$\boxed{\text{Evolution algebras are not power associative (in general!)}}$ ①

4/16/21

Recall, in an evolution algebra A with natural basis $\{e_1, \dots, e_n\} = B$

$$ab = \sum_{i,j} \alpha_i \beta_i \omega_{ji} e_j \qquad a = \sum \alpha_i e_i \quad b = \sum \beta_i e_i$$

where $[\omega_{ji}] = M_B(A)$ is the structure matrix with respect to B

power associative algebra means $a^i(a^j a^k) = (a^i a^j) a^k \quad \forall a \in A$

$i, j, k \in \{1, 2, 3, \dots\}$

def. $a^1 = a \quad a^{k+1} = a^k a$

(i.e. the subalgebra generated by any element a is associative)

$\text{span}\{a, a^2, a^3, \dots\}$

$$e_i^2(e_i e_i) = e_i^2 e_i^2 = \left(\sum_k \omega_{ki} e_k\right)\left(\sum_j \omega_{ji} e_j\right) = \sum_k \omega_{ki}^2 e_k^2$$

$$(e_i^2 e_i) e_i = \left(\left(\sum_k \omega_{ki} e_k\right) e_i\right) e_i = \omega_{ii} e_i^2 e_i = \omega_{ii}\left(\sum_k \omega_{ki} e_k\right) e_i$$

$$= \omega_{ii}\, \omega_{ii} e_i^2 = \omega_{ii}^2 e_i^2$$

so $e_i^2(e_i e_i) \neq (e_i^2 e_i) e_i$ unless $M_B(A)$ is diagonal

$$M_B(A) = \begin{bmatrix} \omega_{11} & & 0 \\ & \ddots & \\ 0 & & \omega_{nn} \end{bmatrix} \qquad \omega_{ki} = 0 \quad k \neq i$$

Exercise 2(b)

$A =$ evolution algebra is power associative $\overset{\Rightarrow \text{ clear}}{\Longleftrightarrow} \exists$ nat basis B $M_B(A)$ diagonal?

$\Leftarrow$?

Exercise 2(a)

If $A =$ any algebra, then

A is power associative $\Longleftrightarrow$ $\boxed{a^{k+l} = a^k a^l \quad \forall\, k, l \geq 1}$

$\boxed{(a^i a^j) a^k = a^i (a^j a^k) \quad \forall\, i, j, k \geq 1}$

All Jordan algebras are power associative

Jordan algebra: $xy = yx \qquad (xy)x^2 = x(x^2 y)$

let (x,y,z) be the associator $(x,y,z) = (xy)z - x(yz)$

Then $(x,y,x^2) = (xy)x^2 - x(yx^2) = 0 \qquad \forall\, x,y$

$\therefore\ (x+\lambda z, y, (x+\lambda z)^2) = 0 \qquad \forall\, x,y,z$ & λ a number $\lambda \neq 0$

$$\big((x+\lambda z)y\big)(x+\lambda z)^2 - (x+\lambda z)\big(y(x+\lambda z)^2\big) = 0$$

(1) $(xy + \lambda zy)(x^2 + 2\lambda xz + \lambda^2 z^2) - (x+\lambda z)\big(yx^2 + 2\lambda y(xz) + \lambda^2 yz^2\big) = 0$

coeffricent of λ in (1)

$C_1 = (zy)x^2 + 2(xy)(xz) - 2x(y(xz)) - z(yx^2)$

Constant term $C_0 = (xy)x^2 - x(yx^2) = 0$ (Jordan axiom)

coefficient of λ^2 $C_2 = (xy)z^2 + 2(zy)(xz) - x(yz^2) - 2z(y(xz))$

coefficient of λ^3 $C_3 = (zy)z^2 - z(yz^2)$

$$C_3\lambda^3 + C_2\lambda^2 + C_1\lambda + \underbrace{C_0}_{=0} = 0 \qquad \forall\, x,y,z$$

divide by λ

$$C_3\lambda^2 + C_2\lambda + C_1 = 0$$

let $\lambda \to 0 \quad \therefore\ C_1 = 0$

$$C_1 = (z,y,x^2) + 2(x,y,xz) = 0$$

$$(z, y, (x+\lambda w)^2) + 2(x+\lambda w, y, (x+\lambda w)z) = 0$$

$\forall\ x, y, z, w$
$\lambda \neq 0$ numbers

(2)
$$\begin{cases} (zy)(x^2 + 2\lambda xw + \lambda^2 w^2) \\ - z(y(x^2 + 2\lambda xw + \lambda^2 w^2) \\ + 2\Big[(xy)(xz) + \lambda(wy)(xz) + \lambda(xy)(wz) + \lambda^2 (wy)(wz) \\ \quad - x(y(xz)) - \lambda x(y(wz)) - \lambda w(y(xz)) - \lambda^2 w(y(wz)) \Big] \\ = 0 \end{cases}$$

coeff of λ in (2)

$$2(zy)(xw) - 2z(y(xw)) + 2(wy)(xz) + 2(xy)(wz)$$
$$- 2x(y(wz)) - 2w(y(xz)) = 0 \quad \text{because}$$

the constant term $= (zy)x^2 - z(yx^2) + 2(xy)(xz) - 2x(y(xz))$
$$= (z, y, x^2) + 2(x, y, xz) = C_1 = 0$$

Thus $(z, y, xw) + (x, y, wz) + (w, y, xz) = 0 \quad \forall\ x, y, z, w$

(3) $\boxed{(zy)(xw) - z(y(xw)) + (xy)(wz) - x(y(wz)) + (wy)(xz) - w(y(xz)) = 0}$

Define the operator $R_x : A \to A$ for $x \in A$
by $R_x a = xa$ (multiplication operator)
$a \in A$

(3) can be rewritten as operators (acting on w)

(4) $R_{zy} R_x - R_z R_y R_x + R_{xy} R_z - R_x R_y R_z + R_{xz} R_y - R_{y(xz)} = 0$

In (4) put $y = x$ $\quad z = x^{i-1}$

$$R_{x^i}R_x - R_{x^{i-1}}(R_x)^2 + R_{x^2}R_{x^{i-1}} - (R_x)^2 R_{x^{i-1}} + R_{x^i}R_x - R_{x^{i+1}} = 0$$

By induction, $R_{x^i} \in G_x$ (i.e. if $R_{x^i} \in G_x$ then $R_{x^{i+1}} \in G_x$)

G_x = the algebra generated by R_x and R_{x^2}

it is commutative! since $R_x R_{x^2} = R_{x^2} R_x$

Hence $R_{x^i} R_{x^j} = R_{x^j} R_{x^i}$ $\quad (x(x^2y) = x^2(xy))$

$\forall\, i,j \geq 1$

Now prove $x^{i+j} = x^i x^j$ by <u>induction on i</u>

assume $x^{i+j} = x^i x^j$ for <u>some</u> i and <u>all</u> j

so $x^i x^{j+1} = x^{i+j+1}$ $\forall\, j$ (replaced j by $j+1$)

Then $x^{i+1}x^j = (x x^i)x^j = R_{x^j}R_{x^i}x = R_{x^i}R_{x^j}x = x^i(x^j x)$

$= x^i x^{j+1} = x^{i+j+1}$

so assuming $x^{i+j} = x^i x^j$ for some i and all j

we find $x^{i+1+j} = x^{(i+1)}x^j$ $\forall\, j$. Thus by induction

we have $x^i x^j = x^{i+j}$ $\forall\, i, j$ $\blacksquare$

By exercise 2a (HW 2)

this implies $x^i(x^j x^k) = (x^i x^j)x^k$ $\forall\, i,j,k \geq 1$