, \beta_4 = 0, \frac{\theta_2}{\gamma_4} = \frac{\alpha_5}{\alpha_3} - \frac{\theta_1}{\gamma_4} + 1, \beta_2 \neq 0$ and $\frac{\theta_2}{\gamma_4} = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{83}$.

- If $\gamma_4 \neq 0, \beta_4 = 0, \frac{\theta_2}{\gamma_4} = \frac{\alpha_5}{\alpha_3} - \frac{\theta_1}{\gamma_4} + 1, \beta_2 \neq 0$ and $\frac{\theta_2}{\gamma_4} \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{50}(\alpha)$.

- If $\gamma_4 \neq 0, \beta_4 = 0, \frac{\theta_2}{\gamma_4} \neq \frac{\alpha_5}{\alpha_3} - \frac{\theta_1}{\gamma_4} + 1$ and $\frac{\theta_2}{\gamma_4} = -\frac{(\frac{\theta_1}{\gamma_4} - \frac{\alpha_5}{\alpha_3})^2}{4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{46}(\alpha)$.

- If $\gamma_4 \neq 0, \beta_4 = 0, \frac{\theta_2}{\gamma_4} \neq \frac{\alpha_5}{\alpha_3} - \frac{\theta_1}{\gamma_4} + 1$ and $\frac{\theta_2}{\gamma_4} \neq -\frac{(\frac{\theta_1}{\gamma_4} - \frac{\alpha_5}{\alpha_3})^2}{4}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{44}(\alpha, \beta)$.

- If $\gamma_4 \neq 0, \beta_4 \neq 0$ and $\frac{\alpha_5}{\alpha_3} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{84}(\alpha, \beta)$.

- If $\gamma_4 \neq 0, \beta_4 \neq 0$ and $\frac{\alpha_5}{\alpha_3} \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{85}(\alpha, \beta, \gamma)$.

Case 1.2.1.2.1.1.2: Let $\beta_6 \neq 0$. If $\gamma_4 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_4 e_1 - \beta_6 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\gamma_4 = 0$. So we can assume $\gamma_4 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5.$$

If $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5.$$

Take $\theta_1 = \frac{\alpha_4\gamma_1 - \alpha_3\gamma_2}{\beta_6\gamma_1}, \theta_2 = \frac{\alpha_5\gamma_1 - \alpha_6\gamma_2}{\beta_6\gamma_1}$ and $\theta_3 = \sqrt{\frac{\beta_6\gamma_1}{\alpha_3\beta_2}}$. The base change $y_1 = \frac{\gamma_1}{\alpha_3} e_1, y_2 = e_2, y_3 = e_3, y_4 = \gamma_1 e_4 + \gamma_2 e_5, y_5 = \frac{\beta_6\gamma_1}{\alpha_3} e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_2] = y_4 + \theta_1 y_5, [y_2, y_1] = \frac{\alpha_5}{\alpha_3} y_4 + \theta_2 y_5, [y_2, y_2] = \frac{\alpha_3\beta_2}{\beta_6\gamma_1} y_5, [y_1, y_3] = \frac{\beta_4}{\beta_6} y_5, [y_3, y_1] = y_5, [y_2, y_3] = y_4.$$

- If $\beta_2 = 0, \theta_2 = 0, \theta_1 = 0, \frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\beta_4}{\beta_6} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{42}(0)$.

- If $\beta_2 = 0, \theta_2 = 0, \theta_1 = 0, \frac{\alpha_5}{\alpha_3} = 0$ and $\frac{\beta_4}{\beta_6} \in \mathbb{C}\backslash\{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(\alpha, 0)$.

- If $\beta_2 = 0, \theta_2 = 0, \theta_1 = 0, \frac{\alpha_5}{\alpha_3} \neq 0$ and $\frac{\beta_4}{\beta_6} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha, 0)$.

- If $\beta_2 = 0, \theta_2 = 0, \theta_1 = 0, \frac{\alpha_5}{\alpha_3} \neq 0$ and $\frac{\beta_4}{\beta_6} \in \mathbb{C}\backslash\{-1, 0\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(\alpha, \beta, 0)$.

- If $\beta_2=0,\theta_2=0,\theta_1\neq 0$ and $\frac{\beta_4}{\beta_6}=-1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{86}$.
- If $\beta_2=0,\theta_2=0,\theta_1\neq 0$ and $\frac{\beta_4}{\beta_6}\neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{87}(\alpha,\beta)$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}=\frac{\beta_4}{\beta_6}$ and $\frac{\alpha_5}{\alpha_3}=1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{29}(\alpha,i,0)$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}=\frac{\beta_4}{\beta_6},\frac{\alpha_5}{\alpha_3}\neq 1$ and $\frac{\beta_4}{\beta_6}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(\alpha,-\alpha^2,\alpha-\alpha^2)(\alpha\in\mathbb{C}\backslash\{0\})$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}=\frac{\beta_4}{\beta_6},\frac{\alpha_5}{\alpha_3}\neq 1$ and $\frac{\beta_4}{\beta_6}=1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{33}(\alpha-\sqrt{\alpha},\alpha,0)$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}=\frac{\beta_4}{\beta_6},\frac{\alpha_5}{\alpha_3}\neq 1$ and $\frac{\beta_4}{\beta_6}\in\mathbb{C}\backslash\{-1,0,1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{34}(\alpha,\sqrt{\beta}+\alpha\beta,\beta,0)(\alpha\in\mathbb{C}\backslash\{-1,0,1\},\beta\in\mathbb{C}\backslash\{0\})$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}\neq\frac{\beta_4}{\beta_6},\frac{\beta_4}{\beta_6}=0,\frac{\alpha_5\theta_1}{\alpha_3\theta_2}=1$ and $\frac{\alpha_5}{\alpha_3}=-1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{86}$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}\neq\frac{\beta_4}{\beta_6},\frac{\beta_4}{\beta_6}=0,\frac{\alpha_5\theta_1}{\alpha_3\theta_2}=1$ and $\frac{\alpha_5}{\alpha_3}\neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{87}(0,\alpha)$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}\neq\frac{\beta_4}{\beta_6},\frac{\beta_4}{\beta_6}=0$ and $\frac{\alpha_5\theta_1}{\alpha_3\theta_2}\neq 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{88}(\alpha,\beta)$.
- If $\beta_2=0,\theta_2\neq 0,\frac{\theta_1}{\theta_2}\neq\frac{\beta_4}{\beta_6}$ and $\frac{\beta_4}{\beta_6}\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{89}(\alpha,\beta,\gamma)$.
- If $\beta_2\neq 0,\frac{\beta_4}{\beta_6}=0,\frac{\theta_1\theta_2}{\theta_3^2}-\frac{\alpha_5}{\alpha_3}(\frac{\theta_1}{\theta_3})^2+1=0$ and $\frac{\alpha_5}{\alpha_3}=-1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{90}(\alpha)$.
- If $\beta_2\neq 0,\frac{\beta_4}{\beta_6}=0,\frac{\theta_1\theta_2}{\theta_3^2}-\frac{\alpha_5}{\alpha_3}(\frac{\theta_1}{\theta_3})^2+1=0$ and $\frac{\alpha_5}{\alpha_3}\neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{87}(\alpha,\beta)(\alpha\in\mathbb{C}\backslash\{0\},\beta\in\mathbb{C}\backslash\{-1\})$.
- If $\beta_2\neq 0,\frac{\beta_4}{\beta_6}=0$ and $\frac{\theta_1\theta_2}{\theta_3^2}-\frac{\alpha_5}{\alpha_3}(\frac{\theta_1}{\theta_3})^2+1\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{91}(\alpha,\beta,\gamma)$.
- If $\beta_2\neq 0,\frac{\beta_4}{\beta_6}\neq 0$ and $\frac{\theta_2}{\theta_3}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{92}(\alpha,\beta,\gamma)$.
- If $\beta_2\neq 0,\frac{\beta_4}{\beta_6}\neq 0,\frac{\theta_2}{\theta_3}\neq 0,\frac{\theta_1}{\theta_3}=\frac{2\beta_4\theta_2}{\beta_6\theta_3},\frac{\beta_4\theta_2^2}{\beta_6\theta_3^2}=-1$ and $\frac{\alpha_5}{\alpha_3}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{85}(0,\alpha,0)$.

- If $\beta_2 \neq 0, \frac{\beta_4}{\beta_6} \neq 0, \frac{\theta_2}{\theta_3} \neq 0, \frac{\theta_1}{\theta_3} = \frac{2\beta_4\theta_2}{\beta_6\theta_3}, \frac{\beta_4\theta_2^2}{\beta_6\theta_3^2} = -1, \frac{\alpha_5}{\alpha_3} \neq 0$ and $\frac{\beta_4}{\beta_6} = -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{85}(0,-1,0)$.

- If $\beta_2 \neq 0, \frac{\beta_4}{\beta_6} \neq 0, \frac{\theta_2}{\theta_3} \neq 0, \frac{\theta_1}{\theta_3} = \frac{2\beta_4\theta_2}{\beta_6\theta_3}, \frac{\beta_4\theta_2^2}{\beta_6\theta_3^2} = -1, \frac{\alpha_5}{\alpha_3} \neq 0$ and $\frac{\beta_4}{\beta_6} \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{93}(\alpha,\beta)$.

- If $\beta_2 \neq 0, \frac{\beta_4}{\beta_6} \neq 0, \frac{\theta_2}{\theta_3} \neq 0, \frac{\theta_1}{\theta_3} = \frac{2\beta_4\theta_2}{\beta_6\theta_3}$ and $\frac{\beta_4\theta_2^2}{\beta_6\theta_3^2} \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{94}(\alpha,\beta,\gamma)$.

- If $\beta_2 \neq 0, \frac{\beta_4}{\beta_6} \neq 0, \frac{\theta_2}{\theta_3} \neq 0, \frac{\theta_1}{\theta_3} \neq \frac{2\beta_4\theta_2}{\beta_6\theta_3}, \frac{\beta_4}{\beta_6} = -1$ and $(\frac{\theta_2}{\theta_3})^2 + \frac{\theta_1\theta_2}{\theta_3^2} + 1 = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{89}(-1,-1,\alpha)(\alpha \in \mathbb{C}\backslash\{-1,0\})$.

- If $\beta_2 \neq 0, \frac{\beta_4}{\beta_6} \neq 0, \frac{\theta_2}{\theta_3} \neq 0, \frac{\theta_1}{\theta_3} \neq \frac{2\beta_4\theta_2}{\beta_6\theta_3}, \frac{\beta_4}{\beta_6} = -1$ and $\frac{\theta_2^2}{\theta_3^2} + \frac{\theta_1\theta_2}{\theta_3^2} + 1 \neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{95}(\alpha,\beta)$.

- If $\beta_2 \neq 0, \frac{\beta_4}{\beta_6} \neq 0, \frac{\theta_2}{\theta_3} \neq 0, \frac{\theta_1}{\theta_3} \neq \frac{2\beta_4\theta_2}{\beta_6\theta_3}$ and $\frac{\beta_4}{\beta_6} \neq -1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{96}(\alpha,\beta,\gamma,\theta)$.

Case 1.2.1.2.1.2: Let $\gamma_6 \neq 0$. If $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1,e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2,e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_2] = \beta_2 e_5, [e_1,e_3] = \beta_4 e_5, [e_3,e_1] = \beta_6 e_5,$$
$$[e_2,e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3,e_2] = \gamma_4 e_5, [e_3,e_3] = \gamma_6 e_5.$$

Take $\theta_1 = \frac{\alpha_4\gamma_1 - \alpha_3\gamma_2}{\alpha_3\gamma_6}$ and $\theta_2 = \frac{\alpha_6\gamma_1 - \alpha_5\gamma_2}{\alpha_3\gamma_6}$. The base change $y_1 = \frac{\gamma_1}{\alpha_3} e_1, y_2 = e_2, y_3 = e_3, y_4 = \gamma_1 e_4 + \gamma_2 e_5, y_5 = \gamma_6 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_2] = y_4 + \theta_1 y_5, [y_2,y_1] = \frac{\alpha_5}{\alpha_3} y_4 + \theta_2 y_5, [y_2,y_2] = \frac{\beta_2}{\gamma_6} y_5, [y_1,y_3] = \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} y_5, [y_3,y_1] = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6} y_5,$$
$$[y_2,y_3] = y_4, [y_3,y_2] = \frac{\gamma_4}{\gamma_6} y_5, [y_3,y_3] = y_5.$$

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 = 0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} = 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6} = 0$ and $\frac{\alpha_5}{\alpha_3} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{31}$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 = 0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} = 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6} = 0$ and $\frac{\alpha_5}{\alpha_3} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{38}$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}=0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=0$ and $\frac{\alpha_5}{\alpha_3}\in\mathbb{C}\backslash\{0,1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{53}(\alpha)$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}=0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0$ and $\frac{\alpha_5}{\alpha_3}=\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(0,\alpha)(\alpha\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}=0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\alpha_5}{\alpha_3}\neq\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\alpha_5}{\alpha_3}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{42}(\alpha)(\alpha\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}=0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\alpha_5}{\alpha_3}\neq\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\alpha_5}{\alpha_3}\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha,\beta)(\alpha,\beta\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{42}(\alpha)(\alpha\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0$ and $\frac{\alpha_5}{\alpha_3}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{77}(\alpha)$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0$ and $\frac{\alpha_5}{\alpha_3}\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(\alpha,\beta)(\alpha\in\mathbb{C}\backslash\{-1\},\beta\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=0$ and $\frac{\alpha_5}{\alpha_3}=0$ then the base change $x_1=y_1, x_2=y_2, x_3=y_3, x_4=y_4, x_5=y_5$ shows that A is isomorphic to $\mathcal{A}_{97}(\alpha)$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=0$ and $\frac{\alpha_5}{\alpha_3}\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{55}(\alpha,\beta)(\alpha,\beta\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0$ and $\frac{\alpha_5}{\alpha_3}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{48}(\alpha,\beta)(\alpha\in\mathbb{C}\backslash\{-1,0\},\beta\in\mathbb{C}\backslash\{0\}$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1=0, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}\neq 0, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq\frac{\alpha_5}{\alpha_3}-\frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}\neq 0$ and $\frac{\alpha_5}{\alpha_3}\neq 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{58}(\alpha,\beta,\gamma)(\alpha,\beta,\gamma\in\mathbb{C}\backslash\{0\})$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1\neq 0, \frac{\alpha_5}{\alpha_3}=\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}=0$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{57}(0,0)$.

- If $\frac{\gamma_4}{\gamma_6}=0, \frac{\beta_2}{\gamma_6}=0, \theta_2=0, \theta_1\neq 0, \frac{\alpha_5}{\alpha_3}=\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}=0$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6}=1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{29}(0,0,0)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} = 0$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6} \in \mathbb{C}\backslash\{0,1\}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(0,\alpha,\alpha)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} = -1$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{29}(0,0,0)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} = -1$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6} \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = \frac{1}{\theta_1}y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{98}(\alpha)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} \in \mathbb{C}\backslash\{-1,0\}$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{32}(0,\alpha,\alpha)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} \in \mathbb{C}\backslash\{-1,0\}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6} \neq 0$ and $\frac{\alpha_5}{\alpha_3} = \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{98}(\alpha)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} \in \mathbb{C}\backslash\{-1,0\}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6} \neq 0, \frac{\alpha_5}{\alpha_3} \neq \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\alpha_5}{\alpha_3} = 1$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{98}(\alpha)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} = \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} \in \mathbb{C}\backslash\{-1,0\}, \frac{\beta_6\gamma_1}{\alpha_3\gamma_6} \neq 0, \frac{\alpha_5}{\alpha_3} \neq \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\alpha_5}{\alpha_3} \neq 1$ then the base change $x_1 = y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = \frac{1}{\theta_1}y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{99}(\alpha,\beta)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} \neq \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\alpha_5}{\alpha_3} = \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\beta_4\gamma_1}{\alpha_3\gamma_6} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{88}(\alpha,0)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} \neq \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\alpha_5}{\alpha_3} = \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} \neq 0$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6} = 0$ then w.s.c.o.b. A is isomorphic to $\mathcal{A}_{88}(\alpha,0)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} \neq \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}, \frac{\alpha_5}{\alpha_3} = \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}, \frac{\beta_4\gamma_1}{\alpha_3\gamma_6} \neq 0$ and $\frac{\beta_6\gamma_1}{\alpha_3\gamma_6} \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = \frac{1}{\theta_1}y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{100}(\alpha,\beta)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0, \theta_2 = 0, \theta_1 \neq 0, \frac{\alpha_5}{\alpha_3} \neq \frac{\beta_6\gamma_1}{\alpha_3\gamma_6}$ and $\frac{\alpha_5}{\alpha_3} \neq \frac{\beta_4\gamma_1}{\alpha_3\gamma_6}$ then the base change $x_1 = y_1, x_2 = \frac{1}{\theta_1}y_2, x_3 = y_3, x_4 = \frac{1}{\theta_1}y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{A}_{101}(\alpha,\beta,\gamma)$.

- If $\frac{\gamma_4}{\gamma_6} = 0, \frac{\beta_2}{\gamma_6} = 0$ and $\theta_2 \neq 0$ then the base change $x_1 = \theta_2 y_1, x_2 = y_2, x_3 = \theta_2 y_3, x_4 = \theta_2 y_4, x_5 = \theta_2^2 y_5$ shows that A is isomorphic to $\mathcal{R}_1$.

- If $\frac{\gamma_4}{\gamma_6} = 0$ and $\frac{\beta_2}{\gamma_6} \neq 0$ then the base change $x_1 = y_1, x_2 = \sqrt{\frac{\gamma_6}{\beta_2}} y_2, x_3 = y_3, x_4 = \sqrt{\frac{\gamma_6}{\beta_2}} y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_2$.
- If $\frac{\gamma_4}{\gamma_6} \neq 0$ then the base change $x_1 = y_1, x_2 = \frac{\gamma_6}{\gamma_4} y_2, x_3 = y_3, x_4 = \frac{\gamma_6}{\gamma_4} y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_3$.

Case 1.2.1.2.2: Let $\alpha_2 \neq 0$. If $(\beta_4 + \beta_6, \gamma_6) \neq (0,0)$ then the base change $x_1 = xe_1 + e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\alpha_2 x^2 + (\beta_4 + \beta_6)x + \gamma_6 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.59). Hence A is isomorphic to $\mathcal{A}_{14}, \mathcal{A}_{16}(\alpha), \mathcal{A}_{26}(\alpha,\beta), \mathcal{A}_{29}(\alpha,\beta,\gamma), \mathcal{A}_{31}, \mathcal{A}_{32}(\alpha,\beta,\gamma), \mathcal{A}_{33}(\alpha,\beta,\gamma), \mathcal{A}_{34}(\alpha,\beta,\gamma,\theta), \mathcal{A}_{35}, \mathcal{A}_{36}, \mathcal{A}_{38}, \mathcal{A}_{42}(\alpha), \mathcal{A}_{44}(\alpha,\beta), \mathcal{A}_{46}(\alpha), \mathcal{A}_{48}(\alpha,\beta), \mathcal{A}_{50}(\alpha), \mathcal{A}_{53}(\alpha), \mathcal{A}_{55}(\alpha,\beta), \mathcal{A}_{57}(\alpha,\beta), \mathcal{A}_{58}(\alpha,\beta,\gamma), \mathcal{A}_{77}(\alpha), \mathcal{A}_{80}, \mathcal{A}_{81}(\alpha,\beta), \mathcal{A}_{82}(\alpha), \mathcal{A}_{83}, \mathcal{A}_{84}(\alpha,\beta), \mathcal{A}_{85}(\alpha,\beta,\gamma), \mathcal{A}_{86}, \mathcal{A}_{87}(\alpha,\beta), \mathcal{A}_{88}(\alpha,\beta), \mathcal{A}_{89}(\alpha,\beta,\gamma), \mathcal{A}_{90}(\alpha), \mathcal{A}_{91}(\alpha,\beta,\gamma), \mathcal{A}_{92}(\alpha,\beta,\gamma), \mathcal{A}_{93}(\alpha,\beta), \mathcal{A}_{94}(\alpha,\beta,\gamma), \mathcal{A}_{95}(\alpha,\beta), \mathcal{A}_{96}(\alpha,\beta,\gamma,\theta), \mathcal{A}_{97}(\alpha), \mathcal{A}_{98}(\alpha), \mathcal{A}_{99}(\alpha,\beta), \mathcal{A}_{100}(\alpha,\beta)$, $\mathcal{A}_{101}(\alpha,\beta,\gamma), \mathcal{R}_1, \mathcal{R}_2$ or $\mathcal{R}_3$. So let $\beta_4 + \beta_6 = 0 = \gamma_6$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2,e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2,e_2] = \beta_1 e_4 + \beta_2 e_5,$$
$$[e_1,e_3] = \beta_4 e_5 = -[e_3,e_1], [e_2,e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3,e_2] = \gamma_4 e_5.$$

Without loss of generality we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = \gamma_1 e_1 - \alpha_5 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Furthermore, if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1,e_1] = \alpha_2 e_5, [e_1,e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2,e_1] = \alpha_6 e_5, [e_2,e_2] = \beta_2 e_5,$$
$$[e_1,e_3] = \beta_4 e_5 = -[e_3,e_1], [e_2,e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3,e_2] = \gamma_4 e_5.$$

Take $\theta = \frac{\alpha_3(\alpha_4\gamma_1 - \alpha_3\gamma_2)}{\alpha_2\gamma_1^2}$. The base change $y_1 = \frac{\gamma_1}{\alpha_3} e_1, y_2 = e_2, y_3 = e_3, y_4 = \gamma_1 e_4 + \gamma_2 e_5, y_5 = \frac{\alpha_2\gamma_1^2}{\alpha_3^2} e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1] = y_5, [y_1,y_2] = y_4 + \theta y_5, [y_2,y_1] = \frac{\alpha_3\alpha_6}{\alpha_2\gamma_1} y_5, [y_2,y_2] = \frac{\beta_2\alpha_3^2}{\alpha_2\gamma_1^2} y_5, [y_1,y_3] = \frac{\alpha_3\beta_4}{\alpha_2\gamma_1} y_5 = -[y_3,y_1],$$
$$[y_2,y_3] = y_4, [y_3,y_2] = \frac{\alpha_3^2\gamma_4}{\alpha_2\gamma_1^2} y_5.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_4$.

Case 1.2.2: Let $\alpha_1 \neq 0$.

Case 1.2.2.1: Let $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. If $\alpha_3 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_3 e_1 - \alpha_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_3 = 0$. So we can assume $\alpha_3 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5. \quad (4.60)$$

If $\alpha_5 \neq 0$ then with the base change $x_1 = \gamma_1 e_1 - \alpha_5 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. So we can assume $\alpha_5 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Furthermore, if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

The base change $y_1 = e_1, y_2 = \frac{\alpha_1}{\gamma_1} e_2, y_3 = e_3, y_4 = \frac{\alpha_1}{\gamma_1}(\gamma_1 e_4 + \gamma_2 e_5), y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4, [y_1, y_2] = \frac{\alpha_1\alpha_4}{\gamma_1} y_5, [y_2, y_1] = \frac{\alpha_1\alpha_6}{\gamma_1} y_5, [y_2, y_2] = \frac{\alpha_1^2\beta_2}{\gamma_1^2} y_5, [y_1, y_3] = \beta_4 y_5,$$
$$[y_3, y_1] = \beta_6 y_5, [y_2, y_3] = y_4, [y_3, y_2] = \frac{\alpha_1\gamma_4}{\gamma_1} y_5, [y_3, y_3] = \gamma_6 y_5.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_5$.

Case 1.2.2.2: Let $\alpha_1\gamma_2 - \alpha_2\gamma_1 \neq 0$. Without loss of generality we can assume $\alpha_3 = 0$ because if $\alpha_3 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_3 e_1 - \alpha_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$

we can make $\alpha_3 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_1 e_4 + \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Furthermore, if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_1 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_4 e_5, [e_2, e_1] = \alpha_5 e_4 + \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Take $\theta_1 = \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_1}$ and $\theta_2 = \frac{\alpha_1(\alpha_6\gamma_2 - \alpha_5\gamma_1)}{\gamma_1^2\theta_1}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_1}{\gamma_1}e_2, y_3 = e_3, y_4 = \frac{\alpha_1}{\gamma_1}(\gamma_1 e_4 + \gamma_2 e_5), y_5 = \theta_1 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4 + y_5, [y_1, y_2] = \frac{\alpha_1\alpha_4}{\gamma_1\theta_1}e_5, [y_2, y_1] = \frac{\alpha_5}{\gamma_1}y_4 + \theta_2 y_5, [y_2, y_2] = \frac{\alpha_1^2\beta_4}{\gamma_1^2\theta_1}y_5, [y_1, y_3] = \frac{\beta_4}{\theta_1}y_5,$$
$$[y_3, y_1] = \frac{\beta_6}{\theta_1}y_5, [y_2, y_3] = y_4, [y_3, y_2] = \frac{\alpha_1\gamma_4}{\gamma_1\theta_1}y_5, [y_3, y_3] = \frac{\gamma_6}{\theta_1}y_5.$$

Note that if $(\beta_4 + \beta_6, \gamma_6) \neq (0, 0)$ then the base change $x_1 = xy_1 + y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$((where $\theta_1 x^2 + (\beta_4 + \beta_6)x + \gamma_6 = 0$) shows that A is isomorphic to an algebra with the nonzero products given by (4.60). Hence A is isomorphic to $\mathcal{R}_5$.
So let $\beta_4 + \beta_6 = 0 = \gamma_6$. Without loss of generality we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = \gamma_1 y_1 - \alpha_5 y_3, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ we can make $\alpha_5 = 0$. Then we have the following products in A:

$$[y_1, y_1] = y_4 + y_5, [y_1, y_2] = \frac{\alpha_1\alpha_4}{\gamma_1\theta_1}e_5, [y_2, y_1] = \theta_2 y_5, [y_2, y_2] = \frac{\alpha_1^2\beta_4}{\gamma_1^2\theta_1}y_5, [y_1, y_3] = \frac{\beta_4}{\theta_1}y_5 = -[y_3, y_1],$$
$$[y_2, y_3] = y_4, [y_3, y_2] = \frac{\alpha_1\gamma_4}{\gamma_1\theta_1}y_5.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_6$.

Case 2: Let $\gamma_3 \neq 0$.

Case 2.1: Let $\beta_3 = 0$.

Case 2.1.1: Let $\alpha_1 = 0$. If $\alpha_3 = 0$(resp. $\alpha_3 \neq 0$) then the base change $x_1 = e_3, x_2 = e_2, x_3 = e_1, x_4 = e_4, x_5 = e_5$(resp. $x_1 = e_1, x_2 = e_2, x_3 = \gamma_3 e_1 - \alpha_3 e_3, x_4 = e_4, x_5 = e_5$) shows that A is isomorphic to an algebra with the nonzero products given by (4.46). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{101}(\alpha, \beta, \gamma), \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ or $\mathcal{R}_6$.

Case 2.1.2: Let $\alpha_1 \neq 0$.

Case 2.1.2.1: Let $\gamma_1 = 0$. Without loss of generality we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_5 e_1 - \alpha_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Similarly we can assume $\beta_1 = 0$ because if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = \gamma_3 e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_2 e_5, [e_3, e_2] = \gamma_3 e_4 + \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

The base change $x_1 = e_1, x_2 = e_3, x_3 = e_2, x_4 = e_4, x_5 = e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.46). Hence A is isomorphic to $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{101}(\alpha, \beta, \gamma), \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ or $\mathcal{R}_6$.

Case 2.1.2.2: Let $\gamma_1 \neq 0$.

Case 2.1.2.2.1: Let $\gamma_1 + \gamma_3 = 0$. If $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = e_1 + x e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\beta_1 x^2 + (\alpha_3 + \alpha_5)x + \alpha_1 = 0$) we can make $\beta_1 = 0$. so we can assume $\beta_1 = 0$. Furthermore we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = \gamma_1 e_1 - \alpha_5 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = -\gamma_1 e_4 + \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Case 2.1.2.2.1.1: Let $\gamma_6 = 0$.

Case 2.1.2.2.1.1.1: Let $\beta_6 = 0$.

Case 2.1.2.2.1.1.1.1: Let $\gamma_2 + \gamma_4 = 0$. Take $\theta_1 = \frac{\alpha_2\gamma_1 - \alpha_1\gamma_2}{\gamma_1}$ and $\theta_2 = \frac{\alpha_4\gamma_1 - \alpha_3\gamma_2}{\gamma_1}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \frac{\alpha_1}{\gamma_1} e_3, y_4 = \frac{\alpha_1}{\gamma_1}(\gamma_1 e_4 + \gamma_2 e_5), y_5 = e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4 + \theta_1 y_5, [y_1, y_2] = \frac{\alpha_3}{\alpha_1} y_4 + \theta_2 y_5, [y_2, y_1] = \alpha_6 y_5, [y_2, y_2] = \beta_2 y_5, [y_1, y_3] = \frac{\alpha_1 \beta_4}{\gamma_1} y_5,$$
$$[y_2, y_3] = y_4 = -[y_3, y_2].$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_7$.

Case 2.1.2.2.1.1.1.2: Let $\gamma_2 + \gamma_4 \neq 0$. Without loss of generality we can assume $\beta_2 = 0$ because if $\beta_2 \neq 0$ then with the base change $x_1 = e_1, x_2 = (\gamma_2 + \gamma_4)e_2 - \beta_2 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_2 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = -\gamma_1 e_4 + \gamma_4 e_5.$$

Take $\theta_1 = \frac{\alpha_2\gamma_1 + \alpha_1\gamma_4}{\alpha_1(\gamma_2+\gamma_4)}$ and $\theta_2 = \frac{\alpha_4\gamma_1 + \alpha_3\gamma_4}{\alpha_1(\gamma_2+\gamma_4)}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \frac{\alpha_1}{\gamma_1} e_3, y_4 = \frac{\alpha_1}{\gamma_1}(\gamma_1 e_4 - \gamma_4 e_5), y_5 = \frac{\alpha_1(\gamma_2+\gamma_4)}{\gamma_1} e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4 + \theta_1 y_5, [y_1, y_2] = \frac{\alpha_3}{\alpha_1} y_4 + \theta_2 y_5, [y_2, y_1] = \frac{\alpha_6 \gamma_1}{\alpha_1(\gamma_2 + \gamma_4)} y_5, [y_1, y_3] = \frac{\beta_4}{\gamma_2 + \gamma_4} y_5,$$
$$[y_2, y_3] = y_4 + y_5, [y_3, y_2] = -y_4.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_8$.

Case 2.1.2.2.1.1.2: Let $\beta_6 \neq 0$. Note that if $\alpha_6 \neq 0$ then with the base change $x_1 = e_1, x_2 = \beta_6 e_2 - \alpha_6 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_6 = 0$. So let $\alpha_6 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5, [e_3, e_1] = \beta_6 e_5,$$
$$[e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = -\gamma_1 e_4 + \gamma_4 e_5.$$

Take $\theta_1 = \frac{\alpha_2\gamma_1 + \alpha_1\gamma_4}{\alpha_1\beta_6}, \theta_2 = \frac{\alpha_4\gamma_1 + \alpha_3\gamma_4}{\alpha_1\beta_6}$ and $\theta_3 = \frac{\gamma_2+\gamma_4}{\beta_6}$. The base change $y_1 = e_1, y_2 = e_2, y_3 =$

$\frac{\alpha_1}{\gamma_1}e_3, y_4 = \frac{\alpha_1}{\gamma_1}(\gamma_1 e_4 - \gamma_4 e_5), y_5 = \frac{\alpha_1\beta_6}{\gamma_1}e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4 + \theta_1 y_5, [y_1, y_2] = \frac{\alpha_3}{\alpha_1}y_4 + \theta_2 y_5, [y_2, y_2] = \frac{\beta_2\gamma_1}{\alpha_1\beta_6}y_5, [y_1, y_3] = \frac{\beta_4}{\beta_6}y_5, [y_3, y_1] = y_5,$$
$$[y_2, y_3] = y_4 + \theta_3 y_5, [y_3, y_2] = -y_4.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_9$.

Case 2.1.2.2.1.2: Let $\gamma_6 \neq 0$. Without loss of generality we can assume $\beta_2 = 0$ because if $\beta_2 \neq 0$ then with the base change $x_1 = e_1, x_2 = xe_2 + e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$(where $\beta_2 x^2 + (\gamma_2 + \gamma_4)x + \gamma_6 = 0$) we can make $\beta_2 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_1, e_3] = \beta_4 e_5, [e_3, e_1] = \beta_6 e_5,$$
$$[e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = -\gamma_1 e_4 + \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Take $\theta_1 = \frac{\alpha_2\gamma_1 + \alpha_1\gamma_4}{\gamma_1\gamma_6}, \theta_2 = \frac{\alpha_1(\alpha_4\gamma_1 + \alpha_3\gamma_4)}{\gamma_1^2\gamma_6}$ and $\theta_3 = \frac{\alpha_1(\gamma_2+\gamma_4)}{\gamma_1\gamma_6}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_1}{\gamma_1}e_2, y_3 = e_3, y_4 = \frac{\alpha_1}{\gamma_1}(\gamma_1 e_4 - \gamma_4 e_5), y_5 = \gamma_6 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4 + \theta_1 y_5, [y_1, y_2] = \frac{\alpha_3}{\gamma_1}y_4 + \theta_2 y_5, [y_2, y_1] = \frac{\alpha_1\alpha_6}{\gamma_1\gamma_6}y_5, [y_1, y_3] = \frac{\beta_4}{\gamma_6}y_5, [y_3, y_1] = \frac{\beta_6}{\gamma_6}y_5,$$
$$[y_2, y_3] = y_4 + \theta_3 y_5, [y_3, y_2] = -y_4, [y_3, y_3] = y_5.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_{10}$.

Case 2.1.2.2.2: Let $\gamma_1 + \gamma_3 \neq 0$.

Case 2.1.2.2.2.1: Let $\gamma_6 = 0$.

Case 2.1.2.2.2.1.1: Let $\beta_6 = 0$.

Case 2.1.2.2.2.1.1.1: Let $\beta_4 = 0$. Without loss of generality we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_5 e_1 - \alpha_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Furthermore if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = (\gamma_1 + \gamma_3)e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$.

Then we have the following products in A:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5,[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_6e_5,[e_2,e_2]=\beta_2e_5,[e_2,e_3]=\gamma_1e_4+\gamma_2e_5,$$
$$[e_3,e_2]=\gamma_3e_4+\gamma_4e_5.$$

Take $\theta_1=\frac{\alpha_2\gamma_3-\alpha_1\gamma_4}{\gamma_3},\theta_2=\frac{\alpha_4\gamma_3-\alpha_3\gamma_4}{\gamma_3}$ and $\theta_3=\frac{\alpha_1(\gamma_2\gamma_3-\gamma_1\gamma_4)}{\gamma_3^2}$. The base change $y_1=e_1,y_2=e_2,y_3=\frac{\alpha_1}{\gamma_3}e_3,y_4=\frac{\alpha_1}{\gamma_3}(\gamma_3e_4+\gamma_4e_5),y_5=e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1]=y_4+\theta_1y_5,[y_1,y_2]=\frac{\alpha_3}{\alpha_1}y_4+\theta_2y_5,[y_2,y_1]=\alpha_6y_5,[y_2,y_2]=\beta_2y_5,[y_2,y_3]=\frac{\gamma_1}{\gamma_3}y_4+\theta_3y_5,$$
$$[y_3,y_2]=y_4.$$

Then the base change $x_1=y_1,x_2=y_2,x_3=y_3,x_4=y_4,x_5=y_5$ shows that A is isomorphic to $\mathcal{R}_{11}$.

Case 2.1.2.2.2.1.1.2: Let $\beta_4\neq 0$. Take $\theta_1=\frac{\alpha_2\gamma_3-\alpha_1\gamma_4}{\alpha_1\beta_4},\theta_2=\frac{\alpha_4\gamma_3-\alpha_3\gamma_4}{\alpha_1\beta_4},\theta_3=\frac{\alpha_5}{\alpha_1},\theta_4=\frac{\alpha_6\gamma_3-\alpha_5\gamma_4}{\alpha_1\beta_4},\theta_5=\frac{\beta_1}{\alpha_1},\theta_6=\frac{\beta_2\gamma_3-\beta_1\gamma_4}{\alpha_1\beta_4}$ and $\theta_7=\frac{\alpha_1(\gamma_2\gamma_3-\gamma_1\gamma_4)}{\alpha_1\beta_4\gamma_3}$. The base change $y_1=e_1,y_2=e_2,y_3=\frac{\alpha_1}{\gamma_3}e_3,y_4=\frac{\alpha_1}{\gamma_3}(\gamma_3e_4+\gamma_4e_5),y_5=\frac{\alpha_1\beta_4}{\gamma_3}e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1]=y_4+\theta_1y_5,[y_1,y_2]=\frac{\alpha_3}{\alpha_1}y_4+\theta_2y_5,[y_2,y_1]=\theta_3y_4+\theta_4y_5,[y_2,y_2]=\theta_5y_4+\theta_6y_5,[y_1,y_3]=y_5,$$
$$[y_2,y_3]=\frac{\gamma_1}{\gamma_3}y_4+\theta_7y_5,[y_3,y_2]=y_4.$$

Without loss of generality we can assume $\theta_1=0$ because if $\theta_1\neq 0$ then with the base change $x_1=y_1-\theta_1y_3,x_2=y_2,x_3=y_3,x_4=y_4,x_5=y_5$ we can make $\theta_1=0$. Also if $\theta_3\neq 0$ then with the base change $x_1=y_1,x_2=\theta_3y_1-y_2,x_3=y_3,x_4=y_4,x_5=y_5$ we can make $\theta_3=0$. So we can assume $\theta_3=0$. Furthermore we can assume $\theta_5=0$ because if $\theta_5\neq 0$ then with the base change $x_1=y_1,x_2=(\frac{\gamma_1}{\gamma_3}+1)y_2-\theta_5y_3,x_3=y_3,x_4=y_4,x_5=y_5$ we can make $\theta_5=0$. Then we have the following products in A:

$$[y_1,y_1]=y_4,[y_1,y_2]=\frac{\alpha_3}{\alpha_1}y_4+\theta_2y_5,[y_2,y_1]=\theta_4y_5,[y_2,y_2]=\theta_6y_5,[y_1,y_3]=y_5,$$
$$[y_2,y_3]=\frac{\gamma_1}{\gamma_3}y_4+\theta_7y_5,[y_3,y_2]=y_4.$$

Then the base change $x_1=y_1,x_2=y_2,x_3=y_3,x_4=y_4,x_5=y_5$ shows that A is isomorphic to $\mathcal{R}_{12}$.

Case 2.1.2.2.2.1.2: Let $\beta_6 \neq 0$. Without loss of generality we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_5 e_1 - \alpha_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Furthermore if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = (\gamma_1 + \gamma_3)e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_3, e_1] = \beta_6 e_5, [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_3 e_4 + \gamma_4 e_5.$$

Take $\theta_1 = \frac{\alpha_2\gamma_3 - \alpha_1\gamma_4}{\alpha_1\beta_6}, \theta_2 = \frac{\alpha_4\gamma_3 - \alpha_3\gamma_4}{\alpha_1\beta_6}$ and $\theta_3 = \frac{\alpha_1(\gamma_2\gamma_3 - \gamma_1\gamma_4)}{\alpha_1\beta_6\gamma_3}$. The base change $y_1 = e_1, y_2 = e_2, y_3 = \frac{\alpha_1}{\gamma_3} e_3, y_4 = \frac{\alpha_1}{\gamma_3}(\gamma_3 e_4 + \gamma_4 e_5), y_5 = \frac{\alpha_1\beta_6}{\gamma_3} e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = y_4 + \theta_1 y_5, [y_1, y_2] = \frac{\alpha_3}{\alpha_1} y_4 + \theta_2 y_5, [y_2, y_1] = \frac{\alpha_6\gamma_3}{\alpha_1\beta_6} y_5, [y_2, y_2] = \frac{\beta_2\gamma_3}{\alpha_1\beta_6} y_5, [y_1, y_3] = \frac{\beta_4}{\beta_6} y_5,$$
$$[y_3, y_1] = y_5, [y_2, y_3] = \frac{\gamma_1}{\gamma_3} y_4 + \theta_3 y_5, [y_3, y_2] = y_4.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_{13}$.

Case 2.1.2.2.2.2: Let $\gamma_6 \neq 0$. Note that if $\beta_6 \neq 0$ then with the base change $x_1 = \gamma_6 e_1 - \beta_6 e_3, x_2 = e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_6 = 0$. So we can assume $\beta_6 = 0$. Without loss of generality we can assume $\alpha_5 = 0$ because if $\alpha_5 \neq 0$ then with the base change $x_1 = e_1, x_2 = \alpha_5 e_1 - \alpha_1 e_2, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\alpha_5 = 0$. Furthermore if $\beta_1 \neq 0$ then with the base change $x_1 = e_1, x_2 = (\gamma_1 + \gamma_3)e_2 - \beta_1 e_3, x_3 = e_3, x_4 = e_4, x_5 = e_5$ we can make $\beta_1 = 0$. So we can assume $\beta_1 = 0$. Then we have the following products in A:

$$[e_1, e_1] = \alpha_1 e_4 + \alpha_2 e_5, [e_1, e_2] = \alpha_3 e_4 + \alpha_4 e_5, [e_2, e_1] = \alpha_6 e_5, [e_2, e_2] = \beta_2 e_5, [e_1, e_3] = \beta_4 e_5,$$
$$[e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5, [e_3, e_2] = \gamma_3 e_4 + \gamma_4 e_5, [e_3, e_3] = \gamma_6 e_5.$$

Take $\theta_1 = \frac{\alpha_2\gamma_3 - \alpha_1\gamma_4}{\gamma_3\gamma_6}, \theta_2 = \frac{\alpha_1(\alpha_4\gamma_3 - \alpha_3\gamma_4)}{\gamma_3^2\gamma_6}$ and $\theta_3 = \frac{\alpha_1(\gamma_2\gamma_3 - \gamma_1\gamma_4)}{\gamma_3^2\gamma_6}$. The base change $y_1 = e_1, y_2 = \frac{\alpha_1}{\gamma_3} e_2, y_3 = e_3, y_4 = \frac{\alpha_1}{\gamma_3}(\gamma_3 e_4 + \gamma_4 e_5), y_5 = \gamma_6 e_5$ shows that A is isomorphic to the following algebra:

$$[y_1,y_1]=y_4+\theta_1y_5,[y_1,y_2]=\frac{\alpha_3}{\gamma_3}y_4+\theta_2y_5,[y_2,y_1]=\frac{\alpha_1\alpha_6}{\gamma_3\gamma_6}y_5,[y_2,y_2]=\frac{\alpha_1^2\beta_2}{\gamma_3^2\gamma_6}y_5,[y_1,y_3]=\frac{\beta_4}{\gamma_6}y_5,$$
$$[y_2,y_3]=\frac{\gamma_1}{\gamma_3}y_4+\theta_3y_5,[y_3,y_2]=y_4,[y_3,y_3]=y_5.$$

Then the base change $x_1=y_1,x_2=y_2,x_3=y_3,x_4=y_4,x_5=y_5$ shows that A is isomorphic to $\mathcal{R}_{14}$.

Case 2.2: Let $\beta_3\neq 0$. Without loss of generality we can assume $\gamma_1=0$ because if $\gamma_1\neq 0$ then with the base change $x_1=e_1,x_2=\gamma_1e_1-\beta_3e_2,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\gamma_1=0$. Then the products in A are the following:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5,[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_5e_4+\alpha_6e_5,[e_2,e_2]=\beta_1e_4+\beta_2e_5,$$
$$[e_1,e_3]=\beta_3e_4+\beta_4e_5,[e_3,e_1]=\beta_6e_5,[e_2,e_3]=\gamma_2e_5,[e_3,e_2]=\gamma_3e_4+\gamma_4e_5,[e_3,e_3]=\gamma_6e_5.$$

Note that if $\beta_1\neq 0$ then with the base change $x_1=e_1,x_2=\gamma_3e_2-\beta_1e_3,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\beta_1=0$. So we can assume $\beta_1=0$. Then the products in A are the following:

$$[e_1,e_1]=\alpha_1e_4+\alpha_2e_5,[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_5e_4+\alpha_6e_5,[e_2,e_2]=\beta_2e_5,$$
$$[e_1,e_3]=\beta_3e_4+\beta_4e_5,[e_3,e_1]=\beta_6e_5,[e_2,e_3]=\gamma_2e_5,[e_3,e_2]=\gamma_3e_4+\gamma_4e_5,[e_3,e_3]=\gamma_6e_5.$$

Furthermore if $\alpha_1\neq 0$ then with the base change $x_1=\beta_3e_1-\alpha_1e_3,x_2=e_2,x_3=e_3,x_4=e_4,x_5=e_5$ we can make $\alpha_1=0$. So let $\alpha_1=0$. Then we have the following products in A:

$$[e_1,e_1]=\alpha_2e_5,[e_1,e_2]=\alpha_3e_4+\alpha_4e_5,[e_2,e_1]=\alpha_5e_4+\alpha_6e_5,[e_2,e_2]=\beta_2e_5,$$
$$[e_1,e_3]=\beta_3e_4+\beta_4e_5,[e_3,e_1]=\beta_6e_5,[e_2,e_3]=\gamma_2e_5,[e_3,e_2]=\gamma_3e_4+\gamma_4e_5,[e_3,e_3]=\gamma_6e_5.$$

Case 2.2.1: Let $\alpha_5=0$. Then the base change $x_1=e_1,x_2=e_3,x_3=e_2,x_4=e_4,x_5=e_5$ shows that A is isomorphic to an algebra with the nonzero products given by (4.46). Hence A is isomorphic to $\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_{101}(\alpha,\beta,\gamma),\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3,\mathcal{R}_4,\mathcal{R}_5$ or $\mathcal{R}_6$.

Case 2.2.2: Let $\alpha_5\neq 0$. Take $\theta_1=\frac{\beta_3(\alpha_4\gamma_3-\alpha_3\gamma_4)}{\alpha_5^2},\theta_2=\frac{\beta_3(\alpha_6\gamma_3-\alpha_5\gamma_4)}{\alpha_5^2}$ and $\theta_3=\frac{\beta_4\gamma_3-\beta_3\gamma_4}{\alpha_5}$. The base change $y_1=\frac{\gamma_3}{\alpha_5}e_1,y_2=\frac{\beta_3}{\alpha_5}e_2,y_3=e_3,y_4=\frac{\beta_3}{\alpha_5}(\gamma_3e_4+\gamma_4e_5),y_5=e_5$ shows that A is isomorphic to the following algebra:

$$[y_1, y_1] = \frac{\alpha_2\gamma_3^2}{\alpha_5^2} y_5, [y_1, y_2] = \frac{\alpha_3}{\alpha_5} y_4 + \theta_1 y_5, [y_2, y_1] = y_4 + \theta_2 y_5, [y_2, y_2] = \frac{\beta_2\beta_3^2}{\alpha_5^2} y_5, [y_1, y_3] = y_4 + \theta_3 y_5,$$
$$[y_3, y_1] = \frac{\beta_6\gamma_3}{\alpha_5} y_5, [y_2, y_3] = \frac{\beta_3\gamma_2}{\alpha_5} y_5, [y_3, y_2] = y_4, [y_3, y_3] = \gamma_6 y_5.$$

Then the base change $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4, x_5 = y_5$ shows that A is isomorphic to $\mathcal{R}_{15}$. □

Now we give the conditions for two Leibniz algebras of the infinite families to be isomorphic for the families obtained in Theorem 4.3.4.

Class	Isomorphism criterion	Class	Isomorphism criterion
$\mathcal{A}_3(\alpha)$	$\alpha_2^2 = \alpha_1^2$	$\mathcal{A}_{45}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$
$\mathcal{A}_4(\alpha)$	$\alpha_2 = \alpha_1$ or $\alpha_2 = \frac{1}{\alpha_1}$	$\mathcal{A}_{46}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_5(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{47}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_6(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{48}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$
$\mathcal{A}_7(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2^2 = \beta_1^2$) or ($\alpha_2 = \frac{1}{\alpha_1}$ and $\beta_2^2 = (\frac{\beta_1}{\alpha_1})^2$)	$\mathcal{A}_{49}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$
$\mathcal{A}_9(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{50}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{13}(\alpha,\beta)$	$(\alpha_2+\beta_2)^2+2(\alpha_2-\beta_2) = (\alpha_1+\beta_1)^2+2(\alpha_1-\beta_1)$	$\mathcal{A}_{51}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{16}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{52}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{17}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{53}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{22}(\alpha)$	$\alpha_2 = \alpha_1$ or $\alpha_2 = \frac{1}{\alpha_1}$	$\mathcal{A}_{55}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$
$\mathcal{A}_{23}(\alpha)$	$\alpha_2 = \alpha_1$ or $\alpha_2 = \frac{1}{\alpha_1}$	$\mathcal{A}_{57}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$
$\mathcal{A}_{24}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{58}(\alpha,\beta,\gamma)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$) or ($\alpha_2 = \frac{1}{\alpha_1}$ and $\beta_2 = -\alpha_1\beta_1$ and $\gamma_2 = -\beta_1 - \alpha_1\beta_1 + \gamma_1$)
$\mathcal{A}_{25}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{60}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = \frac{1}{\alpha_1}$ and $\beta_2 = -1 - \alpha_1 - \alpha_1\beta_1$)
$\mathcal{A}_{26}(\alpha,\beta)$	$\alpha_2^2 = \alpha_1^2$ and $\beta_2 = \beta_1$	$\mathcal{A}_{61}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = -\alpha_1$ and $\beta_2 = -2\alpha_1 + \beta_1$)
$\mathcal{A}_{27}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$	$\mathcal{A}_{62}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$
$\mathcal{A}_{28}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$	$\mathcal{A}_{63}(\alpha,\beta,\gamma)$	hard to compute
$\mathcal{A}_{29}(\alpha,\beta,\gamma)$	$\alpha_2 = \alpha_1$ and $\beta_2^2 = \beta_1^2$ and $\gamma_2 = \gamma_1$	$\mathcal{A}_{64}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{30}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{66}(\alpha)$	$\alpha_2^2 = \alpha_1^2$
$\mathcal{A}_{32}(\alpha,\beta,\gamma)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$	$\mathcal{A}_{67}(\alpha)$	$\alpha_2 = \alpha_1$ or $\alpha_2 = \frac{1}{\alpha_1}$
$\mathcal{A}_{33}(\alpha,\beta,\gamma)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$) or ($\alpha_2 = -\alpha_1$ and $\beta_2 = -\beta_1$ and $\gamma_2 = -2\alpha_1 - 2\beta_1 + \gamma_1$)	$\mathcal{A}_{68}(\alpha)$	$\alpha_2 = \alpha_1$ or $\alpha_2 = -1 - \alpha_1$
$\mathcal{A}_{34}(\alpha,\beta,\gamma,\theta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$ and $\theta_2 = \theta_1$) or ($\alpha_2 = \frac{1}{\alpha_1}$ and $\beta_2 = -\alpha_1\beta_1$ and $\gamma_2 = -\alpha_1\gamma_1$ and $\theta_2 = -\beta_1 - \alpha_1\beta_1 - \gamma_1 - \alpha_1\gamma_1 + \theta_1$)	$\mathcal{A}_{70}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{39}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2^2 - 2\alpha_1\beta_2 - \beta_1^2 + 2\alpha_1\beta_1 = 0$	$\mathcal{A}_{71}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{40}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{72}(\alpha)$	$\alpha_2 = \alpha_1$ or $\alpha_2 = \frac{\alpha_1}{2\alpha_1-1}$
$\mathcal{A}_{42}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{73}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{43}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{74}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = \frac{1}{\alpha_1}$ and $\beta_2 = \frac{\alpha_1\beta_1}{\alpha_1\beta_1+\beta_1-1}$)
$\mathcal{A}_{44}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = \beta_1$ and $\beta_2 = \alpha_1$)	$\mathcal{A}_{75}(\alpha)$	$\alpha_2 = \alpha_1$

$\mathcal{A}_{76}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{91}(\alpha,\beta,\gamma)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$) or ($\alpha_2 = -\alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = -\gamma_1$)
$\mathcal{A}_{77}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{92}(\alpha,\beta,\gamma)$	$\alpha_2^2 = \alpha_1^2$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$
$\mathcal{A}_{78}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{93}(\alpha,\beta)$	$\alpha_2^2 = \alpha_1^2$ and $\beta_2 = \beta_1$
$\mathcal{A}_{81}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = -\alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = \frac{\alpha_1\beta_1-(\beta_1-2)\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$ and $\beta_2 = \frac{\alpha_1^2+2\beta_1+4-\alpha_1\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$) or ($\alpha_2 = \frac{-\alpha_1\beta_1+(\beta_1-2)\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$ and $\beta_2 = \frac{\alpha_1^2+2\beta_1+4-\alpha_1\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$) or ($\alpha_2 = \frac{-\alpha_1\beta_1-(\beta_1-2)\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$ and $\beta_2 = \frac{\alpha_1^2+2\beta_1+4+\alpha_1\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$) or ($\alpha_2 = \frac{\alpha_1\beta_1+(\beta_1-2)\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$ and $\beta_2 = \frac{\alpha_1^2+2\beta_1+4+\alpha_1\sqrt{\alpha_1^2+4\beta_1+4}}{2\beta_1}$)	$\mathcal{A}_{94}(\alpha,\beta,\gamma)$	$\alpha_2^2 = \alpha_1^2$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$
$\mathcal{A}_{82}(\alpha)$	$\alpha_2 = \alpha_1$	$\mathcal{A}_{95}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = \alpha_1$ and $\beta_2 = -\beta_1$
$\mathcal{A}_{84}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$	$\mathcal{A}_{96}(\alpha,\beta,\gamma,\theta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$ and $\theta_2 = \theta_1$) or ($\alpha_2 = -\alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = -\gamma_1$ and $\theta_2 = \theta_1$)
$\mathcal{A}_{85}(\alpha,\beta,\gamma)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$	$\mathcal{A}_{97}(\alpha)$	$\alpha_2 = \alpha_1$
$\mathcal{A}_{87}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$	$\mathcal{A}_{98}(\alpha)$	$\alpha_2^2 = \alpha_1^2$
$\mathcal{A}_{88}(\alpha,\beta)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$	$\mathcal{A}_{99}(\alpha,\beta)$	($\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$) or ($\alpha_2 = -\beta_1$ and $\beta_2 = -\alpha_1$) or ($\alpha_2 = \alpha_1\beta_1$ and $\beta_2 = -\frac{1}{\alpha_1}$) or ($\alpha_2 = \alpha_1\beta_1$ and $\beta_2 = \frac{1}{\beta_1}$) or ($\alpha_2 = -\frac{1}{\beta_1}$ and $\beta_2 = -\alpha_1\beta_1$) or ($\alpha_2 = \frac{1}{\alpha_1}$ and $\beta_2 = -\alpha_1\beta_1$)
$\mathcal{A}_{89}(\alpha,\beta,\gamma)$	$\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$ and $\gamma_2 = \gamma_1$	$\mathcal{A}_{100}(\alpha,\beta)$	hard to compute
$\mathcal{A}_{90}(\alpha)$	$\alpha_2^2 = \alpha_1^2$	$\mathcal{A}_{101}(\alpha,\beta,\gamma)$	hard to compute

Table 4.2: Condition of isomorphism classes

Throughout this work, we use Mathematica program implementing Algorithm 2.6 given in [10] which determines if given two Leibniz algebras are isomorphic. This program also gives the change of basis if given two Leibniz are isomorphic. However we note that even with the help of this computer program it is too difficult to get change of bases for some cases. We don't give change of bases for those cases. Furthermore, for some difficult cases in Theorem 4.3.4, this program cannot decide whether given two Leibniz algebras are isomorphic.

Note that in Theorem 4.3.4 we obtain 101 distinct isomorphism classes $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{101}(\alpha, \beta, \gamma)$; and additional 15 algebras $\mathcal{R}_1, \ldots, \mathcal{R}_{15}$ that are not distinct and can be isomorphic to the isomorphism classes $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{101}(\alpha, \beta, \gamma)$. Next we compare the classification of non-split nilpotent Lie and Leibniz algebras. For this purpose we give the following table. It can be seen from the table that the number of isomorphism classes increases drastically with the dimension for Leibniz algebras.

Dimension	Number of isomorphism classes of non-split nilpotent Lie algebra	Number of isomorphism classes of non-split nilpotent Leibniz algebra
1	-	-
2	-	1 single algebra
3	1 single algebra	4 single algebras and 1 infinite family
4	1 single algebra	23 single algebras and 3 infinite families
5	6 single algebras	149 single algebras and 118 infinite families(plus 15 remained algebras)

Table 4.3: Comparision of classification of nilpotent Lie algebras and Leibniz algebras

REFERENCES

[1] A. O. Abdulkareem, I. S. Rakhimov, and S. K. Said Husain. Isomorphism classes and invariants of low-dimensional filiform Leibniz algebras. *Linear Multilinear Algebra*, 63(11):2254–2274, 2015.

[2] S. Albeverio, B. A. Omirov, and I. S. Rakhimov. Classification of 4-dimensional nilpotent complex Leibniz algebras. *Extracta Math.*, 21(3):197–210, 2006.

[3] Sh. A. Ayupov and B. A. Omirov. On Leibniz algebras. In *Algebra and operator theory (Tashkent, 1997)*, pages 1–12. Kluwer Acad. Publ., Dordrecht, 1998.

[4] Sh. A. Ayupov and B. A. Omirov. On 3-dimensional Leibniz algebras. *Uzbek. Mat. Zh.*, (1):9–14, 1999.

[5] Sh. A. Ayupov and B. A. Omirov. On some classes of nilpotent Leibniz algebras. *Sibirsk. Mat. Zh.*, 42(1):18–29, i, 2001.

[6] J. M. A. Bermúdez and M. Goze. Classification des algèbres de Lie nilpotentes complexes de dimension 7. *Arch. Math. (Basel)*, 52(2):175–185, 1989.

[7] A. Bloh. On a generalization of the concept of Lie algebra. *Dokl. Akad. Nauk SSSR*, 165:471–473, 1965.

[8] É. Cartan. *Sur la structure des groupes de transformations finis et continus*, volume 826. Nony, 1894.

[9] J. M. Casas, M. A. Insua, M. Ladra, and S. Ladra. Algorithm for testing the leibniz algebra structure. In *Computer Aided Systems Theory-EUROCAST 2009*, pages 177–186. Springer, 2009.

[10] J. M. Casas, M. A. Insua, M. Ladra, and S. Ladra. An algorithm for the classification of 3-dimensional complex Leibniz algebras. *Linear Algebra Appl.*, 436(9):3747–3756, 2012.

[11] W. A. de Graaf. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J. Algebra*, 309(2):640–653, 2007.

[12] F. De Terán. Canonical forms for congruence of matrices: a tribute to hw turnbull and ac aitken. In *Actas del II congreso de la red ALAMA, Valencia*, pages 2010–1, 2010.

[13] I. Demir, K. C. Misra, and E. Stitzinger. On some structures of Leibniz algebras. In *Recent advances in representation theory, quantum groups, algebraic geometry, and related topics*, volume 623 of *Contemp. Math.*, pages 41–54. Amer. Math. Soc., Providence, RI, 2014.

[14] I. Demir, K. C. Misra, and E. Stitzinger. Classification of Some Solvable Leibniz Algebras. *Algebr. Represent. Theory*, 19(2):405–417, 2016.

[15] I. Demir, K. C. Misra, and E. Stitzinger. On classification of four dimensional nilpotent leibniz algebras. *Communications in Algebra*, forthcoming.

[16] F. Deraman, I. S. Rakhimov, S. K. Said Husain, E. T. Baskoro, and D. Suprijanto. Isomorphism classes and invariants for a subclass of nine-dimensional filiform leibniz algebras. In *AIP Conference Proceedings-American Institute of Physics*, volume 1450, page 326, 2012.

[17] E. B. Dynkin. The structure of semi-simple algebras. *Uspekhi Matematicheskikh Nauk*, 2(4):59–127, 1947.

[18] M. A. Insua and M. Ladra. Gröbner bases in universal enveloping algebras of Leibniz algebras. *J. Symbolic Comput.*, 44(5):517–526, 2009.

[19] E. E. Levi. Sulla struttura dei gruppi finiti e continui. *Atti Accad. Sci. Torino*, 40:551–565, 1905.

[20] J. L. Loday. *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[21] J. L. Loday. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math. (2)*, 39(3-4):269–293, 1993.

[22] J. L. Loday and T. Pirashvili. Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.*, 296(1):139–158, 1993.

[23] L. Magnin. Determination of 7-dimensional indecomposable nilpotent complex Lie algebras by adjoining a derivation to 6-dimensional Lie algebras. *Algebr. Represent. Theory*, 13(6):723–753, 2010.

[24] A. I. Malcev. Solvable Lie algebras. *Amer. Math. Soc. Translation*, 1950(27):36, 1950.

[25] V. V. Morozov. Classification of nilpotent Lie algebras of sixth order. *Izv. Vysš. Učebn. Zaved. Matematika*, 1958(4 (5)):161–171, 1958.

[26] B. A. Omirov and I. S. Rakhimov. On Lie-like complex filiform Leibniz algebras. *Bull. Aust. Math. Soc.*, 79(3):391–404, 2009.

[27] I. S. Rakhimov and U. D. Bekbaev. On isomorphisms and invariants of finite dimensional complex filiform Leibniz algebras. *Comm. Algebra*, 38(12):4705–4738, 2010.

[28] I. S. Rakhimov and M. A. Hassan. On isomorphism criteria for Leibniz central extensions of a linear deformation of μ_n. *Internat. J. Algebra Comput.*, 21(5):715–729, 2011.

[29] I. S. Rakhimov and M. A. Hassan. On low-dimensional filiform Leibniz algebras and their invariants. *Bull. Malays. Math. Sci. Soc. (2)*, 34(3):475–485, 2011.

[30] I. S. Rakhimov and S. J. Langari. A cohomological approach for classifying nilpotent Leibniz algebras. *Int. J. Algebra*, 4(1-4):153–163, 2010.

[31] I. S. Rakhimov, S. J. Langari, and M. B. Langari. On central extensions of nulfiliform Leibniz algebras. *Int. J. Algebra*, 3(5-8):271–280, 2009.

[32] I. S. Rakhimov and S. K. Said Husain. Classification of a subclass of low-dimensional complex filiform Leibniz algebras. *Linear Multilinear Algebra*, 59(3):339–354, 2011.

[33] I. S. Rakhimov and S. K. Said Husain. On isomorphism classes and invariants of a subclass of low-dimensional complex filiform Leibniz algebras. *Linear Multilinear Algebra*, 59(2):205–220, 2011.

[34] I. S. Rakhimov and J. Sozan. Description of nine dimensional complex filiform Leibniz algebras arising from naturally graded non Lie filiform Leibniz algebras. *Int. J. Algebra*, 3(17-20):969–980, 2009.

[35] I. M. Rikhsiboev, I. S. Rakhimov, E. T. Baskoro, and D. Suprijanto. Classification of three dimensional complex leibniz algebras. In *AIP Conference Proceedings-American Institute of Physics*, volume 1450, page 358, 2012.

[36] M. Romdhani. Classification of real and complex nilpotent Lie algebras of dimension 7. *Linear and Multilinear Algebra*, 24(3):167–189, 1989.

[37] È. N. Safiullina. Classification of nilpotent Lie algebras of order 7. In *Candidates' Works (1964), Math., Mech., Phys. (Russian)*, pages 66–69. Izdat. Kazan. Univ., Kazan, 1964.

[38] C. Seeley. Some nilpotent Lie algebras of even dimension. *Bull. Austral. Math. Soc.*, 45(1):71–77, 1992.

[39] C. Seeley. 7-dimensional nilpotent Lie algebras. *Trans. Amer. Math. Soc.*, 335(2):479–496, 1993.

[40] T. Skjelbred and T. Sund. Sur la classification des algèbres de Lie nilpotentes. *C. R. Acad. Sci. Paris Sér. A-B*, 286(5):A241–A242, 1978.

[41] H. W. Turnbull and A. C. Aitken. *An introduction to the theory of canonical matrices.* Dover Publications, Inc., New York, 1961.

[42] K. A. Umlauf. *Über die zusammensetzung der endlichen continuierlichen transformationsgruppen, insbesondre der gruppen vom range null...* Druck von Breitkopf & Härtel, 1891.

[43] M. Vergne. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes. *Bull. Soc. Math. France*, 98:81–116, 1970.