BOUNDEDNESS OF COMPLETELY ADDITIVE MEASURES WITH APPLICATION TO 2-LOCAL TRIPLE DERIVATIONS

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ABSTRACT. We prove a Jordan version of Dorofeev's boundedness theorem for completely additive measures and use it to show that every (not necessarily linear nor continuous) 2-local triple derivation on a continuous JBW*-triple is a triple derivation.

1. INTRODUCTION AND BACKGROUND

Let $\mathcal{P}(M)$ denote the lattice of projections in a von Neumann algebra M. Let X be a Banach space. A mapping $\mu : \mathcal{P}(M) \to X$ is said to be *finitely additive* when

$$(1.1) \qquad \mu\left(\sum_{i=1}^{n} p_i\right) = \sum_{i=1}^{n} \mu(p_i),$$

for every family $p_1, \ldots, p_n$ of mutually orthogonal projections in M. A mapping $\mu : \mathcal{P}(M) \to X$ is said to be *bounded* when the set

$$\{\|\mu(p)\| : p \in \mathcal{P}(M)\}$$

is bounded.

The celebrated Bunce-Wright-Mackey-Gleason theorem ([10], [11]) states that if M has no summand of type I_2, then every bounded finitely additive mapping $\mu : \mathcal{P}(M) \to X$ extends to a bounded linear operator from M to X.

Answering a question posed by George Mackey, Gleason's original theorem [21] characterizes quantum mechanical states on a separable Hilbert space in terms of density operators, and thus plays an important role in the foundations of quantum mechanics. The interdisciplinary nature of the Bunce-Wright-Mackey-Gleason theorem makes this result very useful in a wide range of topics. Applications can be found in quantum physics and quantum information (cf. [18], [43], [37], [36], [22, Chapter 7], and [16], among many others), and in functional analysis with studies on vector-valued measures on von Neumann

algebras and 2-local maps on von Neumann algebras, JBW*-algebras and JBW*-triples (see [20], [3], [4], [13] [14] and [32]).

According to the terminology employed in [39] and [17], a completely additive mapping $\mu : \mathcal{P}(M) \to \mathbb{C}$—that is, (1.1) holds with $X = \mathbb{C}$ for an arbitrary set of mutually orthogonal projections, is called a *charge*. The Dorofeev–Sherstnev theorem ([39, Theorem 29.5] or [17, Theorem 2]) states that any charge on a von Neumann algebra with no summands of type I_n is bounded.

The Dorofeev-Shertsnev theorem was used in [32] in order to apply the Bunce-Wright-Mackey-Gleason theorem to prove the main result of that paper, namely, that a 2-local triple derivation on a von Neumann algebra is a triple derivation (see the next subsection). In section 3 of this paper, we shall establish the first main result of this paper, namely, a Jordan version of Dorofeev's boundedness theorem (Theorem 3.1). This will be used in section 4 to show that 2-local triple derivations on certain continuous JW*-algebras are triple derivations (Theorem 4.6). Combined with the main result of section 2 (Theorem 2.4), this will prove the second main result of this paper, namely, that every 2-local triple derivation on an arbitrary continuous JBW*-triple is a triple derivation (Theorem 4.7).

Having described the purpose and contents of this paper, we shall now present more background and some preliminary material.

We shall use the term *measure* to denote a complex valued finitely additive function μ on the projections of a von Neumann algebra or a JBW*-algebra. If μ is positive (resp. real) valued, we call it a positive (resp. signed) measure. If countable additivity or complete additivity is assumed, it will be explicitly stated.

Let us recall that a *derivation* is a linear map D from an algebra A to a two sided A-module M over the algebra satisfying the Leibniz identity: $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in A$.

Local derivations were introduced simultaneously in 1990 by Kadison [28] and by Larson-Sourour [33]. A *local derivation* from an algebra into a module is a linear mapping whose value at each point in the algebra coincides with the value of some derivation at that point. Kadison proved that every continuous local derivation of a von Neumann algebra into a dual Banach module is in fact a derivation. Johnson [27] extended Kadison's result to C*-algebras, and moreover showed that the continuity assumption was not necessary. Larson and Sourour showed that a local derivation on the algebra of all bounded linear operators on a Banach space is a derivation.

Let us also recall that a *triple derivation* is a linear map D from a triple system E to an E-module N over the triple system satisfying the triple Leibniz identity: $D(\{abc\}) = \{D(a)bc\} + \{aD(b)c\} + \{abD(c)\}$ for all $a, b, c \in E$, where $\{abc\}$ denotes the triple product. (Jordan triple systems are defined later in this section.)

Local triple derivations were introduced in 2013 by Michael Mackey [34]. A *local triple derivation* on a triple system is a linear mapping whose value at each point in the triple system coincides with the value of some triple derivation at that point. Mackey showed that a continuous local triple derivation on a JBW*-triple (to itself) is a triple derivation, an

exact analog of Kadison's result mentioned above. This result was extended to JB*-triples in 2014 by Burgos, Fernandez-Polo, and Peralta [12], who also showed that the continuity assumption was redundant, an exact analog of Johnson's result also mentioned above.

Since 1997 there has been much interest in the notion of 2-local derivation and more recently, in the notion of 2-local triple derivation. The application of the main theorem of this paper concerns 2-local triple derivations. A *2-local derivation* (respectively, *2-local triple derivation*) from an algebra (respectively, triple system) into itself is a mapping (not necessarily linear) whose values at each pair of points in the algebra (respectively, triple system) coincides with the values of some derivation (respectively, triple derivation) at those two points. 2-local derivations were introduced in 1997 by Semrl [38] and 2-local triple derivations were introduced in 2014 by Kudaybergenov, Oikhberg, Peralta, and Russo [32] although the concept was mentioned by Michael Mackey in a lecture in 2012 at a conference in Hong Kong celebrating the 65th birthday of Cho-Ho Chu. It is now known that, for von Neumann algebras, a 2-local derivation is in fact a derivation (Ayupov-Kudaybergenov [3]) and, as noted above, a 2-local triple derivation is a triple derivation (Kudaybergenov-Oikhberg-Peralta-Russo [32]).

For an elaboration of the above summary, see the forthcoming survey of Ayupov, Kudaybergenov, and Peralta, [4]. Local and 2-local derivations have also been considered on algebras of measurable operators associated with von Neumann algebras. For more details on this, see the forthcoming survey of Ayupov and Kudaybergenov [2].

A complex *Jordan triple* is a complex vector space E equipped with a non-trivial triple product

$$E \times E \times E \to E$$

$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one satisfying the so-called *"Jordan Identity"*:

$$L(a, b)L(x, y) - L(x, y)L(a, b) = L(L(a, b)x, y) - L(x, L(b, a)y),$$

for all a, b, x, y in E, where $L(x, y)z := \{x, y, z\}$.

A subspace F of a Jordan triple E is said to be a *subtriple* if $\{F, F, F\} \subseteq F$ and an *ideal* if $\{E, E, J\} + \{E, J, E\} \subseteq J$.

A *(complex) JB*-triple* is a complex Jordan Banach triple E satisfying the following axioms:

- For each a in E the map $L(a, a)$ is an hermitian operator on E with non negative spectrum;
- $\|\{a, a, a\}\| = \|a\|^3$ for all a in A.

A JB*-algebra is a complex Jordan Banach algebra $(A, \circ)$ equipped with an algebra involution * satisfying $\| \{a, a^*, a\} \| = \|a\|^3$, $a \in A$. (Recall that $\{a, a^*, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$.) JB-algebras are precisely the self adjoint parts of JB*-algebras, and a JBW-algebra is a JB-algebra which is a dual space.

Every C*-algebra (resp., every JB*-algebra) is a JB*-triple with respect to the product $\{a, b, c\} = \frac{1}{2} \left(ab^*c + cb^*a\right)$ (resp., $\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

For the theory of C*-algebras and von Neumann algebras, we shall refer to the monographs [29] and [40]. For the theory of JB*-algebras and JBW*-algebras we refer to [23] and [41]. For basic facts about abstract Jordan triple systems, consult [15, section 1.2]. However, the Jordan triple systems we consider in this paper are concrete, so statements about them can usually be verified directly. For example, a tripotent (defined in the next section) is nothing but a partial isometry.

A complex *JBW*-triple* is a complex JB*-triple which is also a dual Banach space. The structure of JBW*-triples is fairly well understood. Every JBW*-triple is a direct sum of a JBW*-triple of type I and a continuous JBW*-triple (defined below). JBW*-triples of type I have been classified in [25] and continuous JBW*-triples have been classified in [26]. Since it is the continuous JBW*-triples that concern us here, we shall not define type I, but we shall state their classification theorem from [25]: A JBW*-triple of type I is an ℓ^∞-direct sum of JBW*-triples of the form $A \otimes C$, where A is a commutative von Neumann algebra and C is a Cartan factor (for Cartan factors, see [15, Theorem 2.5.9 and page 168]).

A JBW^*-triple $\mathcal{A}$ is said to be *continuous* if it has no type I direct summand. In this case it is known that, up to isometry, $\mathcal{A}$ is a JW^*-triple, that is, a subspace of the bounded operators on a Hilbert space which is closed under the triple product $xy^*z + zy^*x$ and closed in the weak operator topology. More importantly, it has a unique decomposition into weak*-closed ideals, $\mathcal{A} = H(W, \alpha) \oplus pV$, where W and V are continuous von Neumann algebras, p is a projection in V, α is an involution on W commuting with *, that is, a *-antiautomorphism of W order 2, which we shall call henceforth a $\mathbb{C}$-linear *-involution, and $H(W, \alpha) = \{x \in W : \alpha(x) = x\}$ (see [26, (1.20)]). Notice that the triple product in pV is given by $(xy^*z + zy^*x)/2$ and that $H(W, \alpha)$ is a JBW*-algebra with the Jordan product $x \circ y = (xy + yx)/2$.

We shall show in section 4 that for continuous JBW*-triples, every 2-local triple derivation is a derivation. (We are leaving the study of 2-local triple derivations on the JBW*-triples of type I as one of the problems at the end of this paper—see Problem 4.9(a).)

2. 2-LOCAL TRIPLE DERIVATIONS ON RIGHT IDEALS OF VON NEUMANN ALGEBRAS

Recall that a (not necessarily linear) mapping Δ on a Jordan triple E is said to be a 2-local triple derivation if, given two points $x, y \in E$, there is a triple derivation $D_{x,y}$ on E such that $\Delta(x) = D_{x,y}(x)$ and $\Delta(y) = D_{x,y}(y)$. Every 2-local triple derivation $\Delta : E \to E$ is homogeneous. Indeed, for each $a \in E$, $t \in \mathbb{C}$ consider a triple derivation $D_{a,ta}$. Then $\Delta(ta) = D_{a,ta}(ta) = tD_{a,ta}(a) = t\Delta(a)$.

An element e in a Jordan triple E is called a *tripotent* if $\{e, e, e\} = e$. Each tripotent e in E induces a decomposition of E (called *Peirce decomposition*) in the form:

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e),$$

where $E_k(e) = \{x \in E : L(e, e)x = \frac{k}{2}x\}$ for $k = 0, 1, 2$ (compare [15, page 32]).

Lemma 2.1. *Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a 2-local triple derivation on a JB^*-triple. Suppose v is a tripotent in $\mathcal{A}$ such that $\Delta(v) = 0$. Then $\Delta(\mathcal{A}_k(v)) \subseteq \mathcal{A}_k(v)$, for every $k = 0, 1, 2$.*

Proof. Let $x \in \mathcal{A}_k(v)$ with $k = 0, 1, 2$, that is, $\{v, v, x\} = \frac{k}{2}x$. Since

$$\{v, v, \Delta(x)\} = \{v, v, D_{v,x}(x)\} = D_{v,x}(\{v, v, x\}) - \{D_{v,x}(v), v, x\} - \{v, D_{v,x}(v), x\} =$$
$$= D_{v,x}\left(\frac{k}{2}x\right) - \{\Delta(v), v, x\} - \{v, \Delta(v), x\} = \frac{k}{2}D_{v,x}(x) = \frac{k}{2}\Delta(x).$$

The proof is complete. $\qquad\square$

We recall the following result (see [32, Theorem 2.14]).

Theorem 2.2. *[32] Let M be an arbitrary von Neumann algebra and let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then T is a triple derivation.*

Throughout this section $\mathcal{A}$ will denote the JBW^*-triple pM where M is a von Neumann algebra and p is a projection in M. The following is the main result of this section. The proof will be carried out in the next subsections.

Theorem 2.3. *Let M be a von Neumann algebra and let p be a projection in M. Then any 2-local triple derivation Δ on the JBW^*-triple $\mathcal{A} = pM$ is a triple derivation.*

Let a and b be skew-hermitian elements in pMp and M, respectively. Let L_a and R_b be the left and right multiplication operators, i.e.

$$(2.1) \qquad\qquad L_a(x) = ax, \ x \in \mathcal{A}$$

and

$$(2.2) \qquad\qquad R_b(x) = xb, \ x \in \mathcal{A}.$$

It is clear that L_a and R_b both are triple derivations on M, and in particular on $\mathcal{A}$.

Let u be a tripotent in the JBW^*-triple $\mathcal{A} = pM$, and let $(\mathcal{A}_2(u), \cdot_u, {}^{*_u})$ denote the von Neumann algebra whose underlying Banach space is the Pierce-2-space $\mathcal{A}_2(u) = uu^*Mu^*u$, and whose product and involution are given by $x \cdot_u y = xu^*y$ and $x^{*_u} = ux^*u$, respectively.

Let $\{.,.,.\}_1$ denote the triple product associated to $\mathcal{A}_2(u)$, i.e. $\{x, y, z\}_1 = \frac{1}{2}(x \cdot_u y^{*_u} \cdot_u z + z \cdot_u y^{*_u} \cdot_u x)$. By direct calculation, $\{x, y, z\}_1 = \{x, y, z\}$. This also follows since the identity map is a linear isometry, and therefore an isomorphism ([31, Proposition (5.5)]). Therefore a linear mapping $D : \mathcal{A}_2(u) \to \mathcal{A}_2(u)$ is a triple derivation (resp. 2-local triple derivation) for the product $\{.,.,.\}$ if and only if it is a triple derivation (resp. 2-local triple derivation) for the product $\{.,.,.\}_1$.

2.1. **Properly infinite case.** In this subsection we will consider 2-local triple derivations on JBW^*-triples of the form $\mathcal{A} = pM$, where p is a properly infinite projection in a von Neumann algebra M.

Let q be a projection in M and let D be a triple derivation on $\mathcal{A} = pM$. It is easily seen that an operator $D_{(q)}$ on the JBW^*-subtriple pMq defined by

$$D_{(q)}(x) = D(x)q, \ x \in pMq$$

is a triple derivation on pMq. Thus, if Δ is a 2-local triple derivation on $\mathcal{A} = pM$, then the operator $\Delta_{(q)}$ on the JBW^*-subtriple pMq defined by

(2.3) $$\Delta_{(q)}(x) = \Delta(x)q, \ x \in pMq$$

is a 2-local triple derivation on pMq.

The following is the main result of this subsection.

Theorem 2.4. *Let M be a von Neumann algebra and let p be a properly infinite projection in M. Then any 2-local triple derivation Δ on $\mathcal{A} = pM$ is a triple derivation.*

Proof. Since p is properly infinite, by using the halving Lemma five times (see for example [29, Lemma 6.3.3]) we can find mutually orthogonal projections $e_1, \ldots, e_6$ in M such that $p \sim e_1 \sim \ldots \sim e_6$ and $p = e_1 + \cdots + e_6$.

Denote by $r(x)$ and $l(x)$ the right and left supports in M of the element x from M, respectively. Since $r(x) \sim l(x)$ (see [40, Proposition V.1.5]) and $l(x) \leq p$, it follows that $r(x) \preceq p$ for all $x \in \mathcal{A}$.

Let $x, y \in \mathcal{A}$. Denote by $q_1, \ldots, q_6$ the right supports of elements x, y, $x + y$, $\Delta(x)$, $\Delta(y)$ and $\Delta(x + y)$, respectively. Then $q_i \preceq p$ for all $i \in \{1, \ldots, 6\}$. Since $p \sim e_i$ for all i, it follows that $q_i \preceq e_i$ for all $i \in \{1, \ldots, 6\}$. Therefore $\bigvee_{i=1}^{6} q_i \preceq e_1 + \ldots + e_6 = p$ (see [30, Exercise 6.9.3]).

Let us show the existence of a projection $q \in M$ such that $\bigvee_{i=1}^{6} q_i \leq q \sim p = e_1 + \cdots + e_6$. Since p is properly infinite by [30, Exercise 6.9.4] it follows that

$$\left(\bigvee_{i=1}^{6} q_i \right) \vee p \sim p.$$

Then it suffices to take $q = \left(\bigvee_{i=1}^{6} q_i \right) \vee p$.

Since $p \sim q$ there exists a partially isometry $u \in M$ such that $uu^* = p$, $u^*u = q$. As was mentioned before this subsection, $pMq = uu^*Mu^*u$ is a von Neumann algebra with respect to product and involution given by $x \cdot_u y = xu^*y$ and $x^{*_u} = ux^*u$, respectively.

Let $\Delta_{(q)}$ be the 2-local triple derivation on pMq defined by (2.3). Then by Theorem 2.2, $\Delta_{(q)}$ is a triple derivation. By the construction of q it follows that $x, y, x+y, \Delta(x), \Delta(y), \Delta(x+y)$ all belong to pMq. Therefore

$$\begin{aligned}
\Delta(x + y) &= \Delta(x + y)q = \Delta_{(q)}(x + y) = \Delta_{(q)}(x) + \Delta_{(q)}(y) = \\
&= \Delta(x)q + \Delta(y)q = \Delta(x) + \Delta(y).
\end{aligned}$$

Thus Δ is additive and hence linear. Since every (linear) local triple derivation on a JB*-triple is automatically continuous and hence a triple derivation (see [12, Theorem 2.8]), the proof is complete. $\qquad\square$

2.2. **Finite case.** In this subsection we will consider 2-local triple derivations on JBW^*-triples of the form $\mathcal{A} = pM$, where p is a finite projection in a von Neumann algebra M.

Let D be a triple derivation on $\mathcal{A}$. Set, for a tripotent $u \in \mathcal{A}$,

(2.4) $$D^{(u)}(x) = \{u, \{u, D(x), u\}, u\} = uu^*D(x)u^*u, \ x \in \mathcal{A}_2(u).$$

It is easily seen that $D^{(u)}$ is a triple derivation on $\mathcal{A}_2(u)$.

Let Δ be a 2-local triple derivation on $\mathcal{A}$ and let u be a tripotent in $\mathcal{A}$. Then

(2.5) $$uu^*\Delta(u)u^*u = -u\Delta(u)^*u.$$

Indeed, take a triple derivation D on $\mathcal{A}$ with $\Delta(u) = D(u)$. From the equality $\{u, u, u\} = u$, we have that

(2.6) $$uu^*D(u)u^*u = -uD(u)^*u,$$

which implies (2.5).

Lemma 2.5. *Let Δ be a 2-local derivation on $\mathcal{A}$. There exist skew-hermitian elements a_1 in pMp and b_1 in M such that*
$$\Delta(p) = L_{a_1}(p) + R_{b_1}(p).$$

Proof. Set
$$a_1 = \Delta(p)p \ \text{and} \ b_1 = \Delta(p)p^\perp - p^\perp\Delta(p)^*,$$
where $p^\perp = \mathbf{1} - p$. From (2.5) it follows that a_1 is skew-hermitian. It is clear that b_1 is also skew-hermitian. We have
$$L_{a_1}(p) + R_{b_1}(p) = a_1 p + p b_1 = \Delta(p)p + p\Delta(p)p^\perp = \Delta(p)p + \Delta(p)p^\perp = \Delta(p).$$
$\qquad\square$

Lemma 2.6. *Let Δ be a 2-local derivation on $\mathcal{A}$. Suppose that $\Delta(p) = 0$. Then there exists a skew-hermitian element a_2 in pMp such that $\Delta(x) = L_{a_2}(x) - R_{a_2}(x)$ for all $x \in \mathcal{A}_2(p) = pMp$.*

Proof. Since $\Delta(p) = 0$, Lemma 2.1 implies that Δ maps $\mathcal{A}_2(p) = pMp$ into itself.

Let $x, y \in \mathcal{A}_2(p)$. Take a triple derivation $D_{x,y}$ on $\mathcal{A}$ such that
$$\Delta(x) = D_{x,y}(x), \ \Delta(y) = D_{x,y}(y).$$
Let $D_{x,y}^{(p)}$ be the triple derivation defined by (2.4). Then
$$\Delta(x) = D_{x,y}^{(p)}(x), \ \Delta(y) = D_{x,y}^{(p)}(y).$$

This means that the restriction $\Delta|_{\mathcal{A}_2(p)}$ is a 2-local triple derivation on the von Neumann algebra $\mathcal{A}_2(p)$. By Theorem 2.2, $\Delta|_{\mathcal{A}_2(p)}$, is a triple derivation. Since $\Delta(p) = 0$, there exists a skew-hermitian element a_2 in pMp such that $\Delta(x) = a_2 x - x a_2$ for all $x \in \mathcal{A}_2(p) = pMp$ (see [32, beginning of section 2]). $\qquad\square$

Let D be an arbitrary triple derivation (or a 2-local triple derivation) on $\mathcal{A}$. Then D can be decomposed in the form

$$(2.7) \qquad\qquad D = D_1 + D_2,$$

where $D_1 = L_a + R_b$, with a, b skew-hermitian and $D_2|_{\mathcal{A}_2(p)} \equiv 0$.

Indeed, by Lemma 2.5, there exist skew-hermitian elements $a_1 \in pMp$ and $b_1 \in M$ such that $(D - L_{a_1} - R_{b_1})(p) = 0$. By Lemma 2.6, there is a skew-hermitian element $a_2 \in pMp$ such that $(D - L_{a_1} - R_{b_1})(x) = L_{a_2}x - R_{a_2}x$ for all $x \in pMp$. Now it suffices to set

$$D_1 = L_{a_1+a_2} + R_{b_1-a_2} \text{ and } D_2 = D - D_1.$$

Lemma 2.7. *Let D be a triple derivation on $\mathcal{A}$ such that $D|_{\mathcal{A}_2(p)} \equiv 0$. Then*

$$(2.8) \qquad\qquad D(x)y^* + xD(y)^* = 0$$

for all $x, y \in \mathcal{A}$.

Proof. Let us first consider a case $x, y \in \mathcal{A}_1(p) = pM(\mathbf{1} - p)$.

Since $D|_{\mathcal{A}_2(p)} \equiv 0$, it follows from Lemma 2.1 that D maps $\mathcal{A}$ into $\mathcal{A}_1(p)$. Taking into account these properties we have

$$
\begin{aligned}
xD(y)^* &= xD(y)^*p + pD(y)^*x = 2\{x, D(y), p\} = \\
&= 2D\left(\{x, y, p\}\right) - 2\{D(x), y, p\} - 2\{x, y, D(p)\} = \\
&= D(xy^*p + py^*x) - D(x)y^*p - py^*D(x) = \\
&= D(xy^*) - D(x)y^* = -D(x)y^*,
\end{aligned}
$$

i.e. $D(x)y^* + xD(y)^* = 0$ for $x, y \in \mathcal{A}_1(p)$.

Let now $x, y \in \mathcal{A}$ be arbitrary and let $x = x_2 + x_1, y = y_2 + y_1 \in \mathcal{A} = \mathcal{A}_2(p) + \mathcal{A}_1(p)$. We have

$$
\begin{aligned}
D(x)y^* + xD(y)^* &= D(x_2 + x_1)(y_2 + y_1)^* + (x_2 + x_1)D(y_2 + y_1)^* = \\
&= D(x_1)y_2^* + x_1D(y_1)^* + D(x_1)y_1^* + x_2D(y_1)^* \\
&= D(x_1)y_2^* + x_2D(y_1)^* = 0,
\end{aligned}
$$

because $D(x_1)y_2^* = (D(x_1)(\mathbf{1} - p))(y_2p)^* = 0$ and $x_2D(y_1)^* = x_2p(D(y_1)(\mathbf{1} - p))^* = 0$. The proof is complete. □

Since pMp is finite, there exists a faithful center-valued trace τ on pMp, that is, a linear map from pMp into the center, $Z(pMp)$, of pMp such that

(i) $\tau(xy) = \tau(yx)$ for all $x, y \in pMp$;
(ii) $\tau(z) = z$ for all $z \in Z(pMp)$;
(iii) $\tau(xx^*) = 0$ implies $x = 0$.

Define a $Z(pMp)$-valued sesquilinear form on $\mathcal{A}$ by

$$\langle x, y \rangle = \tau(xy^*), \ x, y \in \mathcal{A}.$$

Since τ is faithful it follows that the form $\langle \cdot, \cdot \rangle$ is non-degenerate, i.e. $\langle x, y \rangle = 0$ for all $y \in \mathcal{A}$ implies that $x = 0$.

Lemma 2.8. *Let D be an arbitrary triple derivation on $\mathcal{A}$. Then*

$$\langle D(x), y \rangle = -\langle x, D(y) \rangle$$

for all $x, y \in \mathcal{A}$.

Proof. Let $D = D_1 + D_2$ be a decomposition of D in the form (2.7). For $x, y \in \mathcal{A}$, we have $a^* = -a \in pMp$ and $b^* = -b \in M$ such that

$$
\begin{aligned}
D_1(x)y^* + xD_1(y)^* &= (ax + xb)y^* + x(ay + yb)^* = \\
&= axy^* + xby^* + xy^*a^* + xb^*y^* = \\
&= axy^* + xby^* - xy^*a - xby^* = \\
&= axy^* - xy^*a,
\end{aligned}
$$

i.e.

$$D_1(x)y^* + xD_1(y)^* = axy^* - xy^*a.$$

Since a center-valued trace annihilates commutators we have that

$$\tau\left(D_1(x)y^* + xD_1(y)^*\right) = 0.$$

Thus

$$\langle D_1(x), y \rangle = -\langle x, D_1(y) \rangle.$$

On the other hand, by Lemma 2.7 it follows that

$$\langle D_2(x), y \rangle = -\langle x, D_2(y) \rangle.$$

The proof is complete. $\qquad\square$

The following is the main result of this subsection.

Theorem 2.9. *Let M be a von Neumann algebra and let p be a finite projection in M. Then any 2-local triple derivation Δ on $\mathcal{A} = pM$ is a triple derivation.*

Proof. Let us first show that

$$\langle \Delta(x), y \rangle = -\langle x, \Delta(y) \rangle$$

for all $x, y \in \mathcal{A}$.

Take a triple derivation D on $\mathcal{A}$ such that

$$\Delta(x) = D(x) \text{ and } \Delta(y) = D(y).$$

By Lemma 2.8, we have

$$\langle \Delta(x), y \rangle = \langle D(x), y \rangle = -\langle x, D(y) \rangle = -\langle x, \Delta(y) \rangle.$$

Let now x, y, z be arbitrary elements in $\mathcal{A}$. Then

$$
\begin{aligned}
\langle \Delta(x + y), z \rangle &= -\langle x + y, \Delta(z) \rangle = -\langle x, \Delta(z) \rangle - \langle y, \Delta(z) \rangle = \\
&= \langle \Delta(x), z \rangle + \langle \Delta(y), z \rangle = \langle \Delta(x) + \Delta(y), z \rangle,
\end{aligned}
$$

i.e.

$$\langle \Delta(x + y) - \Delta(x) - \Delta(y), z \rangle = 0.$$

Since z is an arbitrary and the sesquilinear form is non-degenerate it follows that $\Delta(x+y) = \Delta(x) + \Delta(y)$, so Δ is additive, hence linear, hence a triple derivation by [12, Theorem 2.8] (compare the proof of Theorem 2.4). $\qquad\square$

2.3. **General case.** We need the following two Lemmata.

Lemma 2.10. *Let D be a triple derivation on pM. Then D is $\mathcal{P}(Z(M))$-homogeneous, i.e.*

$$D(cx) = cD(x)$$

for any central projection $c \in \mathcal{P}(Z(M))$ and $x \in pM$.

Proof. Let $c \in \mathcal{P}(Z(M))$. Take $x, y, z \in pM$. We have

$$
\begin{aligned}
c\{x, D(cy), z\} &= cD\left(\{x, cy, z\}\right) - c\{D(x), cy, z\} - c\{x, cy, D(z)\} = \\
&= cD\left(\{x, cy, z\}\right) - c\{cD(x), y, z\} - c\{x, y, D(z)\}
\end{aligned}
$$

and

$$
\begin{aligned}
c\{x, D(cy), z\} &= c\{cx, D(cy), z\} = \\
&= cD\left(\{cx, cy, z\}\right) - c\{D(cx), cy, z\} - c\{cx, cy, D(z)\} = \\
&= cD\left(\{x, cy, z\}\right) - c\{D(cx), y, z\} - c\{x, y, D(z)\}.
\end{aligned}
$$

Thus $c\{cD(x), y, z\} = c\{D(cx), y, z\}$. Since c is a central projection we obtain that

$$\{cD(x), y, z\} = \{cD(cx), y, z\}.$$

Since y, z are arbitrary, it follows that

(2.9) $$cD(x) = cD(cx).$$

Thus

$$cD((\mathbf{1} - c)x) = 0.$$

Replacing c by $\mathbf{1} - c$ in the last equality we obtain that

(2.10) $$(\mathbf{1} - c)D(cx) = 0.$$

Thus

$$D(cx) = (c + (\mathbf{1} - c))D(cx) = cD(cx) + (\mathbf{1} - c)D(cx) \overset{(2.10)}{=} cD(cx) \overset{(2.9)}{=} cD(x).$$

The proof is complete. $\qquad\square$

Lemma 2.11. *Let Δ be a 2-local triple derivation on pM. Then Δ is $\mathcal{P}(Z(M))$-homogeneous, i.e.*

$$\Delta(cx) = c\Delta(x)$$

for any central projection $c \in \mathcal{P}(Z(M))$ and $x \in pM$.

Proof. Let $c \in \mathcal{P}(Z(M))$ and $x \in pM$. Let $D_{cx,x} : pM \to pM$ be a triple derivation satisfying $\Delta(cx) = D_{cx,x}(cx)$ and $\Delta(x) = D_{cx,x}(x)$. By Lemma 2.10, we have

$$\Delta(cx) = D_{cx,x}(cx) = cD_{cx,x}(x) = c\Delta(x).$$

$\qquad\square$

Now we are in position to prove Theorem 2.3.

Proof of Theorem 2.3. Let M be a von Neumann algebra, p be a projection in M and Δ be a 2-local triple derivation on the JBW^*-triple $\mathcal{A} = pM$.

Take mutually orthogonal central projections z_1 and z_2 in M with $z_1 + z_2 = \mathbf{1}$ such that $z_1 p$ is finite and $z_2 p$ is properly infinite. Lemma 2.11 implies that Δ maps each $z_i \mathcal{A}$ into itself and hence induces a 2-local triple derivation $\Delta_i = \Delta|_{z_i \mathcal{A}}$ on $z_i \mathcal{A} = z_i pM$ for $i = 1, 2$. Theorems 2.4, 2.9 imply that both Δ_1 and Δ_2 are triple derivations. Since

$$\Delta(x) = z_1 \Delta(x) + z_2 \Delta(x) = \Delta_1(z_1 x) + \Delta_2(z_2 x)$$

for all $x \in \mathcal{A}$, it follows that Δ is also a triple derivation. The proof is complete. $\qquad\square$

A Cartan factor of type 1 is the JBW*-triple $B(H, K)$ of all bounded operators from a Hilbert space H to a Hilbert space K. We thus have:

Corollary 2.12. *Every 2-local triple derivation on a Cartan factor of type 1 is a triple derivation.*

3. BOUNDEDNESS OF COMPLETELY ADDITIVE MEASURES ON CONTINUOUS JW*-ALGEBRAS

In this section we shall establish one of the main results of this note, namely a Jordan version of Dorofeev's boundedness theorem (compare [39, Theorem 29.5] or [17, Theorem 1]). The latter states that any completely additive signed measure on the projections of a continuous von Neumann algebra is bounded.

Theorem 3.1 provides the key tool for the proof of Theorem 4.6, which together with Theorem 2.3 leads to the second main conclusion of this note in Theorem 4.7, namely, that a 2-local triple derivation on a continuous JBW*-triple is a triple derivation.

Assume that M is a continuous von Neumann algebra and $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution (i.e. a *-antiautomorphism of order 2). The subspace $H(M, \beta)$, of all β-fixed points in M, is not, in general, a subalgebra of M. However, $H(M, \beta)$ is a weak* closed Jordan *-subalgebra of M, whenever the latter is equipped with its natural Jordan product

$$x \circ y := \frac{1}{2}(xy + yx).$$

In particular, the self-adjoint part, $H(M, \beta)_{sa}$, of $H(M, \beta)$ is a JBW-subalgebra of M_{sa}.

Theorem 3.1. *Let M be a continuous von Neumann algebra and let $\beta : M \to M$ be a $\mathbb{C}$-linear *-involution. Let $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ be a completely additive (complex) measure. Then Δ is bounded.*

The authors do not know if Theorem 3.1 remains valid when $H(M, \beta)$ is replaced by an arbitrary JBW*-algebra containing no summands of type I_n. See Problem 4.8. However, Theorem 3.1 is sufficient for the purposes of this paper.

We shall show how the arguments in [17] can be adapted to prove the above result. For completeness reasons, we shall present here a draft of the original arguments employed in

the proof of [17, Theorem 1], making the adjustments, some of which are non-trivial, for the Jordan case. The proof of Theorem 3.1 will occupy us throughout this section.

The following Jordan version of the Bunce-Wright-Mackey-Gleason theorem is an instance of a theorem due to Matveĭchuk and has been borrowed from [35]

Theorem 3.2. [35, Theorem 1] *Let M be a continuous von Neumann algebra and let $\beta :$ $M \to M$ be a $\mathbb{C}$-linear *-involution. Let $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ be a bounded finitely additive measure. Then there exists a functional φ in $H(M, \beta)^*$ such that $\Delta(p) = \varphi(p)$, for every $p \in \mathcal{P}(H(M, \beta))$. Furthermore, when Δ is completely additive the functional φ can be assumed to be in $H(M, \beta)_*$.* □

Suppose that M acts on a complex Hilbert space H. Following [17], given two projections $p, q \in \mathcal{P}(M)$, the distance between p and q is defined by

$$d(p, q) = \inf\{\|\xi - \eta\| : \xi \in p(H), \eta \in q(H), \|\xi\| = \|\eta\| = 1\}.$$

Let us take $\xi \in p(H), \eta \in q(H)$ with $\|\xi\| = \|\eta\| = 1$. In this case

$$\|\xi - \eta\| \geq \|\xi - q(\xi)\| - \|q(\xi - \eta)\| \geq \|\xi - q(\xi)\| - \|\xi - \eta\|,$$

which gives $2\|\xi - \eta\| \geq \|\xi - q(\xi)\|$. Therefore

$$2\|\xi - \eta\| \geq \inf\{\|\zeta - q(\zeta)\| : \zeta \in p(H), \|\zeta\| = 1\},$$

and thus

(3.1) $$d(p, q) \geq \frac{1}{2} \inf\{\|\zeta - q(\zeta)\| : \zeta \in p(H), \|\zeta\| = 1\}.$$

Following standard notation, given two projections p, q in a von Neumann algebra M, the symbols $p \vee q$ and $p \wedge q$ will denote the supremum and the infimum of p and q in M, respectively. Let β be a $\mathbb{C}$-linear *-involution on M. It is clear that $\beta(\mathbf{1}) = \mathbf{1}$. Furthermore, $\beta(p \vee q) = \beta(p) \vee \beta(q)$ and $\beta(p \wedge q) = \beta(p) \wedge \beta(q)$. So, if $p, q \in H(M, \beta)$, then $p \vee q$ and $p \wedge q$ both belong to $H(M, \beta)$. Having these comments in mind, the arguments in the proof of [17, Lemma 2] can be slightly adapted to obtain:

Lemma 3.3. *Let M be a continuous von Neumann algebra and let $\beta : M \to M$ be a $\mathbb{C}$-linear *-involution. Let $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ be a completely additive measure. Suppose there exists a constant $C > 0$ and an increasing sequence (q_n) of projections in $H(M, \beta)$ such that $(q_n) \uparrow \mathbf{1}$ and*

$$\sup\{|\Delta(q)| : q \in \mathcal{P}(H(M, \beta)) : q \leq q_n\} \leq C,$$

for every natural n. Then Δ is bounded.

Proof. Let us observe that Δ being a completely additive measure implies that for every increasing (respectively, decreasing) sequence (r_n) in $\mathcal{P}(H(M, \beta))$ with $(r_n) \uparrow r$ (respectively, $(r_n) \downarrow r$), where $r \in \mathcal{P}(H(M, \beta))$, then $\Delta(r_n) \to \Delta(r)$.

We shall show that the set $\{|\Delta(p)| : p \in \mathcal{P}(H(M, \beta))\}$ is bounded. Let us fix $p \in \mathcal{P}(H(M, \beta))$. Since $(p_n) = (\mathbf{1} - q_n) \downarrow 0$ and $p_n \wedge (\mathbf{1} - p) \leq p_n$, we deduce that $(|\Delta(p_n)|)$,

and $(|\Delta(p_n \wedge (\mathbf{1}-p))|)$ tend to 0. We can therefore assume that $|\Delta(p_n)|, |\Delta(p_n \wedge (\mathbf{1}-p))| \leq 1$, for each natural n.

We claim that for each natural n and every projection $r \in \mathcal{P}(H(M,\beta))$ we have

(3.2) $|\Delta(r \vee p_n)| \leq 1 + C.$

Indeed, since $r \vee p_n \geq p_n$, it follows that $r \vee p_n = r \vee p_n - p_n + p_n$ with $r \vee p_n - p_n \perp p_n$. Therefore $\Delta(r \vee p_n) = \Delta(r \vee p_n - p_n) + \Delta(p_n)$. Since $p_n(r \vee p_n - p_n) = 0 = (r \vee p_n - p_n)p_n$, we deduce that $r \vee p_n - p_n \leq \mathbf{1} - p_n = q_n$. It follows from the assumptions that $|\Delta(r \vee p_n)| \leq |\Delta(r \vee p_n - p_n)| + |\Delta(p_n)| \leq C + 1$, as desired.

With p as above, let us denote $q = p + (\mathbf{1}-p) \wedge p_1$. It is easy to check that $(\mathbf{1}-q) \wedge p_1 = 0$, therefore Remark 1 in [17] proves that $q = r(qp_1q) + q \wedge (\mathbf{1}-p_1)$ with $r(qp_1q) \perp q \wedge (\mathbf{1}-p_1)$. Since $p \perp (\mathbf{1}-p) \wedge p_1$, we deduce from the finite additivity of Δ that $\Delta(q) = \Delta(p) + \Delta((\mathbf{1}-p) \wedge p_1)$ and $\Delta(q) = \Delta(r(qp_1q)) + \Delta(q \wedge (\mathbf{1}-p_1))$, and hence

$$|\Delta(p)| \leq |\Delta(q)| + |\Delta((\mathbf{1}-p) \wedge p_1)| \leq |\Delta(r(qp_1q))| + |\Delta(q \wedge (\mathbf{1}-p_1))| + 1$$

$$\leq |\Delta(r(qp_1q))| + C + 1.$$

The sequence $(G_n) = (1_{(0,1-\frac{1}{n})}(qp_1q)) \subseteq \mathcal{P}(H(M,\beta))\}$ grows to the range projection $r(qp_1q)$. We deduce that $(\Delta(G_n)) \uparrow \Delta(r(qp_1q))$, and thus, there exists $n_1 \in \mathbb{N}$ such that $|\Delta(G_{n_1}) - \Delta(r(qp_1q))| < 1$, and consequently,

(3.3) $|\Delta(p)| \leq 2 + C + |\Delta(G_{n_1})|.$

We claim that $G = G_{n_1}$ is "separated" from p_1 in the sense of [17], that is, $d(p_1, G_{n_1}) > 0$. Considering the von Neumann subalgebra generated by the element qp_1q and the functional calculus it is easy to see that $qp_1q \leq (1 - \frac{1}{n_1})G + 1_{[1-\frac{1}{n_1},1]}(qp_1q)$, with $(1 - \frac{1}{n_1})G \perp 1_{[1-\frac{1}{n_1},1]}(qp_1q)$. Then for each normal state $\varphi \in M_*$ with $\varphi(G) = 1 = \|\varphi\|$, we have $\varphi(qp_1q) \leq 1 - \frac{1}{n_1}$. Consequently, for each $\xi \in G(H)$ with $\|\xi\| = 1$, we have $\langle qp_1q(\xi)/\xi \rangle \leq 1 - \frac{1}{n_1}$. Having in mind that $G \leq r(qp_1q) \leq q$, we deduce that $q(\xi) = \xi$, and hence $\langle p_1(\xi)/\xi \rangle \leq 1 - \frac{1}{n_1}$, for every ξ as above. This shows that $\|\xi - p_1(\xi)\| \geq \frac{1}{\sqrt{n_1}}$, for every ξ satisfying the above conditions. The inequality in (3.1) shows that $d(G, p_1) \geq \frac{1}{2\sqrt{n_1}}$, proving that G is separated from p_1. Therefore, $d(G, p_n) \geq \frac{1}{2\sqrt{n_1}}$, for every $n \in \mathbb{N}$. Lemma 1(b) in [17] shows that

$$G \vee p_n \leq \frac{16}{d(G,p_n)^2}(G + p_n) \leq 64 n_1 (G + p_n),$$

for every natural n. We deduce that $\lim_n G \vee p_n \leq \lim_n 64 n_1 (G + p_n) = 64 n_1 G$, which implies that $G \vee p_n \downarrow G$. We can find $n_2 \in \mathbb{N}$ satisfying $|\Delta(G)| \leq 1 + |\Delta(G \vee p_{n_2})|$. Combining (3.3) and (3.2) we obtain

$$|\Delta(p)| \leq 3 + C + |\Delta(G \vee p_{n_2})| \leq 4 + 2C.$$

The conclusion of the lemma follows from the arbitrariness of p. $\square$

The following result for projections in von Neumann algebras is part of the folklore (cf. [17, Lemma 3] or [22, Lemma 6.1.10]. Let us observe that in the latter results the normal state should have been assumed to be faithful). By using the halving lemma for JBW-algebras the same proof holds in the case of JBW*-algebras.

Lemma 3.4. *Let M be a continuous von Neumann algebra and let $\beta : M \rightarrow M$ be a $\mathbb{C}$-linear *-involution. Suppose p is a projection in $H(M, \beta)$, φ is a faithful normal state in $H(M, \beta)_*$ and $0 < \delta < 1$. Then there exists a family of pairwise orthogonal projections $(p_i)_{i=1,\ldots,n}$ in $H(M, \beta)$ satisfying:*

(a) $p = \sum_{i=1}^{n} p_i$;
(b) $\varphi(p_i) \leq \delta$, for every $i = 1, \ldots, n$;
(c) $n \leq 2/\delta$. □

The following result is a crucial point in the proof of the main theorem of this section.

Proposition 3.5. *Let W be a JW-algebra containing no finite Type I part. Then W contains a JW-subalgebra B of Type II$_1$. Furthermore, if M is a (properly infinite) continuous von Neumann algebra and $\beta : M \rightarrow M$ is a $\mathbb{C}$-linear *-involution, then there exists a type II$_1$ von Neumann subalgebra N of M satisfying $\beta(N) = N$.*

The proof of the above proposition will follow from a technical lemma. First, we recall that a real flip α on $B(H)$ is a *-antiautomorphism of order 2 given by

$$\alpha(x) = Jx^*J,$$

where J is a conjugation on H. In this setting

$$B(H)^{\alpha}_{sa} = \{x \in B(H) : \alpha(x) = x = x^*\}$$

is a Type I JW-algebra factor. Since any two conjugations on the same complex Hilbert space are unitarily equivalent (see [23, Lemma 7.5.6]) all factor JW-algebras arising from a real flip on a fixed Hilbert space are isomorphic.

Lemma 3.6. *Let α be a real flip on $B(H)$, where H is a separable and infinite dimensional complex Hilbert space. Then there exists a factor von Neumann algebra N of type II$_1$, such that N is an α-invariant subalgebra of $B(H)$. In particular, $H(N, \alpha)_{sa} = \{x \in N : \alpha(x) = x^* = x\}$ is a finite Type II$_1$ JW-factor contained in $B(H)^{\alpha}_{sa}$. Moreover, $H(N, \alpha)_{sa}$ is not isomorphic to the self-adjoint part of a von Neumann algebra and the enveloping von Neumann algebra of $H(N, \alpha)_{sa}$ coincides with N.*

Proof. Let Π be the group of all permutations of natural numbers leaving all but finite integers fixed. Π is infinite and countable and so we can suppose that $H = \ell^2(\Pi)$. Denote by ξ_t an element in $\ell^2(\Pi)$ that takes value 1 at $t \in \Pi$ and zero otherwise. Then $(\xi_t)_{t \in \Pi}$ forms an orthonormal basis of H. By the remark preceding this lemma, there is no loss of generality in assuming that the real flip α is induced by a conjugation J of the form:

$$J\left(\sum_{t \in \Pi} \alpha_t \xi_t\right) = \sum_{t \in \Pi} \overline{\alpha_t} \xi_t,$$

where $(\alpha_t) \in \ell^2(\Pi)$. Let $\mathcal{L}_G$ be the (left) group von Neumann algebra generated by the unitaries u_t $(t \in \Pi)$, where

$$u_t \xi_s := \xi_{ts}.$$

Since

$$\alpha(u_t)\xi_s = Ju_t^* J\xi_s = Ju_t^*\xi_s = Ju_{t^{-1}}\xi_s = \xi_{t^{-1}s} = u_{t^{-1}}\xi_s = u_t^*\xi_s,$$
$$\alpha(u_t) = u_t^*$$

and consequently, $\mathcal{L}_G$ is α-invariant, and hence $\alpha(\mathcal{L}_G) = \mathcal{L}_G$.

By [29, Example 6.7.7, page 438 and Theorem 6.7.5] $\mathcal{L}_G$ is a Type II$_1$ factor (see also [29, Theorem 6.7.2]).

Now, since $\mathcal{L}_G$ is a continuous von Neumann factor, we conclude, by Theorem 1.5.2 in [5], that the algebra $H(\mathcal{L}_G, \alpha)_{sa} := \{x \in \mathcal{L}_G : \alpha(x) = x = x^*\}$ is a continuous JW-algebra factor which is not isomorphic to the self-adjoint part of a von Neumann algebra and the enveloping von Neumann algebra of $H(\mathcal{L}_G, \alpha)_{sa}$ coincides with $\mathcal{L}_G$. Moreover, Theorem 1.3.2 in [5] implies that $H(\mathcal{L}_G, \alpha)$ is finite. $\qquad\square$

Proof of Proposition 3.5. Let us suppose first that W is infinite and homogeneous Type I$_n$, where n is an infinite cardinal number. Then, according to the structure theory (see [23, Definition 5.3.3(ii)]), we can find an infinite system $(p_j)_{j \in \Lambda}$ of mutually orthogonal abelian projections such that $\sum_j p_j = \mathbf{1}$, the central support projection of each p_j coincides with the unit of W and card$(\Lambda) = n$. We can also conclude that the p_j's are mutually exchangeable by a symmetry (compare [23, Lemma 5.3.2]). Clearly, we can restrict to a countable subfamily. Then, there is a unital JW-subalgebra of W containing (p_j) that is isomorphic to $B(H)_{sa}^\alpha$, where α is a real flip and H has infinite countable dimension (see [23, Theorem 7.6.3 $(iii) \Leftrightarrow (iv)$]). The desired conclusion follows, in this case, from Lemma 3.6.

By [41, Theorem 16] (alternatively, [23, Theorem 5.3.5]) any properly infinite Type I JW-algebra W can be decomposed into a direct sum of infinite homogeneous ones. We can obtain the desired finite type II$_1$ continuous JW-subalgebra B by taking the sum of all type II$_1$ JW-subfactors given by Lemma 3.6 in the corresponding homogeneous summand. Actually it is enough to consider a non-zero type II$_1$ JW-subfactor in any of the corresponding homogeneous summands.

We assume now that W contains no type I part. Let p be a non-zero projection in W. If p is modular then $B = \{p, W, p\}$ is a JW-algebra of type II$_1$, which proves the desired statement. If p is not modular, then $\{p, W, p\}$ contains a copy of $B(H)_{sa}^\alpha$, where H is separable and infinite dimensional, and α is a real flip (see Theorem 7.6.3 $(i) \Leftrightarrow (iv)$ in [23]). Lemma 3.6 implies the existence of a type II$_1$ JW-subfactor of $B(H)_{sa}^\alpha$. This finishes the proof of the first statement in Proposition 3.5.

We consider now the second statement in the proposition. Let M be a continuous von Neumann algebra and suppose $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution. We may assume, without loss of generality, that the type II$_1$ part of M is zero. We consider the JW-algebra $H(M, \beta)_{sa} = \{a \in M : \beta(a) = a = a^*\}$.

We claim that $H(M, \beta)_{sa}$ contains a central projection which is not modular. Let z be a central projection in $H(M, \beta)_{sa}$. If z is not modular the claim is obvious, otherwise

$zH(M,\beta)_{sa}z$ is modular. Let $R(M,\beta) = \{x \in M : \beta(x) = x^*\}$. Clearly, $R(M,\beta)$ is a real von Neumann algebra and $H(M,\beta)_{sa} = \{x \in R(M,\beta) : x = x^*\}$ coincides with the hermitian part of $R(M,\beta)$. We also have $M = R(M,\beta) + iR(M,\beta)$, via $x = (x + \beta(x^*))/2 + i(x - \beta(x^*))/2i)$. We observe that z is a projection in M with $\beta(z) = z^* = z$, zMz is β-invariant, and $zH(M,\beta)_{sa}z = H(zMz,\beta)_{sa} = R(zMz,\beta)_{sa}$. We deduce from Proposition 1.3 in [1] that $R(zMz,\beta)$ is finite. Theorem 2.2 in [1] implies that $zMz = R(zMz,\beta) + iR(zMz,\beta)$ (and hence z) is finite in M. Let $c(z)$ denote the central support projection of z in M, that is, $c(z)$ is the smallest central projection in M majorizing z. Since β is a $\mathbb{C}$-linear *-involution, we deduce that $\beta(c(z)) = c(\beta(z)) = c(z)$, and thus $c(z)$ lies in $H(M,\beta)_{sa}$. Since the type II$_1$ part of M is zero and M is continuous, we deduce that $c(z)$ must be an infinite central projection in M (compare [40, Definition V.1.17]). Thus $c(z)$ must be a non-modular central projection in $H(M,\beta)_{sa}$, which proves the claim. Indeed, if $c(z)H(M,\beta)_{sa}$ were modular, then as shown above, $c(z)M$ would be finite.

Finally, let p be a non-modular central projection in $H(M,\beta)_{sa}$. A new application of [23, Theorem 7.6.3 $(i) \Leftrightarrow (iv)$] implies that $\{p, H(M,\beta)_{sa}, p\} = U_p(H(M,\beta)_{sa})$ contains a copy of $B(H)_{sa}^{\alpha}$, where H is separable and infinite dimensional, and α is a real flip. By Lemma 3.6 there exists a von Neumann algebra N of type II$_1$ such that N is an α-invariant von Neumann subalgebra of $B(H)$, $H(N,\alpha)_{sa}$ is a JBW-subalgebra of $H(M,\beta)_{sa}$ and the enveloping von Neumann algebra of $H(N,\alpha)_{sa}$ coincides with N. Clearly N is a subalgebra of M. Since every $x \in H(N,\alpha)_{sa}$ satisfies $\beta(x) = x = x^*$ and β is a $\mathbb{C}$-linear *-involution, the enveloping von Neumann algebra of $H(N,\alpha)_{sa}$, namely N, must be β-invariant, which concludes the proof. $\qquad\square$

The remaining results in this section are appropriate adaptations of the corresponding lemmas in [17] and [22, §6.1], they are included here for completeness reasons.

Let us observe a simple property.

Remark 3.7. Let M be a von Neumann algebra, let β be a $\mathbb{C}$-linear *-involution on M, and let $\Delta : \mathcal{P}(H(M,\beta)) \to \mathbb{C}$ be a completely additive measure. Suppose we can decompose M as finite direct sum of mutually orthogonal β-invariant von Neumann subalgebras $M_1, \ldots, M_k$, that is $M = M_1 \oplus^{\infty} \ldots \oplus^{\infty} M_k$ with $\beta(M_j) = M_j$, for every j. Then Δ is bounded if and only if $\Delta|_{\mathcal{P}(H(M_j,\beta))} : \mathcal{P}(H(M_j,\beta)) \to \mathbb{C}$ is bounded for every $j = 1, \ldots, k$.

Let us briefly recall some basic notions on σ-finite projections in JBW*-algebras. As in the setting of von Neumann algebras, a JBW*-algebra M is said to be σ-*finite* if every family of mutually orthogonal non-zero projections in M is at most countable. A projection p in M is called σ-*finite* if the JBW*-algebra $U_p(M)$ is σ-finite, where U_p is the operator on M given by $U_p(x) = \{p, x^*, p\} = 2(p \circ x) \circ p - p \circ x$. A projection p in M is σ-finite if and only if it is the support projection of a normal state in M_* (cf. [19, Theorem 3.2]). The supremum of countably many σ-finite projections is again σ-finite, and every projection in a JBW*-algebra can be written as a sum of mutually orthogonal σ-finite projections (see [19, Theorem 3.4]). These facts can be derived from [19] and are explicitly developed in [7].

The following two results will be applied in several arguments (compare [17, Lemma 4]).

Proposition 3.8. *Let M be a continuous von Neumann algebra and let $\beta : M \to M$ be a $\mathbb{C}$-linear *-involution. Suppose that $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ is an unbounded completely additive measure. Then there exists a σ-finite projection $p \in H(M, \beta)$ such that $\Delta|_{\mathcal{P}(H(pMp,\beta))}$ is unbounded.*

Proof. Since Δ is unbounded, there exists a sequence (q_n) in $\mathcal{P}(H(M, \beta))$ satisfying that $\lim_{n\to\infty} |\Delta(q_n)| = \infty$. Each q_n can be written as the sum of a family of mutually orthogonal σ-finite projections in $H(M, \beta)$ (compare [19, Theorem 3.4 (ii)]). Therefore, there exists a family $(p_\lambda^{(n)})_{\lambda \in \Lambda_n}$ of mutually orthogonal σ-finite projections in $H(M, \beta)$ such that $q_n = \sum_{\lambda \in \Lambda_n} p_\lambda^n$. By the complete additivity of Δ, there exists a finite subset $F_n \subseteq \Lambda_n$ such that

$$\left| \Delta(q_n) - \Delta\left(\sum_{\lambda \in F_n} p_\lambda^{(n)} \right) \right| < \frac{1}{n}.$$

Clearly, $p_n = \sum_{\lambda \in F_n} p_\lambda^n$ is a σ-finite projection in $\mathcal{P}(H(M, \beta))$ and $\lim_{n\to\infty} |\Delta(p_n)| = \infty$. Let $p = \bigvee_n p_n \in \mathcal{P}(H(M, \beta))$. Since the supremum of countably many σ-finite projections is again σ-finite (compare [19, Theorem 3.4 (i)]), the projection p is σ-finite, and obviously, $\Delta|_{\mathcal{P}(H(pMp,\beta))}$ is unbounded, which finishes the proof. $\qquad\square$

Proposition 3.9. *Let M be a von Neumann algebra of type II_1, II_∞ or III, and let β a $\mathbb{C}$-linear *-involution on M. Suppose $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ is a completely additive unbounded measure. Then there exists a projection $p_0 \in \mathcal{P}(H(M, \beta))$ such that $p_0 M p_0$ is σ-finite and of type II_1, II_∞ or III, and the measure $\Delta_0 = \Delta|_{\mathcal{P}(H(p_0Mp_0,\beta))} : \mathcal{P}(H(p_0 M p_0, \beta)) \to \mathbb{C}$ satisfies the following property*

(3.4)   *for each $p \in \mathcal{P}(H(p_0 M p_0, \beta))$ with $|\Delta_0(p)| > 1$ the measure*

$$\Delta_0|_{\mathcal{P}(H((p_0-p)M(p_0-p),\beta))} : \mathcal{P}(H((p_0 - p)M(p_0 - p), \beta)) \to \mathbb{C} \text{ is bounded.}$$

Proof. If the pair $(H(M, \beta), \Delta)$ satisfies the desired property then the proof is concluded by taking $p_0 = \mathbf{1}$. Otherwise, there exists a projection $p_1 \in \mathcal{P}(H(M, \beta))$ with $|\Delta(p_1)| > 1$ satisfying that the measure $\Delta|_{\mathcal{P}(H((\mathbf{1}-p_1)M(\mathbf{1}-p_1),\beta))} : \mathcal{P}(H((\mathbf{1} - p_1)M(\mathbf{1} - p_1), \beta)) \to \mathbb{C}$ is unbounded. Since $(\mathbf{1} - p_1)M(\mathbf{1} - p_1)$ doesn't contain type I part, we can decompose $(\mathbf{1} - p_1)M(\mathbf{1} - p_1)$ as a direct sum of von Neumann subalgebras of type II_1, II_∞ or III. We also observe that each summand in the above decomposition must be β-invariant. Therefore, by Remark 3.7, there exists a subprojection $q_1 \leq \mathbf{1} - p_1$ such that $q_1 M q_1$ is of type II_1, II_∞ or III and $\Delta|_{\mathcal{P}(H(q_1Mq_1,\beta))}$ is unbounded.

If the pair $(H(q_1 M q_1, \beta), \Delta|_{\mathcal{P}(H(q_1Mq_1,\beta))})$ satisfies property (3.4) we obtain the desired statement. Otherwise, applying the above argument, there exists $p_2 \leq q_1$ in $H(q_1 M q_1, \beta)$ such that $|\Delta(p_2)| > 1$ and $\Delta|_{\mathcal{P}(H((q_1-p_2)M(q_1-p_2),\beta))} : \mathcal{P}(H((q_1 - p_2)M(q_1 - p_2), \beta)) \to \mathbb{C}$ is unbounded. Thus, there exists $q_2 \leq q_1 - p_2$ such that $q_2 M q_2$ is of type II_1, II_∞ or III and $\Delta|_{\mathcal{P}(H(q_2Mq_2,\beta))}$ is unbounded.

By repeating the above arguments, we find a pair $(q_n M q_n, \Delta|_{\mathcal{P}(H(q_n M q_n, \beta))})$ (with $\beta(q_n) = q_n$, for every $n \in \mathbb{N}$) satisfying the desired statement, or there exists an infinite sequence (p_n) of mutually orthogonal β-symmetric projections in M satisfying $|\Delta(p_n)| > 1$, for every natural n, which contradicts the complete additivity of Δ. $\qquad\square$

Henceforth, up to and including Lemma 3.14, M will denote a σ-finite von Neumann algebra of type II_1, II_∞ or III, β a $\mathbb{C}$-linear *-involution on M, and N a type II_1 von Neumann subalgebra of M satisfying $\beta(N) = N$ (compare Proposition 3.5). We observe that $H(N, \beta)$ is a JBW*-subalgebra of $H(M, \beta)$. From now on, τ will stand for a faithful normal norm-one finite trace on N, whose restriction to $H(N, \beta)$ will be also denoted by τ.

First, we recall some facts about the strong* topology. For each normal positive functional φ in the predual of a von Neumann algebra M, the mapping

$$x \mapsto \|x\|_\varphi = \left(\varphi(\frac{xx^* + x^*x}{2}) \right)^{\frac{1}{2}} \quad (x \in M)$$

defines a prehilbertian seminorm on M. The *strong* topology* of M, denoted by $S^*(M, M_*)$, is the locally convex topology on M defined by all the seminorms $\|.\|_\varphi$, where φ runs in the set of all positive functionals in M_*

Lemma 3.10. *Let $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ be a completely additive unbounded measure. Suppose that the pair $(H(M, \beta), \Delta)$ satisfies property (3.4) in Lemma 3.9. Let N a type II_1 von Neumann subalgebra of M satisfying $\beta(N) = N$, and let τ denote the unital normal faithful finite trace on N. Then there exist a positive constant K and $0 < \delta < 1$ satisfying the following property:*
(3.5)
* For each $q \in H(N, \beta)$ with $\tau(q) \le \delta$, we have $\sup\{|\Delta(p)| : p \in H(M, \beta), p \le q\} \le K$.*

Proof. (compare [17, Lemma 5]) Arguing by reduction to the absurd, we suppose that the desired property does not hold. Then there exists a sequence (q_n) in $H(N, \beta)$ such that $|\tau(q_n)| \le \frac{1}{2^n}$ and

$$\sup\{|\Delta(p)| : p \in H(M, \beta), p \le q_n\} > n,$$

for every natural n. Set $G_n := \vee_{k=n}^\infty q_k$. Since every q_n is β-symmetric, we deduce that G_n also is β-symmetric for all $n \in \mathbb{N}$ (i.e., $(G_n) \subset H(N, \beta)$). Considering strong*-limits of growing sequences, we deduce that $\tau(G_n) \le \sum_{k=n}^\infty \tau(q_k) \le \sum_{k=n}^\infty \frac{1}{2^k}$, which implies, by the faithfulness of τ on N, that $(G_n) \searrow 0$ in the strong*-topology of N (and also in the strong*-topology of M). Since $G_n \ge q_n$ and $G_n \downarrow 0$, we have for every $m \ge n$,

$$\sup\{|\Delta(p)|; p \le G_n\} \ge \sup\{|\Delta(p)|; p \le G_m\} \ge \sup\{|\Delta(p)|; p \le q_n\} > m,$$

and $\Delta|_{\mathcal{P}(H(G_n M G_n, \beta))}$ is unbounded. Since the pair $(H(M, \beta), \Delta)$ satisfies property (3.4), we deduce that $\sup\{|\Delta(p)| : p \in H(M, \beta), p \le \mathbf{1} - G_n\} \le 1$, for every natural n. Since $\mathbf{1} - G_n \nearrow \mathbf{1}$ in the strong*-topology, Lemma 3.3 implies that Δ is bounded, which is impossible. $\qquad\square$

The automatic boundedness of completely additive measure on $\mathcal{P}(H(M, \beta))$ actually relies on the appropriate Jordan version of the Mackey-Gleason theorem stated in Theorem 3.2.

Lemma 3.11. *Let* $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ *be a completely additive unbounded measure. Suppose that the pair* $(H(M, \beta), \Delta)$ *satisfies the property* (3.4) *in Lemma* 3.9. *Let* N *a type* II_1 *von Neumann subalgebra of* M *satisfying* $\beta(N) = N$ *and let* τ *be the unital faithful normal trace on* N. *Then there exists a positive constant* C_0 *satisfying that if* $q \in \mathcal{P}(H(N, \beta))$ *and* $\Delta|_{\mathcal{P}(H(qMq,\beta))}$ *is bounded then*

$$\sup\{|\Delta(p)| : p \in \mathcal{P}(H(qMq, \beta))\} \leq C_0.$$

Proof. The proof given in [17, Lemma 6] or in [22, Lemma 6.1.13] remains valid here when we replace [17, Lemma 5] with our previous Lemma 3.10 and the Bunce-Wright-Mackey-Gleason theorem [10] with Theorem 3.2. We include some details for completeness reasons.

Let $C_0 = \frac{32K}{\delta}$, where K and δ are given by Lemma 3.10. Take a projection q in $H(N, \beta)$ with $\Delta|_{\mathcal{P}(H(qMq,\beta))}$ bounded. Theorem 3.2 implies the existence of a (normal) continuous linear functional $\varphi : H(qMq, \beta) \to \mathbb{C}$ such that $\varphi(p) = \Delta(p)$, for every $p \in \mathcal{P}(H(qMq, \beta))$. By Lemma 3.4 there exists a family of pairwise orthogonal projections $(q_i)_{i=1,...,n}$ in $H(N, \beta)$ such that $q = \sum_{i=1}^{n} q_i$, $\tau(q_i) \leq \frac{\delta}{2}$, for every $i = 1, \ldots, n$, and $n \leq 4/\delta$.

Let us pick an arbitrary projection $p \in H(qMq, \beta)$. We need to show that $|\Delta(p)| \leq C_0$. If we write $p = \sum_{i,j=1}^{n}\{q_i, p, q_j\} = \sum_{i,j=1}^{n} \frac{1}{2}(q_i p q_j + q_j p q_i)$, we observe that, for $i \neq j$, $q_i + q_j$ is a projection in $H(N, \beta)$ and $\tau(q_i + q_j) \leq \delta$. Lemma 3.10 implies that

$$\sup\{|\Delta(r)| : r \in \mathcal{P}(H(M, \beta)), r \leq q_i + q_j\} \leq K,$$

and hence that

$$\sup\{|\varphi(r)| : r \in \mathcal{P}(H(M, \beta)), r \leq q_i + q_j\} \leq K.$$

Considering spectral resolutions, we deduce that $\sup\{|\varphi(a)| : a \in H((q_i + q_j)M(q_i + q_j), \beta), \|a\| \leq 1\} \leq 2K$. Similarly, $\sup\{|\varphi(a)| : a \in H(q_j M q_j, \beta), \|a\| \leq 1\} \leq 2K$, for every $j = 1, \ldots, n$. Therefore, having in mind that $q_i p q_j + q_j p q_i$ lies in $H((q_i + q_j)M(q_i + q_j), \beta)$, we deduce that

$$|\Delta(p)| = |\varphi(p)| \leq \sum_{i,j=1}^{n} \frac{1}{2}|\varphi(q_i p q_j + q_j p q_i)| \leq n^2 2K \leq \frac{16}{\delta^2} 2K = C_0.$$

$\square$

In a similar fashion, replacing [17, Lemmas 2 and 6] with Lemmas 3.3 and 3.11, respectively, the proof of [17, Lemma 7] holds to prove the following result.

Lemma 3.12. *Let* $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ *be a completely additive unbounded measure, where* $H(M, \beta)$ *is a* σ-*finite JBW**-*algebra. Suppose that the pair* $(H(M, \beta), \Delta)$ *satisfies the property* (3.4) *in Lemma* 3.9. *Let* N *a type* II_1 *von Neumann subalgebra of* M *satisfying* $\beta(N) = N$. *Then there exists a projection* q_0 *in* $H(N, \beta)$ *satisfying the following properties:*
(a) $\Delta|_{\mathcal{P}(H(q_0 M q_0, \beta))}$ *is bounded;*

(b) if $q \in \mathcal{P}(H(N, \beta))$, $q \gneqq q_0$ then $\Delta|_{\mathcal{P}(H(qMq, \beta))}$ is unbounded.

Proof. Let $\mathcal{B}$ denote the set of all families $(q_i)_{i \in I}$ of mutually orthogonal projections in $H(N, \beta)$ such that for each finite subset $F \subset I$, the projection $q_F := \sum_{i \in F} q_i$ satisfies that $\Delta|_{\mathcal{P}(H(q_F M q_F, \beta))}$ is bounded. The set $\mathcal{B}$ is an inductive set when it is equipped with the order given by inclusion (by Proposition 3.9, $\mathcal{B} \neq \emptyset$). By Zorn's lemma there exists a maximal element $(q_i^0)_I \in \mathcal{B}$. The set I is at most countable because $H(M, \beta)$ is σ-finite. We claim that the projection $q_0 = \sum_{i \in I} q_i^0 \in H(N, \beta)$ satisfies the desired property. Indeed, defining $q_n := \sum_{i=1}^{n} q_i^0$, we have $q_n \nearrow q_0$. Since $(q_i^0)_I \in \mathcal{B}$, the measure $\Delta|_{\mathcal{P}(H(q_n M q_n, \beta))}$ is bounded for every n. Lemma 3.11 implies the existence of a constant $C_0 > 0$ such that $\sup\{|\Delta(p)| : p \in H(M, \beta), p \leq q_n\} \leq C_0$, for every natural n. Lemma 3.3 proves that $\Delta|_{\mathcal{P}(H(q_0 M q_0, \beta))}$ is bounded.

Finally, the second property follows from the maximality of the element $(q_i^0)_I \in \mathcal{B}$. $\square$

We shall see now that the arguments in the proof of [22, Lemma 6.1.15] are also valid in the Jordan setting. Actually, the proof follows the arguments we gave in Lemma 3.10.

Lemma 3.13. *Let $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ be a completely additive unbounded measure. Suppose that the pair $(H(M, \beta), \Delta)$ satisfies the property (3.4) in Lemma 3.9. Let us assume that $H(M, \beta)$ is σ-finite and let φ be a faithful normal state on $H(M, \beta)$. Then there exists a projection p_0 in $H(M, \beta)$ and $\delta > 0$ such that $\Delta|_{\mathcal{P}(H(p_0 M p_0, \beta))}$ is unbounded and the following property holds:*

$$(3.6) \qquad \textit{If } p \in \mathcal{P}(H(M, \beta)), p \leq p_0 \textit{ and } \varphi(p) \leq \delta, \textit{ then } \Delta|_{\mathcal{P}(H(pMp, \beta))} \textit{ is bounded.}$$

Proof. If the desired property holds for $p_0 = \mathbf{1}$ and some δ, then the Lemma is proved. Otherwise, there exists a projection p_1 in $\mathcal{P}(H(M, \beta))$ such that $\varphi(p_1) \leq \frac{1}{2}$ and $\Delta|_{\mathcal{P}(H(p_1 M p_1, \beta))}$ is unbounded. If p_1 satisfies the desired property the statement is proved. If that is not the case, there exists a projection p_2 in $\mathcal{P}(H(M, \beta))$ such that $p_2 \leq p_1$, $\varphi(p_2) \leq \frac{1}{3}$ and $\Delta|_{\mathcal{P}(H(p_2 M p_2, \beta))}$ is unbounded. Repeating the above argument, we obtain the desired conclusion for a suitable projection, or there exists a decreasing sequence of projections (p_n) in $H(M, \beta)$ satisfying $\varphi(p_n) \leq \frac{1}{n}$ and $\Delta|_{\mathcal{P}(H(p_n M p_n, \beta))}$ is unbounded. The faithfulness of φ implies that $p_n \searrow 0$ in the strong*-topology.

Since the pair $(H(M, \beta), \Delta)$ satisfies property (3.4) in Lemma 3.9, we conclude that

$$\sup\{|\Delta(p)| : p \in \mathcal{P}(H(M, \beta)), p \leq \mathbf{1} - p_n\} \leq 1,$$

for all n. Recalling that $\mathbf{1} - p_n \nearrow \mathbf{1}$ in the strong*-topology, Lemma 3.3 implies that Δ is bounded, which contradicts the hypothesis of the lemma. $\square$

Lemma 3.14. *Let $\Delta : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ be a completely additive unbounded measure. Let p_0 be a projection in $H(M, \beta)$ satisfying that $\Delta|_{\mathcal{P}(H(p_0 M p_0, \beta))}$ and $\Delta|_{\mathcal{P}(H((1-p_0)M(1-p_0), \beta))}$ are bounded. Let $K_n \to \infty$. Then for each natural n, there exists a projection $q = q_n \in H(M, \beta)$ such that $|\Delta(q)| > K_n$ and $d(q, p_0) \geq \frac{1}{8}$.*

Proof. (compare [17, Lemma 9]) Let us take $C > 0$ satisfying $\sup\{|\Delta(p)| : p \in H(M, \beta), p \leq p_0 \text{ or } p \leq \mathbf{1} - p_0\} \leq C$. Given n, by the unboundedness of Δ, we can find a projection p in

$H(M,\beta)$ such that $|\Delta(p)| > 2K_n + 6C$. The projection $r = p + (\mathbf{1} - p) \wedge p_0 \in H(M,\beta)$ satisfies $(\mathbf{1} - r) \wedge p = 0$ and $|\Delta(r)| \geq |\Delta(p)| - |\Delta((\mathbf{1} - p) \wedge p_0)| > 2K_n + 5C$. By [17, Remark 1] we have

$$r = r(rp_0 r) + r \wedge (\mathbf{1} - p_0)$$

in M, as well as in $H(M,\beta)$. The proof of [17, Lemma 9] shows that taking $r_1 = 1_{(0,\frac{1}{2}]}(rp_0 r) \in H(M,\beta)$ and $r_2 = 1_{(\frac{1}{2},1]}(rp_0 r) \in H(M,\beta)$ we have $d(r_1, p_0) \geq \frac{1}{2}$ and $r_1 + r_2 = r - r \wedge (\mathbf{1} - p_0)$. It is further seen that for $r_2' = r_2 \vee (\mathbf{1} - p_0) - r_2 \in H(M,\beta)$ the inequality $d(r_2', p_0) \geq \frac{1}{8}$ holds.

It is also clear that $r_1 \perp r_2$, and since $r - r \wedge (\mathbf{1} - p_0) \perp r \wedge (\mathbf{1} - p_0)$. Therefore

$$|\Delta(r_1)| + |\Delta(r_2)| \geq |\Delta(r_1) + \Delta(r_2)| = |\Delta(r_1 + r_2)| = |\Delta(r - r \wedge (\mathbf{1} - p_0))|$$

$$= |\Delta(r) - \Delta(r \wedge (\mathbf{1} - p_0))| \geq |\Delta(r)| - |\Delta(r \wedge (\mathbf{1} - p_0))| > 2K_n + 4C.$$

It follows that $|\Delta(r_1)| > K_n + 2C$ or $|\Delta(r_2)| > K_n + 2C$. In the first case the projection $q = r_1$ satisfies the desired statement; otherwise, the projection $q = r_2'$ satisfies the conclusion of the lemma. Indeed,

$$|\Delta(r_2 \vee (\mathbf{1} - p_0))| \leq |\Delta(p_0)| + |\Delta(r_2 \vee (\mathbf{1} - p_0) - (\mathbf{1} - p_0))| \leq 2C,$$

because $(r_2 \vee (\mathbf{1} - p_0) - (\mathbf{1} - p_0)) \perp (\mathbf{1} - p_0)$. Thus, we get

$$|\Delta(q)| = |\Delta(r_2')| \geq |\Delta(r_2)| - |\Delta(r_2 \vee (\mathbf{1} - p_0))| > K_n.$$

$\square$

We complete now the proof of our Jordan version of Dorofeev's theorem. The arguments are based on appropriate Jordan adaptations of the proofs in [17, Theorem 1] and [22, Theorem 6.1.16].

Proof of Theorem 3.1. Arguing by contradiction, we shall assume that $\Delta : \mathcal{P}(H(M,\beta)) \to \mathbb{C}$ is an unbounded completely additive measure. By Proposition 3.8 there exists a σ-finite projection $p \in H(M,\beta)$ such that $\Delta|_{\mathcal{P}(H(pMp,\beta))}$ is unbounded.

We can therefore assume that $H(M,\beta)$ is σ-finite. Let φ be a faithful normal state on $H(M,\beta)$. Furthermore, by Remark 3.7, we can also assume that M is of type II_1, II_∞ or III.

Having in mind Proposition 3.9, we can assume that the pair $(H(M,\beta), \Delta)$ satisfies property (3.4) for $p_0 = \mathbf{1}$ in that proposition (otherwise we replace M with $p_0 M p_0$). Applying Lemma 3.13, we may assume that Δ satisfies property (3.6) for $p_0 = \mathbf{1}$, the faithful normal state φ fixed in the above paragraph, and a suitable $\delta > 0$. By Proposition 3.5 there exists a type II_1 subalgebra N of M such that $\beta(N) = N$.

Let q_0 be the projection in $H(N,\beta)$ given by Lemma 3.12, that is, q_0 satisfies the following properties:

(a) $\Delta|_{\mathcal{P}(H(q_0 M q_0,\beta))}$ is bounded;
(b) if $q \in \mathcal{P}(H(N,\beta))$, $q \gneqq q_0$ then $\Delta|_{\mathcal{P}(H(qMq,\beta))}$ is unbounded.

The unboundedness of Δ implies that $q_0 \neq \mathbf{1}$. By the Halving lemma (see [23, Theorem 5.2.14]) there exists an infinite sequence (q_n) of mutually orthogonal nonzero projections in $H(N, \beta)$ such that $q_n \leq \mathbf{1} - q_0$, for every $n \in \mathbb{N}$. Property (b) above implies that $\Delta|_{\mathcal{P}(H((q_0+q_n)M(q_0+q_n),\beta))}$ is unbounded for all natural n.

We claim that $\Delta|_{\mathcal{P}(H((\mathbf{1}-q_0)M(\mathbf{1}-q_0),\beta))}$ is bounded. Indeed, let (r_n) be a sequence of projections in $H(N, \beta)$ such that $(r_n) \searrow 0$ and $r_n \leq \mathbf{1} - q_0$. The above property (b) of q_0 also implies that $\Delta|_{\mathcal{P}(H((q_0+r_n)M(q_0+r_n),\beta))}$ is unbounded for all natural n. Since the pair $(H(M, \beta), \Delta)$ satisfies property (3.4) for $p_0 = \mathbf{1}$ in Proposition 3.9, it follows that

$$\sup\{|\Delta(p)| : p \leq \mathbf{1} - q_0 - r_n\} \leq 1, \ (n \in \mathbb{N}).$$

The boundedness of the previous set together with the condition $\mathbf{1} - q_0 - r_n \nearrow \mathbf{1} - q_0$ imply, via Lemma 3.3 that $\Delta|_{\mathcal{P}(H((\mathbf{1}-q_0)M(\mathbf{1}-q_0),\beta))}$ is bounded, which proves the claim.

We have shown that $\Delta|_{\mathcal{P}(H(q_0Mq_0,\beta))}$ and $\Delta|_{\mathcal{P}(H((\mathbf{1}-q_0)M(\mathbf{1}-q_0),\beta))}$ are bounded measures. Applying Lemma 3.14 to $(q_0+q_n)M(q_0+q_n)$ and the projection $p_0 = q_0$, we find a projection p_n in $(q_0 + q_n)M(q_0 + q_n)$ satisfying $|\Delta(p_n)| > \frac{n2^{12n}}{\delta}$ and $d(p_n, q_0) \geq \frac{1}{8}$ (let us observe that since $q_n \leq \mathbf{1} - q_0$, $\Delta|_{\mathcal{P}(H(q_nMq_n,\beta))}$ is bounded). We define in this way a sequence (p_n) in $\mathcal{P}(H(M, \beta))$.

We shall prove next that, for each natural n, $d(p_n, \vee_{i \neq n}p_i) \geq \frac{1}{8}$. To this end, let us pick norm-one elements $\xi \in p_n(H)$ and $\eta \in \vee_{i \neq n}p_i(H)$ (we regard M as a von Neumann subalgebra of some $B(H)$). Having in mind that $p_n \leq q_0 + q_n$ with $q_n \perp q_0$ ($n \in \mathbb{N}$), we deduce that $\vee_{i \neq n}p_i(H) \subset q_0(H) + \left(\sum_{i \neq n} q_i\right)(H)$, and thus, we can write

$$\eta = \alpha u_1 + \beta u_2,$$

where $\alpha, \beta \geq 0$, $\alpha^2 + \beta^2 = 1$, $u_1 \in q_0(H)$ and $u_2 \in \left(\sum_{i \neq n} q_i\right)(H)$. The images of q_0 and $\left(\sum_{i \neq n} q_i\right)$ are orthogonal in the Hilbert sense, and hence

$$\|\xi - \eta\|^2 = \|\xi - \alpha u_1 - \beta u_2\|^2 = \|\xi - \alpha u_1\|^2 + \|\beta u_2\|^2$$

$$\geq (\|\xi - u_1\| - \|(1 - \alpha)u_1\|)^2 + \beta^2 = (\|\xi - u_1\| - 1 + \alpha)^2 + 1 - \alpha^2.$$

The last expression in the above inequality defines a function $f(\alpha)$, $\alpha \in [0, 1]$, whose extreme values are attained at $\alpha = 0$ or $\alpha = 1$. Taking $\alpha = 0$, we have $\|\xi - \eta\|^2 \geq (\|\xi - u_1\| - 1)^2 + 1 \geq 1$. In the case $\alpha = 1$, we have $\|\xi - \eta\|^2 = \|\xi - u_1\|^2 \geq \frac{1}{8^2}$, because $u_1 \in q_0(H)$ and $d(q_0, p_n) \geq \frac{1}{8}$.

We apply now Lemma 3.4. For each natural n, we can find a finite set $\{p_i^n : i = 1, \ldots, k_n\}$ of mutually orthogonal projections in $H(M, \beta)$ satisfying $p_n = \sum_{i=1}^{k_n} p_i^n$, $\varphi(p_i^n) \leq \frac{\delta}{2^{11n}}$, and $k_n \leq 2\frac{2^{11n}}{\delta}$. The projections in $\{p_i^n : i = 1, \ldots, k_n\}$ are mutually orthogonal, so

$$\frac{n2^{12n}}{\delta} < |\Delta(p_n)| = \left|\sum_{i=1}^{k_n} \Delta(p_i^n)\right| \leq \sum_{i=1}^{k_n} |\Delta(p_i^n)|,$$

and therefore there exists $i_n \in \{1, \ldots, k_n\}$ such that $|\Delta(p_{i_n}^n)| > n$. So, replacing p_n with $p_{i_n}^n$, it may be assumed that $\varphi(p_n) \leq \frac{\delta}{2^{11n}}$ and $|\Delta(p_n)| > n$.

Now, we take $\varepsilon = \frac{1}{2^{10}}$. Lemma 1($b$) in [17] asserts that

$$p_1 \vee \ldots \vee p_n \le \frac{1}{\varepsilon}(p_1 + p_2 \vee \ldots \vee p_n) \le \frac{1}{\varepsilon}p_1 + \frac{1}{\varepsilon^2}(p_2 + p_3 \vee \ldots \vee p_n) \le \ldots \le \sum_{k=1}^{n} \frac{1}{\varepsilon^k} p_k.$$

Therefore,

$$\varphi(p_1 \vee \ldots \vee p_n) \le \sum_{k=1}^{n} \frac{1}{\varepsilon^k} \varphi(p_k) \le \sum_{k=1}^{n} \frac{1}{\varepsilon^k} \frac{\delta}{2^{11k}} = \sum_{k=1}^{n} 2^{10k} \frac{\delta}{2^{11k}} < \delta.$$

This shows that for $r = \vee_{n=1}^{\infty} p_n$, $\varphi(r) \le \delta$ and $\Delta|_{\mathcal{P}(H(rMr,\beta))}$ is unbounded, which contradicts that Δ satisfies property (3.6) for $p_0 = 1$ and $\delta > 0$. $\square$

4. 2-LOCAL TRIPLE DERIVATIONS ON CONTINUOUS JBW*-TRIPLES

Recall that a JBW^*-triple $\mathcal{A}$ is said to be *continuous* if it has no type I direct summand, and that in this case, up to isometry, $\mathcal{A}$ is a JW^*-triple with unique decomposition, $\mathcal{A} = H(W, \alpha) \oplus pV$, where W and V are continuous von Neumann algebras, p is a projection in V, α is a *-antiautomorphism of W of order 2, and $H(W, \alpha) = \{x \in W : \alpha(x) = x\}$ (see [26, (1.20)]).

We have shown in section 2 that every 2-local triple derivation on pV is a triple derivation. In this section we show that every 2-local triple derivation on $H(W, \alpha)$ is a triple derivation, and hence that every 2-local triple derivation on a continuous JBW*-triple is a triple derivation.

4.1. **Triple derivations on** $H(M, \beta)$. Assume that M is a continuous von Neumann algebra and $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution (i.e. a *-antiautomorphism of order 2). In this subsection we shall show that every 2-local triple derivation on the subspace $H(M, \beta)$ of all β-fixed points in M is a triple derivation.

We begin by taking advantage of the Jordan structure of $H(M, \beta)$ (see the beginning of section 3) to provide a precise description of triple derivations on it.

Let $\delta : H(M, \beta) \to H(M, \beta)$ be a triple derivation. By [24, Lemma 1 and its proof],

(4.1) $\qquad \delta(\mathbf{1})^* = -\delta(\mathbf{1})$, and $M_{\delta(\mathbf{1})} = \delta\left(\frac{1}{2}\delta(\mathbf{1}), \mathbf{1}\right)$ is a triple derivation.

This implies that $D = \delta - M_{\delta(\mathbf{1})} = \delta - \delta\left(\frac{1}{2}\delta(\mathbf{1}), \mathbf{1}\right)$ is a triple derivation satisfying $D(\mathbf{1}) = 0$. Lemma 2 in [24] implies that D is a Jordan *-derivation on $H(M, \beta)$. Thus, $D|_{H(M,\beta)_{sa}} : H(M, \beta)_{sa} \to H(M, \beta)_{sa}$ is a Jordan derivation on the continuous JBW-algebra $H(M, \beta)_{sa}$. Theorem 3.5 in [42] assures that $D|_{H(M,\beta)_{sa}}$ is an inner derivation, that is, there exist $a_1, \ldots, a_m$ $b_1, \ldots, b_m$ in $H(M, \beta)_{sa}$ satisfying

(4.2) $\qquad D(x) = \sum_{j=1}^{m} \left[M_{a_j}, M_{b_j} \right](x) = \sum_{j=1}^{m} a_j \circ (b_j \circ x) - b_j \circ (a_j \circ x)$

$$= \frac{1}{4} \sum_{j=1}^{m} (a_j b_j - b_j a_j) x - x(a_j b_j - b_j a_j) = \sum_{j=1}^{m} \left[\frac{(a_j b_j - b_j a_j)}{4}, x \right] = \left[\sum_{j=1}^{m} \frac{(a_j b_j - b_j a_j)}{4}, x \right],$$

for every $x \in H(M, \beta)_{sa}$. If we denote $a = \sum_{j=1}^{m} \frac{(a_j b_j - b_j a_j)}{4} \in M$, then $\beta(a) = -a$ and $a^* = -a$ (just observe that $\beta(a_j) = a_j$, $a_j^* = a_j$, $\beta(b_j) = b_j$, and $b_j^* = b_j$, for every j), and, by (4.2),

$$\delta(x) = [a, x] + \delta(\mathbf{1}) \circ x,$$

for every $x \in H(M, \beta)_{sa}$. The following proposition summarizes the above facts.

Proposition 4.1. *Let M be a continuous von Neumann algebra and let $\beta : M \to M$ be a $\mathbb{C}$-linear $*$-involution. Then for every triple derivation δ on the JBW*-algebra $H(M, \beta)$, of all β-fixed points in M, there exist $a, b \in M$ with $a^* = -a$, $b^* = -b$, $\beta(a) = -a$ and $\beta(b) = b = \delta(\mathbf{1})$, satisfying*

$$\delta(x) = [a, x] + b \circ x,$$

for every $x \in H(M, \beta)$. Consequently, every triple derivation on $H(M, \beta)$ admits an extension to a triple derivation on M. ☐

4.2. **2-local triple derivations on** $H(M, \beta)$**.** Let J be a JBW*-subalgebra of a von Neumann algebra M. Suppose that J contains the unit of M. Given a self-adjoint element $z \in J$, the JBW*-subalgebra, $\mathcal{W}^*(z)$, of J generated by z and the unit element is an associative JBW*-algebra isometrically isomorphic to a commutative von Neumann algebra (cf. [23, Lemma 4.1.11]). It is known that $\mathcal{W}^*(z)$ coincides with the abelian von Neumann subalgebra of M generated by the element z and the unit element.

Let $\Delta : H(M, \beta) \to H(M, \beta)$ be a (not necessarily linear nor continuous) 2-local triple derivation. By (4.1) we deduce that $\Delta(\mathbf{1})^* = -\Delta(\mathbf{1})$ and $M_{\Delta(\mathbf{1})} = \delta \left(\frac{1}{2}\Delta(\mathbf{1}), \mathbf{1} \right)$ is a triple derivation. Replacing Δ with $\Delta - \delta \left(\frac{1}{2}\Delta(\mathbf{1}), \mathbf{1} \right)$ we can assume that our 2-local triple derivation satisfies $\Delta(\mathbf{1}) = 0$. Having in mind the description provided by the above Proposition 4.1, the arguments given in [32, Lemmas 2.2, 2.3, and 2.6] can be literally adapted to prove the following:

Lemma 4.2. *Let M be a continuous von Neumann algebra and let $\beta : M \to M$ be a $\mathbb{C}$-linear $*$-involution. Suppose that $\Delta : H(M, \beta) \to H(M, \beta)$ is a (not necessarily linear nor continuous) 2-local triple derivation. Then the following statements hold:*

(a) *If $\Delta(\mathbf{1}) = 0$, then $\Delta(x) = \Delta(x)^*$ for all $x \in H(M, \beta)_{sa}$;*

(b) *If $\Delta(\mathbf{1}) = 0$, then for every $x, y \in H(M, \beta)_{sa}$ there exists a skew-hermitian element $a_{x,y} \in M$ with $\beta(a_{x,y}) = -a_{x,y}$ such that $\Delta(x) = [a_{x,y}, x]$, and $\Delta(y) = [a_{x,y}, y]$;*

(c) *Let $z \in H(M, \beta)$ be a self-adjoint element and let $\mathcal{W}^*(z) = \{z\}''$ be the abelian von Neumann subalgebra of M generated by the element z and the unit element. Then there exist skew-hermitian elements $a_z, b_z \in M$, depending on z, such that*

$$\Delta(x) = [a_z, x] + b_z \circ x = a_z x - x a_z + \frac{1}{2}(b_z x + x b_z)$$

for all $x \in \mathcal{W}^*(z) \subseteq H(M, \beta)$. In particular, Δ is linear and continuous on $\mathcal{W}^*(z)$.

$\square$

The results in Lemma 4.2 will be now applied to obtain a Jordan version of [32, Proposition 2.7]. Given a JBW*-algebra J whose lattice of projections is denoted by $\mathcal{P}(J)$, and a Banach space X, a *finitely additive X-valued measure* on $\mathcal{P}(J)$ is defined in the same way as in the case of a von Neumann algebra, namely, a mapping $\mu : \mathcal{P}(J) \to X$ satisfying

$$\mu \left(\sum_{i=1}^{n} p_i \right) = \sum_{i=1}^{n} \mu(p_i),$$

for every family $p_1, \ldots, p_n$ of mutually orthogonal projections in J.

Let $(p_i)_{i \in I}$ be a family of mutually orthogonal projections in a JBW*-algebra J. The series $\sum_{i \in I} p_i$ is summable with respect to the strong* topology of J, and we further know that the limit $p = \text{strong}^*\text{-}\sum_{i \in I} p_i$ is another projection in J (cf. [23, remark 4.2.9]). In particular, $\sum_{i \in I} p_i$ is summable with respect to the weak* topology of J and $\text{strong}^*\text{-}\sum_{i \in I} p_i = \text{weak}^*\text{-}\sum_{i \in I} p_i$.

Let J_1 and J_2 be JBW*-algebras, and let τ denote the norm, the weak* or the strong* topology of J_1. As in the case of von Neumann algebras, a mapping $\mu : J_1 \to J_2$ is said to be τ-*completely additive* (respectively, *countably or sequentially τ-additive*) when

(4.3)
$$\mu \left(\sum_{i \in I} p_i \right) = \tau\text{-}\sum_{i \in I} \mu(p_i)$$

for every family (respectively, sequence) $\{p_i\}_{i \in I}$ of mutually orthogonal projections in J_1.

We can easily obtain now a Jordan version of [32, Proposition 2.7].

Proposition 4.3. *Let M be a continuous von Neumann algebra and let $\beta : M \to M$ be a $\mathbb{C}$-linear *-involution. Let $\Delta : H(M, \beta) \to H(M, \beta)$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then the following statements hold:*

(a) *The restriction $\Delta|_{\mathcal{P}(J)}$ is sequentially strong*-additive, and consequently sequentially weak*-additive;*

(b) *$\Delta|_{\mathcal{P}(J)}$ is weak*-completely additive, i.e.,*

(4.4)
$$\Delta \left(\text{weak}^*\text{-}\sum_{i \in I} p_i \right) = \text{weak}^*\text{-}\sum_{i \in I} \Delta(p_i)$$

for every family $(p_i)_{i \in I}$ of mutually orthogonal projections in J.

Proof. (a) Let $(p_n)_{n \in \mathbb{N}}$ be a sequence of mutually orthogonal projections in $H(M, \beta)$. Let us consider the element $z = \sum_{n \in \mathbb{N}} \frac{1}{n} p_n$. Let $\mathcal{W}^*(z)$ be the JBW*-subalgebra of $H(M, \beta)$

generated by z. By Lemma 4.2(c), there exist skew-hermitian elements $a_z, b_z \in M$ with $\beta(a_z) = -a_z$ and $\beta(b_z) = b_z$, satisfying

$$T(x) = [a_z, x] + b_z \circ x,$$

for all $x \in \mathcal{W}^*(z)$.

The elements $\sum_{n=1}^{\infty} p_n$, and p_m belong to $\mathcal{W}^*(z)$, for all $m \in \mathbb{N}$. The reader should be warned that a_z might not belong to $H(M, \beta)$. In any case, the product of M is jointly strong* continuous on bounded sets, and by [8, Corollary] $S^*(M, M_*)|_{H(M,\beta)} \equiv S^*(H(M, \beta), H(M, \beta)_*)$. Therefore,

$$\Delta\left(S^*(M, M_*)\text{-}\sum_{n=1}^{\infty} p_n\right) = \left[a_z, S^*(M, M_*)\text{-}\sum_{n=1}^{\infty} p_n\right] + b_z \circ S^*(M, M_*)\text{-}\left(\sum_{n=1}^{\infty} p_n\right)$$

$$= S^*(M, M_*)\text{-}\sum_{n=1}^{\infty}[a_z, p_n] + S^*(M, M_*)\text{-}\sum_{n=1}^{\infty} b_z \circ p_n = S^*(M, M_*)\text{-}\sum_{n=1}^{\infty}\Delta(p_n),$$

i.e. $\Delta|_{\mathcal{P}(M)}$ is a countably or sequentially strong* additive mapping.

(b) As we have commented above, the strong*-topology of the JBW*-algebra $H(M, \beta)$ coincides with the restriction to $H(M, \beta)$ of the strong*-topology of M. When in the proof of [32, Proposition 2.7](b), we replace [32, Lemmas 2.2 and 2.3] with Lemma 4.2 (and having in mind the conclusion of Proposition 4.1), the arguments remain valid to obtain the desired statement here. $\square$

Let $\Delta : H(M, \beta) \to H(M, \beta)$ be a (not necessarily linear nor continuous) 2-local triple derivation, where M is a continuous von Neumann algebra and $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution. For each normal state $\phi \in H(M, \beta)_*$ (or $\phi \in M_*$), Proposition 4.3 implies that the mapping $\phi \circ \Delta|_{\mathcal{P}(H(M,\beta))} : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ is a completely additive measure. We conclude from Theorem 3.1, and from the arbitrariness of ϕ together with the uniform boundedness principle, that $\Delta|_{\mathcal{P}(H(M,\beta))} : \mathcal{P}(H(M, \beta)) \to \mathbb{C}$ is a bounded weak*-completely additive measure. An appropriate Jordan version of the Bunce-Wright-Mackey-Gleason theorem (see Theorem 3.2) implies the existence of a bounded linear operator $G : H(M, \beta) \to H(M, \beta)$ satisfying that $G(p) = \Delta(p)$ for every $p \in \mathcal{P}(H(M, \beta))$.

Let us pick a self-adjoint element z in $H(M, \beta)$. By Lemma 4.2(c), there exist skew-hermitian elements $a_z, b_z \in M$, with $\beta(a_z) = -a_z$ and $\beta(b_z) = b_z$, such that $\Delta(x) = [a_z, x] + b_z \circ x$, for every $x \in \mathcal{W}^*(z)$, the JBW*-subalgebra of $H(M, \beta)$ generated by z. Since $G|_{\mathcal{W}^*(z)}$ and $\Delta|_{\mathcal{W}^*(z)}$ are bounded linear operators from $\mathcal{W}^*(z)$ to M, which coincide on the set of projections of $\mathcal{W}^*(z)$, and every self-adjoint element in $\mathcal{W}^*(z)$ can be approximated in norm by finite linear combinations of mutually orthogonal projections in $\mathcal{W}^*(z)$, we conclude that $\Delta(x) = G(x)$ for every $x \in \mathcal{W}^*(z)$, and hence

$$\Delta(z) = G(z), \text{ for every } z \in H(M, \beta)_{sa},$$

in particular, Δ is additive on $H(M, \beta)_{sa}$. This proves the following Proposition.

Proposition 4.4. *Let* $\Delta : H(M, \beta) \to H(M, \beta)$ *be a* (*not necessarily linear nor continuous*) *2-local triple derivation, where M is a continuous von Neumann algebra and $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution. Then the restriction $\Delta|_{H(M,\beta)_{sa}}$ is additive.* □

Lemma 4.5. *Let* $\Delta : H(M, \beta) \to H(M, \beta)$ *be a* (*not necessarily linear nor continuous*) *2-local triple derivation, where M is a continuous von Neumann algebra and $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution. Suppose $\Delta(\mathbf{1}) = 0$. Then there exists a skew-hermitian element $a \in M$ such that $\beta(a) = -a$, and $\Delta(x) = [a, x]$, for all $x \in H(M, \beta)_{sa}$.*

Proof. Let $x \in M_{sa}$. By Lemma 4.2(c) there exist a skew-hermitian element $a_{x,x^2} \in M$ such that $\beta(a_{x,x^2}) = -a_{x,x^2}$, and $\Delta(x) = [a_{x,x^2}, x]$, $\Delta(x^2) = [a_{x,x^2}, x^2]$.

Thus,

$$(4.5) \qquad \Delta(x^2) = [a_{x,x^2}, x^2] = [a_{x,x^2}, x]x + x[a_{x,x^2}, x] = 2\Delta(x) \circ x.$$

By Proposition 4.4 and Lemma 4.2(a), $\Delta|_{H(M,\beta)_{sa}} : H(M, \beta)_{sa} \to H(M, \beta)_{sa}$ is a real linear mapping. Now, we consider the linear extension $\hat{\Delta}$ of $\Delta|_{H(M,\beta)_{sa}}$ to $H(M, \beta)$ defined by

$$\hat{\Delta}(x_1 + ix_2) = T(x_1) + iT(x_2), \ x_1, x_2 \in H(M, \beta)_{sa}.$$

Taking into account the homogeneity of Δ, Proposition 4.4 and the identity (4.5), we deduce that $\hat{\Delta}$ is a Jordan *-derivation (and hence, a triple derivation) on $H(M, \beta)$. Proposition 4.1 implies the existence of a skew-symmetric element $a \in M$ such that $\beta(a) = -a$ and $\hat{\Delta}(x) = [a, x]$ for all $x \in H(M, \beta)$. In particular, $\Delta(x) = [a, x]$ for all $x \in H(M, \beta)_{sa}$, which completes the proof. □

We now prove the main result of this section.

Theorem 4.6. *Let* $\Delta : H(M, \beta) \to H(M, \beta)$ *be a* (*not necessarily linear nor continuous*) *2-local triple derivation, where M is a continuous von Neumann algebra and $\beta : M \to M$ is a $\mathbb{C}$-linear *-involution. Then Δ is a linear and continuous triple derivation.*

Proof. From (4.1) we know that $\Delta(\mathbf{1})^* = -\Delta(\mathbf{1})$, and $M_{\Delta(\mathbf{1})} = \delta\left(\frac{1}{2}\Delta(\mathbf{1}), \mathbf{1}\right)$ is a triple derivation. Replacing Δ with $\Delta - \delta\left(\frac{1}{2}\Delta(\mathbf{1}), \mathbf{1}\right)$ we can assume that $\Delta(\mathbf{1}) = 0$. By Lemma 4.5 there exists a skew-hermitian element $a \in M$ such that $\beta(a) = -a$, and $\Delta(x) = [a, x]$, for all $x \in H(M, \beta)_{sa}$. Observe that the mapping $\widehat{\Delta} = \Delta - [a, .]$ is a 2-local triple derivation on $H(M, \beta)_{sa}$ satisfying $\widehat{\Delta}|_{H(M,\beta)_{sa}} \equiv 0$.

We shall finally prove that $\widehat{\Delta} = 0$. This result follows from a direct adaptation of the arguments in [32, Lemma 2.16], we include here a sketch of the proof for completeness reasons.

Let $x \in H(M, \beta)$ be an arbitrary element and let $x = x_1 + ix_2$, where $x_1, x_2 \in H(M, \beta)_{sa}$. Since $\widehat{\Delta}$ is homogeneous, by passing to the element $(1 + \|x_2\|)^{-1}x$ if necessary, we can suppose that $\|x_2\| < 1$. In this case the element $y = \mathbf{1} + x_2$ is positive and invertible. Take skew-hermitian elements $a_{x,y}, b_{x,y} \in M$ such that $\beta(a_{x,y}) = -a_{x,y}$, $\beta(b_{x,y}) = b_{x,y}$, and

$$\widehat{\Delta}(x) = [a_{x,y}, x] + b_{x,y} \circ x, \ \text{and} \ \widehat{\Delta}(y) = [a_{x,y}, y] + b_{x,y} \circ y.$$

Since $\widehat{\Delta}(y) = 0$, we get $[a_{x,y}, y] + b_{x,y} \circ y = 0$. Lemma 2.4 in [32] implies that $[a_{x,y}, y] = 0$ and $ib_{x,y} \circ y = 0$. Having in mind that y is positive and invertible, and that $ib_{x,y}$ is hermitian, [32, Lemma 2.5] proves that $b_{x,y} = 0$.

The condition $0 = [a_{x,y}, y] = [a_{x,y}, \mathbf{1} + x_2] = [a_{x,y}, x_2]$, implies

$$\widehat{\Delta}(x) = [a_{x,y}, x] + b_{x,y} \circ x = [a_{x,y}, x_1 + ix_2] = [a_{x,y}, x_1],$$

which shows that

$$\widehat{\Delta}(x)^* = [a_{x,y}, x_1]^* = [x_1, a_{x,y}^*] = [x_1, -a_{x,y}] = [a_{x,y}, x_1] = \widehat{\Delta}(x).$$

The arbitrariness of $x \in H(M, \beta)$ implies that $\widehat{\Delta}(x) = 0$, as desired. $\square$

Since every element in a closed ideal of a JB*-triple can be written as a cube of an element in that ideal, it is clear that a triple derivation leaves closed ideals invariant. Hence the same is true for 2-local triple derivations. Thus, by invoking the structure theorem of continuous JBW*-triples stated at the beginning of this section, and combining Theorems 2.3 and 4.6, we obtain the second main result of this paper.

Theorem 4.7. *Let* $\Delta : A \rightarrow A$ *be a* (*not necessarily linear nor continuous*) *2-local triple derivation, where* A *is a continuous JBW*-triple. Then* Δ *is a linear and continuous triple derivation.*

Problem 4.8. *Does Theorem 3.1 remain valid when* $H(M, \beta)$ *is replaced by an arbitrary JBW*-algebra without summands of type* I_n?

Problem 4.9. *Is Theorem 4.7 valid for*

(a): *JBW*-triples of type I?* (See Corollary 2.12)
(b): *reversible JBW*-algebras?*
(c): *2-local triple derivations with values in a Jordan triple module?*
(d): *2-local triple derivations on various algebras of measurable operators?*
(e): *real JBW*-triples?*
(f): *complex and real JB*-triples?*

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