PART I

DEFINITIONS, ILLUSTRATIONS AND ELEMENTARY THEOREMS

1. Arithmetical definition of ordinary complex numbers. The following purely arithmetical theory of couples (a, b) of real numbers differs only in unessential points from the initial theory of W. R. Hamilton*. Two couples (a, b) and (c, d) are called equal if and only if $a = c$, $b = d$. Addition, subtraction and multiplication of two couples are defined by the formulas†

$$\left.\begin{aligned}(a, b) + (c, d) &= (a + c,\ b + d)\\ (a, b) - (c, d) &= (a - c,\ b - d)\\ (a, b)\,(c, d) &= (ac - bd,\ ad + bc)\end{aligned}\right\} \qquad (1).$$

Addition is seen to be commutative and associative:

$$x + x' = x' + x, \quad (x + x') + x'' = x + (x' + x'') \qquad (2),$$

where x, x', x'' are any couples. Multiplication is commutative, associative and distributive:

$$xx' = x'x, \quad (xx')\,x'' = x\,(x'x'') \qquad (3),$$

$$x\,(x' + x'') = xx' + xx'', \quad (x' + x'')\,x = x'x + x''x \qquad (4).$$

* *Trans. Irish Acad.*, vol. 17 (1837), p. 293; *Lectures on Quaternions*, 1853, Preface.

† Each couple (a, b) uniquely determines a vector from the origin O to the point A with the rectangular coordinates a, b. The sum of two vectors from O to A and the point $C = (c, d)$ is defined to be the vector from O to the fourth vertex S of the parallelogram having the lines OA and OC as two sides. The coordinates of S are $a + c$, $b + d$. Subtraction of vectors is the operation inverse to addition; thus $OS - OA = OC$. To define the product of the vectors from O to A and C, we employ initially the polar coordinates r, θ and r', θ' of A and C. Then $OA \cdot OC$ is defined to be the vector from O to the point P with the polar coordinates rr', $\theta + \theta'$. Since A has the rectangular coordinates $a = r\cos\theta$, $b = r\sin\theta$, and similarly for C and P, the expansions of $\cos(\theta + \theta')$ and $\sin(\theta + \theta')$ lead to the third relation (1) between the rectangular coordinates of A, C, P.

Division is defined as the operation inverse to multiplication. Division except by (0, 0) is possible and unique:

$$\frac{(c, d)}{(a, b)} = \left(\frac{ac + bd}{a^2 + b^2}, \frac{ad - bc}{a^2 + b^2}\right) \qquad (5).$$

In particular, we have

$$(a, 0) \pm (c, 0) = (a \pm c, 0), \quad (a, 0)(c, 0) = (ac, 0),$$

$$\frac{(c, 0)}{(a, 0)} = \left(\frac{c}{a}, 0\right) \quad \text{if } a \neq 0.$$

Hence the couples $(a, 0)$ combine under the above defined addition, multiplication, etc., exactly as the real numbers a combine under ordinary addition, multiplication, etc. Without introducing any contradiction, we may and shall impose upon our system of couples (a, b), subject to the above definitions of addition, etc., the further assumption * that the couple $(a, 0)$ shall be the real number a. For brevity write i for (0, 1). Then

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1,$$

$$(a, b) = (a, 0) + (0, b) = a + (b, 0)(0, 1) = a + bi.$$

The resulting symbol $a + bi$ is called a complex number. Relations (1) and (5) now take the familiar forms

$$\left.\begin{aligned} (a+bi) \pm (c+di) &= (a \pm c) + (b \pm d)\, i \\ (a+bi)(c+di) &= (ac - bd) + (ad + bc)\, i \\ \frac{c+di}{a+bi} &= \frac{ac+bd}{a^2+b^2} + \frac{ad-bc}{a^2+b^2}\, i \end{aligned}\right\} \qquad (6),$$

where, for the last, $a + bi \neq 0$, i.e. a and b are not both zero.

2. Number fields. A set of complex numbers is called a number field (domain of rationality or Körper) if the rational operations can always be performed unambiguously within the set. In other words, the sum, difference, product and quotient (the divisor not being zero) of any two equal or distinct numbers of the set must be numbers belonging to the set.

In view of (6), all complex numbers $a + bi$ form a field. Again, all real numbers form a field. The set of all rational numbers is a field, but the set of all integers is not.

* Just as the natural numbers are included among the signed integers, the integers among the rational numbers, and the latter among the real numbers defined by means of them. In the same train of ideas, 1 is often used to denote the principal unit (§ 7, § 11), and the number e for the scalar matrix S_e (§ 4, second foot-note).

3. Matrices. The concept matrix* affords an excellent introduction to the subject of this tract and, moreover, is of special importance in the general theory. We shall consider only square matrices of n rows each containing n elements. For example, if $n=2$,

$$m=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mu=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{7}$$

are matrices, the elements of the first matrix m being a, b, c, d. Each element may be any number of a given number field F. We shall say that m and μ are equal if and only if their corresponding elements are equal, $a=\alpha$, etc. Addition and multiplication are defined by

$$m+\mu=\begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix}, \quad m\mu=\begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{pmatrix} \tag{8}.$$

The element in the ith row and jth column of the product is the sum of the products of each element of the ith row of the first matrix by the corresponding element of the jth column of the second matrix, i.e. first by first, second by second, etc. This rule holds also for matrices of n^2 elements. Of the four possible rules for expressing the product of two determinants of order n as a determinant of order n, the above is the only rule which holds also for matrices.

With the exception of (3_1), the laws (2)—(4) for addition and multiplication hold for matrices. Since the product (8_2) is in general altered when the Roman and Greek letters are interchanged, matric multiplication is usually not commutative. Accordingly we shall see that we must distinguish between two distinct kinds of division of matrices. To this end, note that

$$\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} m = m \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} \tag{9}.$$

In particular, the *unit* matrix

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

has the property that $Im=mI=m$, for every matrix m. By the *inverse* of a matrix m whose determinant $\Delta=|m|$ is not zero is meant

* Cayley, *Phil. Trans. London*, vol. 148 (1858), pp. 17—37 (=*Coll. Math. Papers*, II, pp. 475, 604).

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$$m^{-1}=\begin{pmatrix}\frac{d}{\Delta} & \frac{-b}{\Delta}\\ \frac{-c}{\Delta} & \frac{a}{\Delta}\end{pmatrix} \qquad (11),$$

if $n=2$, while if $n \gtreqless 2$, we employ as the element in the ith row and jth column the quotient of the co-factor of the element in the jth row and ith column of Δ by Δ. Then

$$mm^{-1}=m^{-1}m=I \qquad (12).$$

Given two matrices m and p such that $|m| \neq 0$, we can find one and only one matrix $\mu=m^{-1}p$ such that $m\mu=p$, also one and only one matrix $\nu=pm^{-1}$ such that $\nu m=p$. These respective kinds of division by p by m shall be called *right-hand and left-hand division.*

On the contrary, if $|m|=0$, there is no matrix μ for which $m\mu=I$, since this would imply $0\,|\mu|=|I|=1$. Likewise, there is no matrix ν for which $\nu m=I$.

Thus right- and left-hand division by m are each always possible and unique if and only if the determinant of m is not zero.

Addition, subtraction, multiplication and division of matrices with elements in a field F lead to matrices with elements in F. Accordingly we shall speak of the matric algebra over the field F. When F is the field of all complex numbers, the field of all real numbers, or that of all rational numbers, we have the complex, real or rational matric algebra of square matrices of n^2 elements.

4. A matric algebra viewed as a linear algebra*. Taking $n=2$, we shall make use of the particular matrices

$$e_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad e_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad e_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad e_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix} \qquad (13).$$

Their sixteen products by twos are

$$e_{ij}e_{jk}=e_{ik},\quad e_{ij}e_{tk}=0 \quad (t\neq j) \qquad (14).$$

If m is a matrix and e is a number, we shall define the product† em

* For references, see § 13.

† In the product (9) we may therefore replace the "scalar matrix" $\begin{pmatrix}e&0\\0&e\end{pmatrix}=S_e$ by the number e. This becomes intuitive if we note that $S_e=eI$. Since $S_e+S_f=S_{e+f}$, $S_eS_f=S_{ef}$, etc., the algebra of all scalar matrices over a field F is abstractly identical with F. This replacement of S_e by e is similar to that of $(a, 0)$ by a in § 1.

or me to be the matrix each of whose elements is the product of e by the corresponding element of m:

$$e\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} e = \begin{pmatrix} ea & eb \\ ec & ed \end{pmatrix} \qquad (15).$$

In view of (13), matrices (7) and (8) may be expressed in the form

$$\left.\begin{aligned} m &= ae_{11} + be_{12} + ce_{21} + de_{22} \\ \mu &= \alpha e_{11} + \beta e_{12} + \gamma e_{21} + \delta e_{22} \end{aligned}\right\} \qquad (16),$$

$$\left.\begin{aligned} m + \mu &= (a + \alpha)\, e_{11} + (b + \beta)\, e_{12} + (c + \gamma)\, e_{21} + (d + \delta)\, e_{22} \\ m\mu &= (a\alpha + b\gamma)\, e_{11} + (a\beta + b\delta)\, e_{12} + (c\alpha + d\gamma)\, e_{21} + (c\beta + d\delta)\, e_{22} \end{aligned}\right\} \qquad (17).$$

The last may also be found from (16) by use of relations (14).

The set of hyper-complex numbers $ae_{11} + \dots + de_{22}$, in which $a, \dots, d$ range independently over a field F, and for which addition and multiplication are defined by (17), is called a linear associative algebra over F with the four units $e_{11}, \dots, e_{22}$ subject to the multiplication table (14).

For any n, let e_{ij} be the square matrix of n^2 elements all zero except that in the ith row and jth column which is unity. Then relations (14) hold. We obtain a linear associative algebra with n^2 units e_{ij}.

5. General definition of hyper-complex numbers and linear algebras*. We shall generalize the notion of couples in § 1 and, with a change of notation, the notion of quadruples (7). Consider the set of all *n-tuples* $(x_1, \dots, x_n)$, whose *coordinates* $x_1, \dots, x_n$ range independently over a given number field F.

Two n-tuples are called equal if and only if their corresponding coordinates are equal.

Addition and subtraction of n-tuples are defined by

$$(x_1, \dots, x_n) \pm (x_1', \dots, x_n') = (x_1 \pm x_1', \dots, x_n \pm x_n') \qquad (18).$$

The product of any number ρ of the field F and any n-tuple

$$x = (x_1, \dots, x_n)$$

is defined to be

$$\rho x = x\rho = (\rho x_1, \dots, \rho x_n) \qquad (19).$$

* Hamilton's *Lectures on Quaternions*, 1853, Introduction. For definitions by independent postulates, see Dickson, *Trans. Amer. Math. Soc.*, vol. 4 (1903), p. 21; vol. 6 (1905), p. 344.

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The n *units* are defined to be

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ e_n = (0, \ldots, 0, 1).$$

Hence any n-tuple x can be expressed in the form

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n.$$

We shall call x a *hyper-complex* number, or briefly a number. In view of the definition of equality of n-tuples, x and

$$x' = x_1' e_1 + \ldots + x_n' e_n$$

are equal if and only if $x_1 = x_1', \ldots, x_n = x_n'$. In particular, $x = 0$ implies that each $x_i = 0$. Hence the units $e_1, \ldots, e_n$ are linearly independent with respect to the field F.

It is assumed that any two such numbers x and x' can be combined by an operation called multiplication subject to the distributive laws (4):

$$xx' = \sum_{i,j=1}^{n} x_i x_j' e_i e_j,$$

and such that the product xx' is a number $\Sigma z_i e_i$ with coordinates in F. Necessary conditions for the latter property are

$$e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \qquad (i, j = 1, \ldots, n\,;\ \gamma\text{'s in } F) \qquad (20).$$

These are sufficient conditions, since they imply

$$xx' = y \equiv \Sigma y_k e_k, \quad y_k = \sum_{i,j=1}^{n} x_i x_j' \gamma_{ijk} \qquad (k = 1, \ldots, n) \qquad (21).$$

Properties (18) and (19) of n-tuples give

$$x \pm x' = \sum_{i=1}^{n} (x_i \pm x_i') e_i, \quad \rho x = x\rho = \sum_{i=1}^{n} (\rho x_i) e_i \qquad (22),$$

if ρ is in F. The set of all numbers $\Sigma x_i e_i$, with coordinates in F, combined under multiplication as in (21), under addition and subtraction as in (22_1), and under multiplication by a number ρ of F as in (22_2), shall be said to form a *linear algebra* (or system of hyper-complex numbers) over the field F, with the units $e_1, \ldots, e_n$ (linearly independent with respect to F) and the multiplication table (20). The n^3 numbers γ_{ijk} are called the constants of multiplication. Neither the commutative nor the associative law of multiplication is assumed.