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2.3 The free Leibniz algebra $\overline{T}(V)$ defined by a R -module V (see [47] for details)

2.4 The Hochschild Complex

A is an associative algebra over R

$b: A \otimes A \otimes A \rightarrow A \otimes A$ is the map

$$b(x \otimes y \otimes z) = xy \otimes z - x \otimes yz + zx \otimes y$$

$\text{Im } b =$ the span of $\{b(x \otimes y \otimes z) : x, y, z \in A\} \subset A \otimes A$

On the vector space (or R -module) $A \otimes A / \text{Im } b$

define $[a \otimes b, c \otimes d] = (ab - ba) \otimes (cd - dc)$.

this really means $a \otimes b + \text{Im } b$, the coset etc.

It satisfies the Leibniz identity (L)

So $A \otimes A / \text{Im } b$ is a Leibniz algebra

Remark If we make A into a Lie algebra

via $[a, b] = ab - ba$, then the map

$$b: A \otimes A / \text{Im } b \rightarrow A \quad b(x \otimes y) = xy - yx$$

Not the same b as above

is a Leibniz algebra homomorphism.

2.5 Low dimensions

Suppose a Lie algebra \mathfrak{g} (over \mathbb{K}) has dimension 1

Then $[x, x] = \alpha x$ for some $\alpha \in \mathbb{K}$.

(L) $\Rightarrow \alpha^2 = 0$ $[x, \alpha x] - \alpha[x, x] + \alpha[x, x] = 0$
 $\alpha[x, x] = 0 \quad \alpha^2 x = 0$

So if \mathbb{K} has no divisors of zero \mathfrak{g} is an abelian Lie algebra of dimension 1.

Suppose \mathfrak{g} has dimension 2, with generators (basis) x, y and suppose \mathbb{K} is a field. There are three types of Lie algebras.

• If $\mathfrak{g}^{\text{ann}} = 0$ (recall from 2.0 that $\mathfrak{g}^{\text{ann}}$ is the kernel

of $\mathfrak{g} \rightarrow \mathfrak{g}_{\text{Lie}} = \mathfrak{g}/\mathfrak{I}$ i.e. $\mathfrak{g}^{\text{ann}} = \mathfrak{I} =$ the 2-sided ideal generated by $\{[x, x] : x \in \mathfrak{g}\}$.)

In this case \mathfrak{g} is a Lie algebra

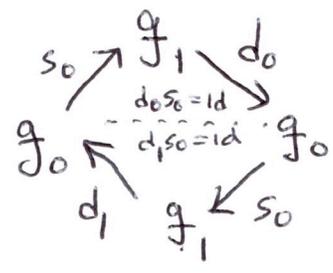
• If $\dim \mathfrak{g}^{\text{ann}} = 1$ there are 2 cases

1° \mathfrak{g} is isomorphic to the algebra defined by $[x, x] = [x, y] = [y, x] = 0$ and $[y, y] = x$
($\mathfrak{g}^{\text{ann}}$ is a trivial module over $\mathfrak{g}_{\text{Lie}}$)

2° \mathfrak{g} is isomorphic to the algebra defined by $[x, x] = [y, x] = 0$ and $[x, y] = x, [y, y] = x$
($\mathfrak{g}^{\text{ann}}$ is a non-trivial module over $\mathfrak{g}_{\text{Lie}}$)

2.6 Partial Lie algebras

Consider two Lie algebras \mathfrak{g}_1 and \mathfrak{g}_0 and homomorphisms $d_0: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$, $d_1: \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$, $s_0: \mathfrak{g}_0 \rightarrow \mathfrak{g}_1$ satisfying $d_0 s_0 = \text{id} = d_1 s_0$



Let $\mathfrak{g} = \ker d_1 \subset \mathfrak{g}_1$

$$[x, y]'_{\mathfrak{g}} = [x, s_0 d_0(y)]_{\mathfrak{g}_1}$$

Then $(\mathfrak{g}, [', \cdot]')$ is a Leibniz algebra

Remark This construction appears naturally in reference [B-C] and is a special case of Example 2.1

2.7 An example from reference [K2] *Hamiltonian Mechanics*

2.8 An example from reference [B] *Differential Forms*

3. Derivations and biderivations

3.1 If \mathfrak{g} is a Leibniz algebra, a derivation $d: \mathfrak{g} \rightarrow \mathfrak{g}$ is a \mathbb{k} -linear map satisfying $d([x, y]) = [dx, y] + [x, dy]$

An anti-derivation satisfies (by definition!) $D: \mathfrak{g} \rightarrow \mathfrak{g}$

$$D([x, y]) = [Dx, y] - [Dy, x] \quad (\text{these are the same for Lie algebras})$$

A biderivation is a pair (d, D) of a derivation d and an anti-derivation D satisfying $[x, dy] = [x, Dy]$.

3.2 Inner derivations, anti-derivations, and biderivations.

inner derivation : $\text{ad}(x)(y) = -[y, x]$

inner anti-derivation : $\text{Ad}(x)(y) = [x, y]$

inner biderivation : $(\text{ad}(x), \text{Ad}(x))$

(denoted $\text{Bider}(\mathfrak{g})$)

3.3 The set of biderivations of a Leibniz algebra \mathfrak{g} can be equipped with a "bracket"

$$[(d, D), (d', D')] := (dd' - d'd, Dd' - d'D)$$

FACTS : • The right side is a biderivation of \mathfrak{g} .

• $\text{Bider}(\mathfrak{g})$ is a Leibniz algebra

• $\mathfrak{g} \rightarrow \text{Bider}(\mathfrak{g}), x \rightarrow (\text{ad}x, \text{Ad}(x))$ is a morphism of Leibniz algebras. □