Homework 2—Math199C-due April 30, 2021

1. Let f_1, f_2, f_3 be a basis for the zygotic algebra:

$$f_1f_1 = f_1, \quad f_1f_2 = f_2f_1 = \frac{1}{2}(f_1 + f_2), \quad f_1f_3 = f_3f_1 = f_2$$

$$f_2f_2 = \frac{1}{2}f_1 + \frac{1}{4}f_3 + \frac{1}{2}f_2, \quad f_2f_3 = f_3f_2 = \frac{1}{2}(f_3 + f_2), \quad f_3f_3 = f_3$$

Define, for $i = 1, 2, 3$, $e_i = \alpha_{1i}f_1 + \alpha_{2i}f_2 + \alpha_{3i}f_3$, and assume that e_1, e_2, e_3 is a natural basis, that is, assume that the zygotic algebra is an evolution algebra.

(a) If $e_ie_j = Af_1 + Bf_2 + Cf_3$, then calculate A, B and C for $i \neq j$.

(b) Let M be the matrix with respect to f_1, f_2, f_3 of the linear transformation T defined by $Tf_i = e_i$. Why is M a non-singular matrix?

2. Let A be any algebra

(a) Show that A is power associative, that is, $(a^ia^j)a^k = a^i(a^ja^k)$ for all $i, j, k \geq 1$ and all $a \in A$, if and only if $a^{k+\ell} = a^ka^\ell$ for all $k, \ell \geq 1$ and all $a \in A$.

(b) (Revised) For an evolution algebra A, define $A^2 = \{\sum_{i=1}^n a_ib_i : a_i, b_i \in A\}$

Prove

(a) A^2 is the linear span of e_i^2, where $e_1, e_2, \ldots, e_m$ is any natural basis.

(b) $A^2 = A$ if and only if $\det M_B(A) \neq 0$

(c) If $\det M_B(A) \neq 0$ for some natural basis B, then $\det M'_B(A) \neq 0$ for any other natural basis

(d) If A is power associative and $\det M_B(A) \neq 0$ for some natural basis B, then $M_B(A)$ is diagonal

3. Let A be an evolution algebra with natural basis e_1, e_2, e_3, with $e_1^2 = e_1 + e_2$, $e_2^2 = -e_1 - e_2$, $e_3^2 = -e_2 + e_3$. Let A' be the linear span of the two vectors $u_1 = e_1 + e_2$ and $u_2 = e_1 + e_3$. Show that A' is a two dimensional subalgebra of A but it is not an evolution algebra, that is, it does not have a natural basis. Hence it is not an evolution subalgebra.

4. Let A be an evolution algebra with natural basis e_1, e_2, e_3 with $e_1^2 = e_2 + e_3$, $e_2^2 = e_1 + e_2$, $e_3^2 = -e_1 - e_2$. Let I be the linear span of the two vectors $u_1 = e_1^2 = e_2 + e_3$ and $u_2 = e_2^2 = e_1 + e_2$. Show that I is a two dimensional ideal of A, but it is not an evolution algebra, that is, it does not have a natural basis. Hence it is not an evolution ideal.

5. Let A be an evolution algebra with natural basis e_1, e_2, e_3, with $e_1^2 = e_3$, $e_2^2 = e_1 + e_2$, $e_3^2 = e_3$. Let I be the linear span of the two vectors $e_1 + e_2$ and e_3. Show that I is an evolution ideal but that it does not have the extension property, that is, no natural basis of I can be extended to a natural basis of A. Explicitly, if $u = \alpha(e_1+e_2)+\beta e_3$ and $v = \gamma(e_1 + e_2) + \delta e_3$ is a natural basis for I, then $\{u, v, w\}$, where $w = \lambda e_1 + \mu e_2 + \rho e_3$ cannot be a natural basis for A.