# NON-ASSOCIATIVE ALGEBRAS, YANG-BAXTER EQUATIONS AND QUANTUM COMPUTERS 

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#### Abstract

Non-associtive algebras is a research direction gaining much attention these days. New developments show that associative algebras and some not-associative structures can be unified at the level of Yang-Baxter structures. In this paper, we present a unification for associative algebras, Jordan algebras and Lie algebras.

The (quantum) Yang-Baxter equation and related structures are interesting topics, because they have applications in many areas of mathematics, physics and computer science. Several new interpretations and results are presented below.


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## 1 INTRODUCTION

The current paper emerged after our participation at the International Conference "Mathematics Days in Sofia", July 7-10, 2014, Sofia, Bulgaria. It
presents our scientic contributions for that conference, as well as new results and directions of research.

Prof. Radu Iordanescu contributed with a paper on Jordan algebras ([5]) to the library of the Institute of Mathematics and Informatics from Sofia. Non-associtive algebras (which include Jordan algebras, Lie algebras and associative algebras) are currently a research direction in fashion (see [18, and the references therein). New developments show that associative algebras and Lie algebras can be unified at the level of Yang-Baxter structures (see [4]). In this paper, we present a unification for associative algebras, Jordan algebras and Lie algebras.

The other authors of this paper presented a poster on combinatorial logical circuits and solutions to the set-theoretical Yang-Baxter equation from Boolean algebras (following the works [15, 13]). Dr. Violeta Ivanova (see [6]) was interested in the applications of these problems in computer science. The Yang-Baxter equation can be interpretated in terms of combinatorial logical structures, and, in logic, it represents some kind of compatibility condition, when working with many logical sentences in the same time. This equation is also related to the theory of universal quantum gates and to the quantum computers (see, for example, [1]).

Florin F. Nichita gave a talk on the Yang-Baxter equation presenting resuts from [9, 14]. In his talk he referred to several papers of Prof. Vladimir Dobrev and to the work of Prof. Tatiana Gateva-Ivanova (one of her questions, arising at that time, will be partially answered in this paper). As an observation, our Yang-Baxter operator $R_{\alpha, \beta, \alpha}^{A}$ is related to a universal quantum gate.

The organization of our paper is the following. The next section introduces the mathematical terminology needed in this paper, and it presents results about the Yang-Baxter equation. Section 3 deals with the unification of Jordan algebras, Lie algebras and associative algebras. Section 4 is a conlusions section and an update on quantum computers.

## 2 THE QYBE

An introduction to the (quantum) Yang-Baxter equation (QYBE) could be found in the paper [9]. Several special sessions on it followed at the openacces journal AXIOMS, explaining its role in areas of mathematics, physics and computer science.

We will work over the field $k$, and the tensor products will be defined over $k$. For $V$ a $k$-space, we denote by $\tau: V \otimes V \rightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w)=w \otimes v$, and by $I: V \rightarrow V$ the identity map of the space
V. For $R: V \otimes V \rightarrow V \otimes V$ a $k$-linear map, let $R^{12}=R \otimes I, R^{23}=I \otimes R, R^{13}=$ $(I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 2.1. A Yang-Baxter operator is defined as an invertible $k$-linear map $R: V \otimes V \rightarrow V \otimes V$ which satisfies the equation:

$$
\begin{equation*}
R^{12} \circ R^{23} \circ R^{12}=R^{23} \circ R^{12} \circ R^{23} \tag{2.1}
\end{equation*}
$$

$R$ satisfies (2.1) if and only if $R \circ \tau($ respectively $\tau \circ R)$ satisfies the QYBE:

$$
\begin{equation*}
R^{12} \circ R^{13} \circ R^{23}=R^{23} \circ R^{13} \circ R^{12} \tag{2.2}
\end{equation*}
$$

Remark 2.2. For $A$ be a (unitary) associative $k$-algebra, and $\alpha, \beta, \gamma \in k$, 2 ] defined the $k$-linear map

$$
R_{\alpha, \beta, \gamma}^{A}: A \otimes A \rightarrow A \otimes A, \quad R_{\alpha, \beta, \gamma}^{A}(a \otimes b)=\alpha a b \otimes 1+\beta 1 \otimes a b-\gamma a \otimes b
$$

which is a Yang-Baxter operator if and only if one of the following holds:
(i) $\alpha=\gamma \neq 0, \quad \beta \neq 0$;
(ii) $\beta=\gamma \neq 0, \quad \alpha \neq 0$;
(iii) $\alpha=\beta=0, \quad \gamma \neq 0$.

Remark 2.3. Using Turaev's general scheme to derive an invariant of oriented links from a Yang-Baxter operator (see [17]), $R_{\alpha, \beta, \gamma}^{A}$ leads to the Alexander polynomial of knots (see [8]).

Remark 2.4. In dimension two, $R_{\alpha, \beta, \alpha}^{A} \circ \tau$, can be expressed as:

$$
\left(\begin{array}{cccc}
1 & 0 & 0 & 0  \tag{2.3}\\
0 & 1 & 0 & 0 \\
0 & 1-q & q & 0 \\
\eta & 0 & 0 & -q
\end{array}\right)
$$

where $\eta \in\{0,1\}$, and $q \in k-\{0\}$. For $\eta=0$ and $q=1$, it can be related to the universal quantum gate CNOT.

Definition 2.5. A Lie superalgebra is a (nonassociative) $Z_{2}$-graded algebra, or superalgebra, over a field $k$ with the Lie superbracket, satisfying the two conditions:

$$
\begin{gathered}
{[x, y]=-(-1)^{|x||y|}[y, x} \\
(-1)^{|z||x|}[x,[y, z]]+(-1)^{|x||y|}[y,[z, x]]+(-1)^{|y||z|}[z,[x, y]]=0
\end{gathered}
$$

where $x, y$ and $z$ are pure in the $Z_{2}$-grading. Here, $|x|$ denotes the degree of $x$ (either 0 or 1 ). The degree of $[x, y]$ is the sum of degree of $x$ and $y$ modulo 2 .

Remark 2.6. For $(L,[]$,$) a Lie superalgebra over k, z \in Z(L)=\{z \in L$ : $[z, x]=0 \quad \forall x \in L\},|z|=0$ and $\alpha \in k$, 7] (and [16]) defined

$$
\phi_{\alpha}^{L}: L \otimes L \quad \longrightarrow \quad L \otimes L, \quad x \otimes y \mapsto \alpha[x, y] \otimes z+(-1)^{|x||y|} y \otimes x,
$$

which is a YB operator.

Remark 2.7. The Remarks 2.2 and 2.6 lead to some kind of unification of associative algebras and structures that are not associative at the level of Yang-Baxter structures (see [4, 10]. For example, the first isomorfism theorem for groups (algebras) and the first isomorfism theorem for Lie algebras, can be unified as an isomorphism theorem for Yang-Baxter structures (see [11]). The fact that Therem 7.2.3 from [11] (The fundamental isomorphism theorem for YB structures) unifies not only associative algebras and coalgebras, but also Lie algebras is a new result.

Remark 2.8. Following a question of Prof. Tatiana Gateva-Ivanova, we can construct an algebra structure associated to the operator
$R=R_{1,1,1}^{A}: A \otimes A \rightarrow A \otimes A, \quad R(a \otimes b)=a b \otimes 1+1 \otimes a b-a \otimes b$, if we use Theorem 3.1 (i) from [14].

For $a, b \in T^{1}(A)=A$, we have: $\mu(a \otimes b)=a b \otimes 1+1 \otimes a b-a \otimes b \in T^{2}(A)$.
For $a \otimes a^{\prime} \in T^{2}(A)=A \otimes A$ and $b \in T^{1}(A)=A$, we have:
$\mu\left(\left(a \otimes a^{\prime}\right) \otimes b\right)=R^{12} \circ R^{23}\left(a \otimes a^{\prime} \otimes b\right)=a a^{\prime} b \otimes 1 \otimes 1+1 \otimes a a^{\prime} b \otimes 1-a \otimes a^{\prime} b \otimes 1+$ $1 \otimes a \otimes a^{\prime} b-a a^{\prime} \otimes 1 \otimes b-1 \otimes a a^{\prime} \otimes b+a \otimes a^{\prime} \otimes b \in T^{3}(A)$.

For $a \in T^{1}(A)=A$ and $b \otimes b^{\prime} \in T^{2}(A)$, we have:
$\mu\left(a \otimes\left(b \otimes b^{\prime}\right)=R^{23} \circ R^{12}\left(a \otimes b \otimes b^{\prime}\right) \in T^{3}(A)\right.$.
In the same manner, we compute other products.

## 3 NON-ASSOCIATIVE ALGEBRAS

Jordan algebras emerged in the early thirties, and their applications are in differential geometry, ring geometries, physics, quantum groups, analysis, biology, etc (see [5, 3]). A poster presented at the 11-th International Workshop on Differential Geometry and its Applications, in September 2013, at the Petroleum-Gas University from Ploiesti led to the paper [12]. That paper presents new results on Jordan algebras, Jordan coalgebras, and how they are related to the QYBE.

One of the main results of [12] is the following theorem, which explaines when the Jordan identity implies associativity. It is an intrinsic result.

Theorem 3.1. Let $V$ be a vector space spanned by a and $b$, which are linearly independent. Let $\theta: V \otimes V \rightarrow V, \quad \theta(x \otimes y)=x y$, be a linear map which is a commutative operation with the property

$$
\begin{equation*}
a^{2}=b, \quad b^{2}=a \tag{3.4}
\end{equation*}
$$

Then: $(V, \theta)$ is a Jordan algebra $\Longleftrightarrow(V, \theta)$ is a non-unital commutative (associative) algebra.

The next remark finds a relationship between Jordan algebras, Lie algebras and associative algebras. In this case, we have an extrinsic result about non-associative structures.

Remark 3.2. For the vector space V , let $\eta: V \otimes V \rightarrow V, \quad \eta(x \otimes y)=x y$, be a linear map which satisfies:

$$
\begin{align*}
(a b) c+(b c) a+(c a) b & =a(b c)+b(c a)+c(a b)  \tag{3.5}\\
\left(a^{2} b\right) a & =a^{2}(b a) \tag{3.6}
\end{align*}
$$

Then, $(V, \eta)$ is a structure which unifies (non-unital) associative algebras, Lie algebras and Jordan algebras.

Indeed, the associativity and the Lie identity are unified by relation (3.5). Also, the commutativity of a Jordan algebra implies (3.5). But, the Jordan identity, (3.6), which appears in the definition of Jordan algebras, holds true in any associative algebra and Lie algebra.

This unifying approach is different from the unification proposed in Re$\operatorname{mark} 2.7$

## 4 CONCLUSIONS AND QUANTUM COMPUTERS

The first quantum computer (which uses principles of quantum mechanics) was sold to the aerospace and security of defense company Lockheed Martin. The manufacturing company, D-Wave, founded in 1999 and called "a company of quantum computing" promised to perform professional services for the computer maintenance as well.

The Yang-Baxter equation has applications in quantum computing, and it can be viewed as an unifying structure for non-associative structures.

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