Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces

Michael K. Kinyon, Alan Weinstein

American Journal of Mathematics, Volume 123, Number 3, June 2001, pp. 525-550 (Article)

Published by Johns Hopkins University Press
DOI: *https://doi.org/10.1353/ajm.2001.0017*

LEIBNIZ ALGEBRAS, COURANT ALGEBROIDS, AND MULTIPLICATIONS ON REDUCTIVE HOMOGENEOUS SPACES

By Michael K. Kinyon and Alan Weinstein

Abstract. We show that the skew-symmetrized product on every Leibniz algebra $\mathcal{E}$ can be realized on a reductive complement to a subalgebra in a Lie algebra. As a consequence, we construct a nonassociative multiplication on $\mathcal{E}$ which, when $\mathcal{E}$ is a Lie algebra, is derived from the integrated adjoint representation. We apply this construction to realize the bracket operations on the sections of Courant algebroids and on the "omni-Lie algebras" recently introduced by the second author.

1. Introduction. Skew-symmetric bilinear operations which satisfy weakened versions of the Jacobi identity arise from a number of constructions in algebra and differential geometry. The purpose of this paper is to show how certain of these operations, in particular the Courant brackets on the doubles of Lie bialgebroids, can be realized in a natural way on the tangent spaces of reductive homogeneous spaces. We use our construction to take steps toward finding group-like objects which "integrate" these not-quite-Lie algebras.

The main ideas behind our construction come from work of K. Nomizu, K. Yamaguti, and M. Kikkawa. Nomizu [18] showed that affine connections with parallel torsion and curvature are locally equivalent to invariant connections on reductive homogeneous spaces, and that each such space has a canonical connection for which parallel translation along geodesics agrees with the natural action of the group. Yamaguti [22] characterized the torsion and curvature tensors of Nomizu's canonical connection as pairs of algebraic operations, one bilinear and the other trilinear, satisfying axioms defining what he called a "general Lie triple system," and what Kikkawa later called a "Lie triple algebra." In this paper, we will call these objects *Lie-Yamaguti algebras*. When the trilinear operation is zero, the bilinear operation is a Lie algebra operation, and the homogeneous space is locally a Lie group on which the connection is the one which makes left-invariant vector fields parallel. Finally, Kikkawa [7] showed how to "integrate" Lie-Yamaguti algebras to nonassociative multiplications on reductive homogeneous spaces, and he characterized these multiplications axiomatically. Unfortunately, Kikkawa's construction when applied in our setting does not quite reproduce the multiplication on a Lie group when the curvature is zero; rather it

Manuscript received June 13, 2000.

Research supported in part by NSF Grants DMS-96-25512 and DMS-99-71505 and the Miller Institute for Basic Research.

American Journal of Mathematics 123 (2001), 525–550.

gives the loop operation $(x, y) \mapsto x + \exp(\mathrm{ad}\, x/2)y$ on the Lie algebra itself. This limitation extends to our own work, so that we do not finally succeed in finding the group-like object which we seek.

The starting point of our investigations was a skew-symmetric but non-Lie bracket introduced by T. Courant [2]. It is defined on the direct sum $\mathcal{E} = \mathcal{X}(P) \oplus \Omega^1(P)$ of the smooth vector fields and 1-forms on a differentiable manifold P by

$$\llbracket(\xi_1, \theta_1), (\xi_2, \theta_2)\rrbracket = \left([\xi_1, \xi_2], \mathcal{L}_{\xi_1}\theta_2 - \mathcal{L}_{\xi_2}\theta_1 - \frac{1}{2}d(i_{\xi_1}\theta_2 - i_{\xi_2}\theta_1)\right), \tag{1.1}$$

where $\mathcal{L}_\xi$ and i_ξ are the operations of Lie derivative and interior product by the vector field ξ. The term $-\frac{1}{2}d(i_{\xi_1}\theta_2 - i_{\xi_2}\theta_1)$ will be especially important to our discussion. It distinguishes the bracket from that on the semidirect product of the vector fields acting on the 1-forms by Lie derivation, and it spoils the Jacobi identity. On the other hand, with this term (this was Courant's original motivation for introducing this bracket), the graph of every Poisson structure $\Gamma\pi\colon\ \Omega^1(P) \to \mathcal{X}(P)$ and every closed 2-form $\Gamma\omega\colon\ \mathcal{X}(P) \to \Omega^1(P)$ is a subalgebra of $\mathcal{E}$. Thus, although $\mathcal{E}$ is not a Lie algebra, it contains many Lie algebras. Is there a "group-like" object associated to $\mathcal{E}$ which contains the (infinite-dimensional, possibly local) Lie groups associated to the Lie subalgebras of $\mathcal{E}$?

Courant's brackets live on infinite dimensional spaces, but we may obtain a simplified, finite-dimensional bracket by "linearization." Namely, we let P be the dual V^* of a vector space V, and we consider the first derivatives of vector fields vanishing at 0, along with the values of 1-forms at 0. The resulting bracket on the finite dimensional space $\mathcal{E} = \mathfrak{gl}(V) \times V$ is given by

$$\llbracket(X, u), (Y, v)\rrbracket = \left([X, Y], \frac{1}{2}(Xv - Yu)\right) \tag{1.2}$$

for $X, Y \in \mathfrak{gl}(V)$ and $u, v \in V$. The factor of $1/2$ is now the "spoiler" of the Jacobi identity, but it yields the following nice property of the bracket. If $\mathrm{ad}_B\colon\ V \to \mathfrak{gl}(V)$ is the adjoint representation of any skew-symmetric operation $B\colon\ V \times V \to V$, then (V, B) is a Lie algebra if and only if the graph of ad_B is a subalgebra of $(\mathfrak{gl}(V) \times V, \llbracket\cdot,\cdot\rrbracket)$, which is then isomorphic to (V, B) under the projection onto the second factor [21]. The problem at the end of the last paragraph now becomes: is there a group-like object associated to $(\mathfrak{gl}(V) \times V, \llbracket\cdot,\cdot\rrbracket)$ which contains Lie groups associated to all the Lie algebra structures on V?

In this paper, we give a partial solution to these problems. We show that each of the algebras denoted by $\mathcal{E}$ above can be embedded in a Lie algebra $\mathcal{D}$ of roughly twice the size. There is a direct sum decomposition $\mathcal{D} = \overline{\mathcal{E}} \oplus \mathcal{E}$ invariant under the adjoint action of $\overline{\mathcal{E}}$, a subalgebra. The bracket on $\mathcal{E}$ is obtained from the bracket in $\mathcal{D}$ by projection along $\overline{\mathcal{E}}$.

Denoting by G and H the groups associated to $\mathcal{D}$ and $\overline{\mathcal{E}}$ respectively, we may identify $\mathcal{E}$ with the tangent space at the basepoint of the reductive homogeneous space $S(\mathcal{E}) = G/H$. If $\mathcal{L} \subseteq \mathcal{E}$ is any Lie subalgebra, such as the graph of a Poisson structure or of an adjoint representation, then there is a reductive homogeneous space $S(\mathcal{L})$ which sits naturally inside $S(\mathcal{E})$. This is as close as we have come so far to solving our problem. It is not a complete solution since $S(\mathcal{L})$ is not a group. It does, however, carry a multiplication which "partially integrates" the Lie algebra structure on $\mathcal{L}$. The general procedure for constructing such multiplications is as follows.

Any reductive homogeneous space G/H corresponds to a reductive Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$; i.e., $\mathfrak{h}$ is a subalgebra with $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. (Note that this is "reductive" in the sense of Nomizu [18], not in the sense of having a completely reducible adjoint representation.) In general $\mathfrak{m}$ is not a subalgebra, but a neighborhood of the identity in $M = \exp(\mathfrak{m}) \subseteq G$ can still be identified with a neighborhood of the base point in G/H. Ignoring for simplicity of exposition the restriction to neighborhoods of the identity (which is in fact unnecessary in many cases), we may now take two elements x and y of G/H, multiply their representatives in M to get a result in G, and project to get a product xy in G/H. Unless M is a subgroup, this multiplication will not, in general, be associative, but it will satisfy the axioms which make G/H into a group-like object called a *homogeneous Lie loop* [7]. Each Lie-Yamaguti subalgebra $\mathfrak{l}$ of $\mathfrak{m}$ will then correspond to a subloop of G/H.

In the example $\mathcal{E} = \mathfrak{gl}(V) \times V$, the Lie algebra $\mathcal{D}$ may be taken to be simply the semidirect product of $\mathfrak{gl}(V)$ acting on the vector space $\mathfrak{gl}(V) \times V$ by the direct sum of the adjoint and standard representations. The subalgebra $\mathcal{E}$ is the first (i.e. the nonabelian) copy of $\mathfrak{gl}(V)$, and the reductive complement is the graph of the mapping $(X, u) \mapsto X/2$ from the last two factors of $\mathfrak{gl}(V) \times \mathfrak{gl}(V) \times V$ to the first one. The construction for the original Courant bracket is similar and is described in detail in §4 below.

The construction of the enveloping Lie algebra $\mathcal{D}$ was in fact worked out in a more general setting. It was observed in [11] that adding a symmetric term to the skew-symmetric Courant bracket "improved" some of its algebraic properties. Y. Kosmann-Schwarzbach and P. Xu, as well as P. Ševera (all unpublished), observed that this unskewsymmetrized operation, which we will denote by $\cdot$, satisfies the derivation identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + y \cdot (x \cdot z)$$

defining what Loday [12] has called a *Leibniz algebra*. The central result in our paper (Theorems 2.6 and 2.9) is that the skew-symmetrization of every Leibniz algebra structure can be extended in a natural way to a Lie-Yamaguti structure and hence can be realized as the projection of a Lie algebra bracket onto a reductive complement of a subalgebra. Thus, the Leibniz algebra is "integrated" to a homogeneous left loop. Whether this loop will enable us to lasso Loday's elusive "coquecigrue" [12] remains to be seen.

In the last section of the paper, we discuss further directions for research. One is to attempt to take into account the fact that the algebra $\mathcal{X}(P)\oplus\Omega^1(P)$ is also the space of sections of a vector bundle, and to try to build a corresponding structure into its enveloping Lie algebra. A second goal is to construct a group-like object for a Leibniz algebra which is actually a group in the case of a Lie algebra. We end the paper with an idea for constructing such an object as a quotient of a path space.

Acknowledgments. We would like to thank the many people from whom we have learned important things about Courant algebroids, path spaces, and nonassociative algebras, among them Anton Alekseev, Hans Duistermaat, Nora Hopkins, Johannes Huebschmann, Atsushi Inoue, Yvette Kosmann-Schwarzbach, Zhang-ju Liu, Kirill Mackenzie, Hala Pflugfelder, Jon Phillips, Dmitry Roytenberg, Arthur Sagle, Pavol Ševera, Jim Stasheff, and Ping Xu. The second author would also like to thank Setsuro Fujiie and Yoshitsugu Takei for their invitation to speak on "Omni-Lie algebras" and to publish a report in the proceedings of the RIMS Workshop on Microlocal Analysis of Schrödinger Operators. Although that report had little to do with the subject of the workshop, the stimulus to write a manuscript led to the posting of a preprint [21] on the arXiv server and the subsequent collaboration which has resulted in the present paper.

2. Leibniz algebras. All vector spaces, algebras, etc. in this section will be over a ground field $\mathbb{K}$ of characteristic 0. Most results extend in obvious ways to positive characteristic (not 2), or even to commutative rings with unit. By an *algebra* $(\mathcal{E},\cdot)$ we will mean a vector space $\mathcal{E}$ over $\mathbb{K}$ with a not necessarily associative bilinear operation $\cdot\colon\ \mathcal{E}\times\mathcal{E}\to\mathcal{E}$. For $x\in\mathcal{E}$, let $\lambda(x)\colon\ \mathcal{E}\to\mathcal{E}; y\mapsto x\cdot y$ denote the left multiplication map. Let $\mathrm{Der}(\mathcal{E})$ denote the Lie subalgebra of $\mathfrak{gl}(\mathcal{E})$ consisting of the derivations of $\mathcal{E}$. A linear map $\xi\in\mathfrak{gl}(\mathcal{E})$ is a derivation of $(\mathcal{E},\cdot)$ if and only if

$$[\xi,\lambda(x)]=\lambda(\xi x) \tag{2.1}$$

for all $x\in\mathcal{E}$. For the class of algebras of interest to us, the left multiplication maps have a stronger compatibility with the derivations.

Definition 2.1. An algebra $(\mathcal{E},\cdot)$ is called a *Leibniz algebra* if

$$x\cdot(y\cdot z)=(x\cdot y)\cdot z+y\cdot(x\cdot z) \tag{2.2}$$

for all $x,y,z\in L$.

Clearly, an algebra $(\mathcal{E},\cdot)$ is a Leibniz algebra if and only if $\lambda(\mathcal{E})\subseteq\mathrm{Der}(\mathcal{E})$, or equivalently, $\lambda\colon\ (\mathcal{E},\cdot)\to(\mathfrak{gl}(\mathcal{E}),[\cdot,\cdot])$ is a homomorphism. Thus we have a homomorphism $\lambda\colon\ (\mathcal{E},\cdot)\to(\mathrm{Der}(\mathcal{E}),[\cdot,\cdot])$ when $(\mathcal{E},\cdot)$ is a Leibniz algebra.

Leibniz algebras were introduced by Loday [12]. (For this reason, they have also been called "Loday algebras" [10].) A skew-symmetric Leibniz algebra structure is a Lie bracket; in this case, (2.2) is just the Jacobi identity. In particular, given a Leibniz algebra $(\mathcal{E}, \cdot)$, any subalgebra on which $\cdot$ is skew-symmetric is a Lie algebra, as is any skew-symmetric quotient. The skew-symmetrization of a Leibniz algebra $(\mathcal{E}, \cdot)$ is an interesting structure in its own right. We will denote the skew-symmetrized operation by

$$[\![x, y]\!] = \frac{1}{2}(x \cdot y - y \cdot x) \tag{2.3}$$

for $x, y \in \mathcal{E}$. In general, $(\mathcal{E}, [\![\cdot, \cdot]\!])$ is not a Lie algebra, i.e. $(\mathcal{E}, \cdot)$ is not Lie-admissible [16]. (In particular, Leibniz algebras are not the same as a related class of algebras arising in differential geometry which is known by such names as pre-Lie algebras [5], Vinberg algebras [17], or left-symmetric algebras.)

From (2.2), we have that $\lambda(x) \in \mathrm{Der}(\mathcal{E}, [\![\cdot, \cdot]\!])$ for all $x \in \mathcal{E}$, and that $\lambda\colon (\mathcal{E}, [\![\cdot, \cdot]\!]) \to \mathrm{Der}(\mathcal{E})$ is a homomorphism of anticommutative algebras.

Let

$$\mathcal{J} = \langle x \cdot x \mid x \in \mathcal{E} \rangle$$

be the two-sided ideal of $(\mathcal{E}, \cdot)$ generated by all squares. Then $\mathcal{J}$ contains all symmetric products $x \cdot y + y \cdot x$ for $x, y \in \mathcal{E}$. Since $\lambda(x \cdot x) = [\lambda(x), \lambda(x)]$ for all $x \in \mathcal{E}$, it follows that $\mathcal{J} \subseteq \ker(\lambda)$. Let $\mathcal{M} \subseteq \mathcal{E}$ be any ideal containing $\mathcal{J}$. Since $x \cdot y + \mathcal{M} = -y \cdot x + (x \cdot y + y \cdot x) + \mathcal{M} = -y \cdot x + \mathcal{M}$ for $x, y \in \mathcal{E}$, the Leibniz product in $\mathcal{E}$ descends to a Lie bracket $[\cdot, \cdot]$ in $\mathcal{E}/\mathcal{M}$. Conversely, if $\mathcal{M} \subseteq \mathcal{E}$ is an ideal such that the quotient $\mathcal{E}/\mathcal{M}$ is a Lie algebra, then for $x \in \mathcal{E}$, we have $x \cdot x + \mathcal{M} = \mathcal{M}$, and thus $\mathcal{J} \subseteq \mathcal{M}$. In particular, $\mathcal{J}$ itself is the smallest two-sided ideal of $\mathcal{E}$ such that $\mathcal{E}/\mathcal{J}$ is a Lie algebra.

We now introduce one of our principal examples, which can be viewed as a non-skew-symmetrized semidirect product of Lie algebras.

Example 2.2. Let $(\mathfrak{h}, [\cdot, \cdot])$ be a Lie algebra, and let V be an $\mathfrak{h}$-module with left action $\mathfrak{h} \times V \to V\colon (\xi, x) \mapsto \xi x$. The induced left action of $\mathfrak{h}$ on $\mathfrak{h} \times V$ is just the restricted adjoint action of $\mathfrak{h}$ on the semidirect product $\mathfrak{h} \ltimes V$:

$$\xi(\eta, y) := \big[(\xi, 0), (\eta, y)\big] = ([\xi, \eta], \xi y) \tag{2.4}$$

for $\xi, \eta \in \mathfrak{h}$, $y \in V$. Define a binary operation $\cdot$ on $\mathcal{E} = \mathfrak{h} \times V$ by

$$(\xi, x) \cdot (\eta, y) = \xi(\eta, y)$$

for $\xi, \eta \in \mathfrak{h}$, $x, y \in V$; i.e.

$$(\xi, x) \cdot (\eta, y) = ([\xi, \eta], \xi y). \tag{2.5}$$

Then $(\mathcal{E}, \cdot)$ is a Leibniz algebra, and if $\mathfrak{h}$ acts nontrivially on V, then $(\mathcal{E}, \cdot)$ is not a Lie algebra. We call $\mathcal{E}$ with this Leibniz algebra structure the *hemisemidirect product* of $\mathfrak{h}$ with V, and denote it by $\mathfrak{h} \ltimes_H V$.

The skew-symmetrized product in $(\mathcal{E}, \cdot)$ is

$$[\![(\xi, x), (\eta, y)]\!] = \left([\xi, \eta], \frac{1}{2}(\xi y - \eta x)\right) \tag{2.6}$$

for $\xi, \eta \in \mathfrak{h}$, $x, y \in V$. We call $\mathcal{E}$ with the bracket $[\![\cdot, \cdot]\!]$ the *demisemidirect product* of $\mathfrak{h}$ with V, and we denote it by $\mathfrak{h} \ltimes_D V$. As we noted in §1, the factor of $1/2$ generally spoils the Jacobi identity for this bracket. We have

$$\begin{aligned} \mathcal{J} &= \{0\} \times \mathfrak{h}V \\ \ker(\lambda) &= \{\xi \in \mathfrak{z}(\mathfrak{h}) \mid \xi V = 0\} \times V \end{aligned}$$

where $\mathfrak{z}(\mathfrak{h})$ is the center of $\mathfrak{h}$. If the representation of $\mathfrak{h}$ on V is faithful and if $\mathfrak{h}V = V$, then $\mathcal{J} = \ker(\lambda)$. For example, if $\mathfrak{h} = \mathfrak{gl}(V)$, then $\mathcal{J} = \ker(\lambda) = \{0\} \times V$.

Let $\pi_{\mathfrak{h}}\colon\ \mathcal{E} \to \mathfrak{h}$ denote the projection onto the first factor. Then $\pi_{\mathfrak{h}}$ is $\mathfrak{h}$-equivariant, i.e.

$$\left[\xi, \pi_{\mathfrak{h}}(\eta, x)\right] = \pi_{\mathfrak{h}}\left(\xi(\eta, x)\right) \tag{2.7}$$

for all $\xi, \eta \in \mathfrak{h}$, $x \in V$. The homomorphism $\lambda\colon\ \mathcal{E} \to \mathrm{Der}(\mathcal{E})$ factors through $\pi_{\mathfrak{h}}$ and the action (2.4):

$$\lambda(\xi, x)(\eta, y) = \pi_{\mathfrak{h}}(\xi, x)(\eta, y). \tag{2.8}$$

The preceding paragraph motivates the following definition.

Definition 2.3. Let $(\mathcal{E}, \cdot)$ be a Leibniz algebra. Let $\mathfrak{h}$ be a Lie algebra with a derivation action $\mathfrak{h} \to \mathrm{Der}(\mathcal{E})$, i.e. $\xi(x \cdot y) = \xi x \cdot y + x \cdot \xi y$ for all $\xi \in \mathfrak{h}$, $x, y \in \mathcal{E}$. Let $f\colon\ \mathcal{E} \to \mathfrak{h}$ be an $\mathfrak{h}$-equivariant linear map such that the diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\lambda} & \mathrm{Der}(\mathcal{E}) \\ f\downarrow & \nearrow & \\ \mathfrak{h} & & \end{array} \tag{2.9}$$

commutes. Let $\mathfrak{g} = \mathfrak{h} \ltimes \mathcal{E}$ be the semidirect product Lie algebra, $\mathcal{E}$ being considered as a Lie algebra with the zero bracket. We say that the triple $(\mathfrak{g}, \mathfrak{h}, f)$ is an *enveloping Lie algebra* of $(\mathcal{E}, \cdot)$. We will justify this name with Theorem 2.9 below. (Note that our notion of enveloping Lie algebra of a Leibniz algebra is not the same as Loday's universal enveloping algebra [12]; the latter is not a Lie algebra, but rather an algebra with two associative multiplications satisfying some compatibility conditions.)

In terms of equations, the $\mathfrak{h}$-equivariance of f and the commuting of (2.9) are expressed by

$$[\xi, f(x)] = f(\xi x) \tag{2.10}$$
$$f(x)y = \lambda(x)y \tag{2.11}$$

for all $x, y \in \mathcal{E}$, $\xi \in \mathfrak{h}$. (The $\mathfrak{h}$-equivariance of f implies that $f(\mathcal{E})$ is a Lie ideal of $\mathfrak{h}$.) Properties (2.10) and (2.11) imply that

$$f(x \cdot y) = f(f(x)y) = [f(x), f(y)]$$

for $x, y \in \mathcal{E}$; that is, f is a homomorphism of Leibniz algebras. In particular, $f(x \cdot x) = 0$ for all $x \in \mathcal{E}$, and

$$f([\![x, y]\!]) = [f(x), f(y)].$$

By (2.11), if $f(x) = 0$, then $\lambda(x) = 0$. Therefore we have the inclusions

$$\mathcal{J} \subseteq \ker(f) \subseteq \ker(\lambda).$$

Just as (2.7) and (2.8) motivated the definition of enveloping Lie algebra, so do they imply the following.

Proposition 2.4. $(\mathfrak{h} \ltimes \mathcal{E}, \mathfrak{h}, \pi_{\mathfrak{h}})$ *is an enveloping Lie algebra for the hemisemidirect product* $\mathcal{E} = \mathfrak{h} \ltimes_H V$.

Every Leibniz algebra $(\mathcal{E}, \cdot)$ has enveloping Lie algebras. By (2.2), $\lambda(\mathcal{E})$ is a Lie subalgebra of $\mathrm{Der}(\mathcal{E})$. Since left multiplication maps are $\mathrm{Der}(\mathcal{E})$-equivariant (see (2.1)), the following holds.

Proposition 2.5. *Let* $(\mathcal{E}, \cdot)$ *be a Leibniz algebra, and let* $\mathfrak{h}$ *be a Lie algebra satisfying* $\lambda(\mathcal{E}) \subseteq \mathfrak{h} \subseteq \mathrm{Der}(\mathcal{E})$. *Then* $(\mathfrak{h} \ltimes \mathcal{E}, \mathfrak{h}, \lambda)$ *is an enveloping Lie algebra for* $(\mathcal{E}, \cdot)$.

If $(\mathfrak{g}, \mathfrak{h}, f)$ is an enveloping Lie algebra of a Leibniz algebra $(\mathcal{E}, \cdot)$, then so is $(\tilde{\mathfrak{g}}, f(\mathcal{E}), f)$ where $\tilde{\mathfrak{g}} = f(\mathcal{E}) \ltimes \mathcal{E}$. For our purposes, the case where $f(\mathcal{E}) = \mathfrak{h}$ is of most interest. In this case we have

$$\mathfrak{h} \cong \mathcal{E} / \ker(f).$$

Conversely, let $\mathcal{M}$ be an ideal of $(\mathcal{E}, \cdot)$ satisfying $\mathcal{J} \subseteq \mathcal{M} \subseteq \ker(\lambda)$. Then $\mathfrak{h} := \mathcal{E}/\mathcal{M}$ is a Lie algebra, and the Leibniz algebra homomorphism $\lambda\colon\ \mathcal{E} \to \mathrm{Der}(\mathcal{E})$

descends to a Lie algebra homomorphism $\overline{\lambda}$: $\mathfrak{h} \to \mathrm{Der}(\mathcal{E})$ such that

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\lambda} & \mathrm{Der}(\mathcal{E}) \\ q\downarrow & \nearrow_{\bar{\lambda}} & \\ \mathfrak{h} & & \end{array}$$

commutes, where q: $\mathcal{E} \to \mathfrak{h}$ is the natural projection. The following result, which is our main construction of enveloping Lie algebras, is an immediate consequence of these considerations.

THEOREM 2.6. *Let* $(\mathcal{E}, \cdot)$ *be a Leibniz algebra, let* $\mathcal{M}$ *be an ideal of* $\mathcal{E}$ *satisfying* $\mathcal{J} \subseteq \mathcal{M} \subseteq \ker(\lambda)$*, and let* $\mathfrak{h} = \mathcal{E}/\mathcal{M}$*. Then* $(\mathfrak{h} \ltimes \mathcal{E}, \mathfrak{h}, q)$ *is an enveloping Lie algebra for* $(\mathcal{E}, \cdot)$*.*

Example 2.7. Let $\mathcal{E} = \mathfrak{h} \ltimes_H V$ be the hemisemidirect product, and assume that the representation of $\mathfrak{h}$ on V is faithful and that $\mathfrak{h}V = V$. Then $\mathcal{J} = \ker(\lambda)$, and $\mathfrak{h} \cong \mathcal{E}/\mathcal{J}$. We may identify the natural projection q: $\mathcal{E} \to \mathcal{E}/\mathcal{J}$ with the projection $\pi_{\mathfrak{h}}$: $\mathcal{E} \to \mathfrak{h}$ onto the first factor. In this case, the enveloping Lie algebra of Proposition 2.4 is that of Theorem 2.6.

Remark 2.8. Let $\mathfrak{h}$ be a Lie algebra, let $\mathcal{E}$ be a left $\mathfrak{h}$-module, and let f: $\mathcal{E} \to \mathfrak{h}$ be a $\mathfrak{h}$-equivariant linear map. Define a binary operation $\cdot$ on $\mathcal{E}$ by $x \cdot y = f(x)y$. Then $(\mathcal{E}, \cdot)$ is clearly a Leibniz algebra. We will discuss this point further in §3. Loday and Pirashvili [13] have shown that f: $\mathcal{E} \to \mathfrak{h}$ can be considered to be a Lie algebra object in what they call the infinitesimal tensor category $\mathcal{LM}$ of linear mappings. The assignments $(f\colon\ \mathcal{E} \to \mathfrak{h}) \rightsquigarrow (\mathcal{E}, \cdot)$ and $(\mathcal{E}, \cdot) \rightsquigarrow (q\colon\ \mathcal{E} \to \mathcal{E}/\mathcal{J})$ are adjoint functors between the category of Lie algebra objects in $\mathcal{LM}$ and the category of Leibniz algebras.

We now move on to our main result, which is to show how to recover the skew-symmetrized structure $(\mathcal{E}, [\![\cdot,\cdot]\!])$ of a Leibniz algebra $(\mathcal{E}, \cdot)$ from the Lie algebra structure of an enveloping Lie algebra $(\mathfrak{g}, \mathfrak{h}, f)$. Let $\pi_{\mathcal{E}}$: $\mathfrak{g} \to \mathcal{E}$ denote the projection onto the second factor. For each $s \in \mathbb{K}$, define a section σ_s: $\mathcal{E} \to \mathfrak{g}$ of $\pi_{\mathcal{E}}$ by

$$\sigma_s(x) = (sf(x), x) \tag{2.12}$$

for $x \in \mathcal{E}$. The image

$$\mathcal{E}_s = \sigma_s(\mathcal{E}) = \{(sf(x), x) \mid x \in \mathcal{E}\}$$

is a copy of $\mathcal{E}$ which is a complement of $\mathfrak{h} = \ker(\pi_{\mathcal{E}})$. We will write the corresponding vector space decomposition of $\mathfrak{g}$ as

$$\mathfrak{g} \cong \mathfrak{h} \oplus \mathcal{E}_s. \tag{2.13}$$

Note that the case $s = 0$ is just the semidirect product of $\mathfrak{h}$ with $\mathcal{E}$.

Since $\mathcal{E}_s$ is essentially the graph of the $\mathfrak{h}$-equivariant map $sf\colon\ \mathcal{E} \to \mathfrak{h}$, σ_s itself is equivariant for the adjoint action (2.4) of $\mathfrak{h}$ on $\mathfrak{g}$. Indeed, for $x \in \mathcal{E}$, $\xi \in \mathfrak{h}$, we have

$$\begin{aligned}\xi\ \sigma_s(x) &= ([\xi, sf(x)], \xi x)\\ &= (sf(\xi x), \xi x)\\ &= \sigma_s(\xi x).\end{aligned}$$

This shows that the bracket of an element of $\mathfrak{h}$ with an element of $\mathcal{E}$ relative to the decomposition (2.13) agrees with the action of $\mathfrak{h}$ on $\mathcal{E}$, independently of the value of s:

$$[\xi, x] = \xi x.$$

Since

$$[\mathfrak{h}, \mathcal{E}] \subseteq \mathcal{E} \tag{2.14}$$

the decomposition (2.13) is *reductive* [18].

Now we consider the bracket of two elements of $\mathcal{E} \cong \mathcal{E}_s$ relative to (2.13). Here, unlike (2.14), we expect the result to depend on s. For $x, y \in \mathcal{E}$, we have

$$\begin{aligned}\left[\sigma_s(x), \sigma_s(y)\right] &= \left[(sf(x), x), (sf(y), y)\right]\\ &= (s^2[f(x), f(y)], s(f(x)y - f(y)x))\\ &= (s^2 f([\![x, y]\!]), 2s[\![x, y]\!])\\ &= (-s^2 f([\![x, y]\!]), 0) + (2s^2 f([\![x, y]\!]), 2s[\![x, y]\!])\\ &= (-s^2 f([\![x, y]\!]), 0) + \sigma_s(2s[\![x, y]\!]).\end{aligned}$$

We use the components of the result of this computation to define skew-symmetric bilinear maps $[\![\cdot, \cdot]\!]_s\colon\ \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and $\Delta_s\colon\ \mathcal{E} \times \mathcal{E} \to \mathfrak{h}$ by

$$[\![x, y]\!]_s = \pi_{\mathcal{E}}\left([\sigma_s(x), \sigma_s(y)]\right) = 2s[\![x, y]\!] \tag{2.15}$$

$$\Delta_s(x, y) = \pi_{\mathfrak{h}}\left([\sigma_s(x), \sigma_s(y)]\right) = -s^2 f([\![x, y]\!]) \tag{2.16}$$

for $x, y \in \mathcal{E}$.

Observe that the choice $s = 1/2$ recovers the original skew-symmetrized Leibniz product on $\mathcal{E}$:

$$[\![x, y]\!]_{1/2} = [\![x, y]\!] \tag{2.17}$$

for $x, y \in \mathcal{E}$. This gives us the following result.

THEOREM 2.9. *Let $(\mathcal{E}, \cdot)$ be a Leibniz algebra with enveloping Lie algebra $(\mathfrak{g}, \mathfrak{h}, f)$, and let $\sigma_{1/2}\colon \mathcal{E} \to \mathfrak{g}$ be defined by (2.12). Then the skew-symmetrized product (2.3) is given in terms of the Lie bracket in $\mathfrak{g}$ by*

$$[\![x, y]\!] = \pi_{\mathcal{E}}\left(\left[\sigma_{1/2}(x), \sigma_{1/2}(y)\right]\right)$$

for $x, y \in \mathcal{E}$.

In §5, we will show that the map $\Delta_s\colon \mathcal{E} \times \mathcal{E} \to \mathfrak{h}$ defined by (2.16) introduces additional structure into $\mathcal{E}$.

Example 2.10. Let $\mathcal{E} = \mathfrak{h} \ltimes_H V$ be the hemisemidirect product, and let $(\mathfrak{g}, \mathfrak{h}, \pi_{\mathfrak{h}})$ be the enveloping Lie algebra obtained in Proposition 2.4. The section $\sigma_s\colon \mathcal{E} \to \mathfrak{g}$ is given by

$$\sigma_s(\xi, x) = (s\xi, \xi, x)$$

for $\xi \in \mathfrak{h}$, $x \in V$. A calculation shows that the skew-symmetric maps $[\![\cdot, \cdot]\!]_s\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ and $\Delta_s\colon \mathcal{E} \times \mathcal{E} \to \mathfrak{h}$ are given by

$$[\![(\xi, x), (\eta, y)]\!]_s = 2s\left([\xi, \eta], \frac{1}{2}(\xi y - \eta x)\right) \tag{2.18}$$

$$\Delta_s((\xi, x), (\eta, y)) = -s^2 [\xi, \eta] \tag{2.19}$$

for $\xi, \eta \in \mathfrak{h}$, $x, y \in V$. If we take $s = 1/2$, (2.18) reduces to the demisemidirect product bracket (2.6).

3. Omni-Lie and Omni-Leibniz algebras. In this section we will show that *every* Leibniz algebra can be embedded in a hemisemidirect product Leibniz algebra.

Let $(\mathcal{E}, \cdot)$ be a Leibniz algebra with enveloping Lie algebra $(\mathfrak{g}, \mathfrak{h}, f)$, and let $\sigma_s\colon \mathcal{E} \to \mathfrak{g}$ be the section of $\pi_{\mathcal{E}}$ defined by (2.12). The vector space $\mathfrak{g} = \mathfrak{h} \times \mathcal{E}$ has (at least) four natural algebra structures which are related to the structure of $\mathcal{E}$. First, $\mathfrak{g} = \mathfrak{h} \ltimes \mathcal{E}$ is a semidirect product Lie algebra. We have shown in Theorem 2.9 that the section $\sigma_{1/2}$ can be used to recover the skew-symmetrized structure $(\mathcal{E}, [\![\cdot, \cdot]\!])$ from the semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathcal{E}$. Second, $\mathfrak{g} = \mathfrak{h} \times \mathcal{E}$ also has the structure of a direct product of Leibniz algebras. As noted before, each $\mathcal{E}_s$ is the graph of the map $sf\colon \mathcal{E} \to \mathfrak{h}$. Since each sf is a homomorphism of

Leibniz algebras, each $\mathcal{E}_s$ is trivially a subalgebra of the direct product $\mathfrak{g} = \mathfrak{h} \times \mathcal{E}$, which, for $s \neq 0$, is isomorphic to $(\mathcal{E}, \cdot)$ under the map $(1/s)\pi_{\mathcal{E}}|_{\mathcal{E}_s}$.

Third and fourth, $\mathfrak{g}$ also has the hemisemidirect product structure $\mathfrak{g} = \mathfrak{h} \ltimes_H \mathcal{E}$ and the demisemidirect product structure $\mathfrak{g} = \mathfrak{h} \ltimes_D \mathcal{E}$. The next result shows how these are related to the Leibniz and skew-symmetrized Leibniz algebra structures, respectively, on $\mathcal{E}$.

PROPOSITION 3.1. (1) $\sigma_1\colon (\mathcal{E}, \cdot) \to \mathfrak{h} \ltimes_H \mathcal{E}$ *is a monomorphism of Leibniz algebras.*

(2) $\sigma_1\colon (\mathcal{E}, [\![\cdot,\cdot]\!]) \to \mathfrak{h} \ltimes_D \mathcal{E}$ *is a monomorphism of skew-symmetrized Leibniz algebras.*

Proof. For $x, y \in \mathcal{E}$, we compute

$$\begin{aligned} \sigma_1(x) \cdot \sigma_1(y) &= (f(x), x) \cdot (f(y), y) \\ &= \left([f(x), f(y)], f(x)y \right) \\ &= (f(x \cdot y), x \cdot y) \\ &= \sigma_1(x \cdot y) \end{aligned}$$

where in the penultimate equality, we have used the fact that f is a homomorphism of Leibniz algebras and (2.11). This establishes (1), and (2) follows from (1). □

Since every Leibniz algebra has enveloping Lie algebras, we have the following.

COROLLARY 3.2. (1) *Every Leibniz algebra can be embedded as a subalgebra in a hemisemidirect product.*

(2) *Every skew-symmetrized Leibniz algebra can be embedded as a subalgebra in a demisemidirect product.*

By Propositions 2.5 and 3.1, the monomorphism $x \mapsto (\lambda(x), x)$ embeds a given Leibniz algebra $(\mathcal{E}, \cdot)$ as a subalgebra of the hemisemidirect product $\mathrm{Der}(\mathcal{E}, \cdot) \ltimes_H \mathcal{E}$, which in turn can be embedded as a subalgebra of the hemisemidirect product $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$. While different Leibniz algebra structures on the vector space $\mathcal{E}$ can lead to different derivation algebras, $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$ contains *all* Leibniz algebra structures on $\mathcal{E}$. We now show that, in fact, this exactly characterizes the Leibniz algebras among all algebra structures on $\mathcal{E}$.

Let $(\mathcal{E}, \cdot)$ be an algebra, and let

$$\mathcal{G}_\lambda = \{(\lambda(x), x) \mid x \in \mathcal{E}\}$$

denote the graph of λ as a subspace of $\mathfrak{gl}(\mathcal{E}) \times \mathcal{E}$.

PROPOSITION 3.3. *An algebra* $(\mathcal{E}, \cdot)$ *is a Leibniz algebra if and only if* $\mathcal{G}_\lambda$ *is a subalgebra of* $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$. *If these conditions hold, then* $\pi_{\mathcal{E}}|_{\mathcal{G}_\lambda}$ *is an isomorphism from* $(\mathcal{G}_\lambda, \cdot)$ *to* $(\mathcal{E}, \cdot)$.

Proof. For $x, y \in V$, we have

$$(\lambda(x), x) \cdot (\lambda(y), y) = ([\lambda(x), \lambda(y)], x \cdot y). \tag{3.1}$$

Thus $\mathcal{G}_\lambda$ is closed under the product $\cdot$ if and only if λ is a homomorphism. The remaining assertions are clear. □

We see from Proposition 3.3 that the class of all Leibniz algebra structures on $\mathcal{E}$ corresponds to the class of all linear maps from $\mathcal{E}$ to $\mathfrak{gl}(\mathcal{E})$ whose graphs are subalgebras of the hemisemidirect product. Recalling that a skew-symmetric subalgebra of a Leibniz algebra is a Lie algebra, we see that $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$ also contains all Lie algebra structures on $\mathcal{E}$. From (3.1), it is immediate that the operation $\cdot$ in $\mathcal{G}_\lambda$ is skew-symmetric if and only if the operation $\cdot$ in $\mathcal{E}$ is skew-symmetric.

COROLLARY 3.4. *An algebra* $(\mathcal{E}, \cdot)$ *is a Lie algebra if and only if* $\mathcal{G}_\lambda$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$. *If these conditions hold, then* $\pi_{\mathcal{E}}|_{\mathcal{G}_\lambda}$ *is an isomorphism from* $(\mathcal{G}_\lambda, \cdot)$ *to* $(\mathcal{E}, \cdot)$.

In case $\mathcal{E} = \mathbb{R}^n$, the demisemidirect product $(\mathfrak{gl}(\mathcal{E}) \ltimes_D \mathcal{E}, [\![\cdot, \cdot]\!])$ is the "omni-Lie algebra" with bracket (1.2) of [21]. Symmetrizing the Leibniz product in $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$ defines a commutative product $\circ$ by

$$(\xi, x) \circ (\eta, y) = \left(0, \frac{1}{2}(\xi y + \eta x)\right),$$

which was called an "$\mathcal{E}$-valued bilinear form" in [21]. The hemisemidirect product Leibniz algebra $\mathfrak{gl}(\mathcal{E}) \ltimes_H \mathcal{E}$ combines both of these structures. The following restatement of Corollary 3.4 encompasses both Proposition 1 in [21] and the subsequent discussion.

COROLLARY 3.5. *An algebra* $(\mathcal{E}, \cdot)$ *is a Lie algebra if and only if (i) for all* $(\xi, x), (\eta, y) \in \mathcal{G}_\lambda$, $(\xi, x) \circ (\eta, y) = 0$, *and (ii)* $(\mathcal{G}_\lambda, [\![\cdot, \cdot]\!])$ *is a Lie subalgebra of* $\mathfrak{gl}(\mathcal{E}) \ltimes_D \mathcal{E}$.

In particular, if $(\mathcal{E}, \cdot)$ is an anticommutative algebra, then it is a Lie algebra if and only if the graph of its adjoint representation is a subalgebra of the demisemidirect product $\mathfrak{gl}(\mathcal{E}) \ltimes_D \mathcal{E}$.

4. Courant algebroids. In this section, we will apply the methods of §2 to Courant algebroids, which include as special cases the doubles of Lie bialgebras and the bundles $TP \oplus T^*P$ with the bracket on sections given by (1.1). The following definition was introduced in [11].

Definition 4.1. A *Courant algebroid* is a vector bundle $E \to P$ equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the bundle, a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ on the space $\mathcal{E} = \Gamma(E)$ of smooth sections of E, and a bundle map $\rho\colon E \to TP$ such that, for any $x, y, z \in \mathcal{E}$ and $f, g \in \mathcal{A} = C^\infty(P)$:

(1) $\sum_{(x,y,z)} [\![[\![x,y]\!],z]\!] = \mathcal{D}T(x,y,z)$;

(2) $\rho[\![x,y]\!] = [\rho x, \rho y]$;

(3) $[\![x, fy]\!] = f[\![x,y]\!] + (\rho(x)f)y - \langle x,y\rangle\mathcal{D}f$;

(4) $\rho\circ\mathcal{D} = 0$; i.e., for any f, g, $\langle\mathcal{D}f, \mathcal{D}g\rangle = 0$;

(5) $\rho(x)\langle y,z\rangle = \langle[\![x,y]\!] + \mathcal{D}\langle x,y\rangle, z\rangle + \langle y, [\![x,z]\!] + \mathcal{D}\langle x,z\rangle\rangle$.

Here, $\sum_{(x,y,z)}$ denotes the sum over cyclic permutations of x, y, and z, $T(x,y,z)$ is the function on P defined by:

$$T(x,y,z) = \frac{1}{3}\sum_{(x,y,z)} \langle[\![x,y]\!],z\rangle$$

and $\mathcal{D}\colon \mathcal{A} \to \mathcal{E}$ is the map defined by $\mathcal{D} = \frac{1}{2}\beta^{-1}\rho^* d$, where β is the isomorphism between E and E^* given by the bilinear form. In other words,

$$\langle\mathcal{D}f, x\rangle = \frac{1}{2}\rho(x)f.$$

It was already noted in [11] that adding the symmetric term $(x,y) \mapsto \mathcal{D}\langle x,y\rangle$ to the bracket of a Courant algebroid leads to an operation $\cdot$ with nicer properties. In fact, as noted independently by Kosmann-Schwarzbach, Ševera, and Xu, the operation $\cdot$ makes $\mathcal{E}$ into a Leibniz algebra. (Details may be found in §2.6 of [19].)

This time, we will use our main construction of Theorem 2.6 to find an enveloping Lie algebra for $\mathcal{E}$. By the definition of the operation $\cdot$, the image $\mathcal{D}(\mathcal{A})$ contains the ideal $\mathcal{J}$ generated by squares. In fact, by the identity $x \cdot \mathcal{D}f = \mathcal{D}\langle x, \mathcal{D}f\rangle$ (Lemma 2.6.2 of [19]), $\mathcal{D}(\mathcal{A})$ is an ideal in $\mathcal{E}$, so it can play the role of $\mathcal{M}$ in our general construction; i.e. $\mathcal{E}/\mathcal{D}(\mathcal{A})$ is a Lie algebra acting on $\mathcal{E}$. We may therefore form the semidirect product Lie algebra $\mathfrak{g} = \mathcal{E}/\mathcal{D}(\mathcal{A}) \ltimes \mathcal{E}$, which functions as an enveloping Lie algebra. Continuing with the application of our general construction in Theorem 2.9, we see that the Courant algebroid bracket on $\mathcal{E}$ is obtained from the Lie algebra bracket on $\mathfrak{g} = \mathcal{E}/\mathcal{D}(\mathcal{A}) \times \mathcal{E}$ by identification of $\mathcal{E}$ with the graph of $\frac{1}{2}$ times the quotient map from $\mathcal{E}$ to $\mathcal{E}/\mathcal{D}(\mathcal{A})$, by projection along the subalgebra $\mathcal{E}/\mathcal{D}(\mathcal{A}) \times \{0\}$.

Example 4.2. If P is a single point, $\mathcal{E}$ is just a Lie algebra with an invariant symmetric bilinear form, and $\mathcal{D} = 0$. The enveloping Lie algebra is then $\mathcal{E} \ltimes \mathcal{E}$. The bracket on $\mathcal{E}$ is recovered by projection along the first factor onto the subspace (not a subalgebra!) $\{(\frac{1}{2}x, x) \mid x \in \mathcal{E}\}$.

For the original Courant bracket (1.1) on $\mathcal{X}(P) \oplus \Omega^1(P)$, the symmetric bilinear form is

$$\langle(\xi_1, \theta_1), (\xi_2, \theta_2)\rangle = \frac{1}{2}(i_{\xi_1}\theta_2 + i_{\xi_2}\theta_1),$$

ρ: $TP \oplus T^*P \to TP$ is projection on the first factor, and $\mathcal{D}$ is the operator $f \mapsto (0, df)$, so the Leibniz product is

$$(\xi_1, \theta_1) \cdot (\xi_2, \theta_2) = ([\xi_1, \xi_2], \mathcal{L}_{\xi_1}\theta_2 - i_{\xi_2} d\theta_1).$$

The Lie algebra $\mathcal{E}/\mathcal{D}(\mathcal{A})$ is thus $\mathcal{X}(P) \oplus \Omega^1(P)/dC^\infty(P)$, on which the bracket, since we can add to $i_{\xi_2} d\theta_1$ the exact form $di_{\xi_2}\theta_1$, is just the semidirect product bracket of $\mathcal{X}(P)$ acting on $\Omega^1(P)/dC^\infty(P)$ by Lie derivation, i.e. $\mathcal{E}/\mathcal{D}(\mathcal{A}) = \mathcal{X}(P) \ltimes \Omega^1(P)/dC^\infty(P)$.

The enveloping Lie algebra is thus a "double semidirect product"

$$(\mathcal{X}(P) \ltimes \Omega^1(P)/dC^\infty(P)) \ltimes (\mathcal{X}(P) \times \Omega^1(P)).$$

The action of $\mathcal{X}(P) \ltimes \Omega^1(P)/dC^\infty(P)$ on $\mathcal{X}(P) \times \Omega^1(P)$ may be described as follows. Elements of $\mathcal{X}(P)$ act by Lie derivation on both the vector fields and the 1-forms. A 1-form ϕ (modulo exact 1-forms) acts by the nilpotent operation $(\xi, \theta) \mapsto (0, -i_\xi d\phi)$.

Remark 4.3. Pavol Ševera has pointed out to us a nice interpretation of the action just described. First of all, we pass from the Lie algebra to the group which is the semidirect product of the diffeomorphisms and the abelian group of 1-forms modulo exact forms. The diffeomorphisms act in the obvious way. To understand the action of the 1-forms, we think of the action not just on the product $\mathcal{X}(P) \times \Omega^1(P)$, but on the space of subspaces of $\mathcal{X}(P) \times \Omega^1(P)$ which are graphs of 2-forms. Then the action of a 1-form ϕ on a 2-form is simply to add $-d\phi$. These two operations on 2-forms—transformation by diffeomorphisms and the addition of exact forms—are precisely the operations which may be considered as symmetries of the variational problem defined by integration of the 2-form over 2-dimensional submanifolds of P.

Remark 4.4. When P is a compact, oriented manifold of dimension n, the space $\Omega^1(P)/dC^\infty(P)$ is in natural duality, by integration, with the space of closed $n-1$–forms on P. If P carries a volume element, then the latter space may be identified with the Lie algebra of volume-preserving vector fields. The Lie algebra $\mathcal{X}(P) \ltimes \Omega^1(P)/dC^\infty(P)$ may then be seen as an enlargement of the Lie algebra of the cotangent bundle of the group of volume preserving diffeomorphisms. It would be interesting to relate this interpretation to other aspects of the material in this paper.

Remark 4.5. Finally, we note that the constructions in this section can be carried out equally well in the setting of $(R, \mathcal{A})$ C-algebras. These algebraic objects, introduced in [21], include as special cases both the spaces of sections of Courant algebroids and the omni-Lie algebras of §3.

5. Lie–Yamaguti structures. Let $(\mathcal{E}, \cdot)$ be a Leibniz algebra, and let $(\mathfrak{g}, \mathfrak{h}, f)$ be an enveloping Lie algebra. As in §2, we have the reductive decomposition $\mathfrak{g} \cong \mathfrak{h} \oplus \mathcal{E}_{1/2}$. We found that (2.17) recovers the original skew-symmetrized operation $[\![\cdot, \cdot]\!]$ in $\mathcal{E}$. We will now show that the skew-symmetric bilinear map $\Delta_{1/2}\colon \mathcal{E} \times \mathcal{E} \to \mathfrak{h}$ induces an additional operation and additional structure on $\mathcal{E}$.

More generally, let $\mathfrak{g}$ be a Lie algebra with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, i.e. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. On $\mathfrak{m}$, define bilinear maps $[\![\cdot, \cdot]\!]\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and $\Delta\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$ by the projections of the Lie bracket:

$$[\![x, y]\!] = \pi_{\mathfrak{m}}\left([x, y]\right) \tag{5.1}$$

$$\Delta(x, y) = \pi_{\mathfrak{h}}\left([x, y]\right) \tag{5.2}$$

for $x, y \in \mathfrak{m}$. Then define a ternary product on $\mathfrak{m}$ by

$$\{x, y, z\} := [\Delta(x, y), z]$$

for $x, y, z \in \mathfrak{m}$. It is straightforward to show that $(\mathfrak{m}, [\![\cdot, \cdot]\!], \{\cdot, \cdot, \cdot\})$ satisfies the following definition.

Definition 5.1. A *Lie-Yamaguti algebra* $(\mathfrak{m}, [\![\cdot, \cdot]\!], \{\cdot, \cdot, \cdot\})$ is a vector space $\mathfrak{m}$ together with a bilinear operation $[\![\cdot, \cdot]\!]\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ and a trilinear operation $\{\cdot, \cdot, \cdot\}\colon \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ such that, for all $x, y, z, u, v, w \in \mathfrak{m}$:

(LY1) $[\![x, y]\!] = -[\![y, x]\!]$;
(LY2) $\{x, y, z\} = -\{y, x, z\}$;
(LY3) $\sum_{(x,y,z)} \left([\![[\![x, y]\!], z]\!] + \{x, y, z\}\right) = 0$;
(LY4) $\sum_{(x,y,z)} \{[\![x, y]\!], z, u\} = 0$;
(LY5) $\{x, y, [\![u, v]\!]\} = [\![\{x, y, u\}, v]\!] + [\![u, \{x, y, v\}]\!]$;
(LY6) $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + \{u, \{x, y, v\}, w\} + \{u, v, \{x, y, w\}\}$.

The properties of the binary and ternary operations of a Lie-Yamaguti algebra can be found in the work of Nomizu [18] as properties satisfied by the torsion and curvature tensors, respectively, in a reductive homogeneous space; we will discuss this further in §6. The notion of a Lie-Yamaguti algebra is a natural abstraction made by K. Yamaguti [22], who called these algebras "general Lie triple systems." M. Kikkawa [7] dubbed them "Lie triple algebras."

Notice that, if the trilinear product in a Lie-Yamaguti algebra is trivial, i.e., $\{\cdot,\cdot,\cdot\} \equiv 0$, then (LY2), (LY4), (LY5), and (LY6) are trivial, and (LY1) and (LY3) define a Lie algebra. On the other hand, if the binary product is trivial, i.e., $[\![\cdot,\cdot]\!] \equiv 0$, then (LY1), (LY4), and (LY5) are trivial, and (LY2), (LY3), and (LY6) define a Lie triple system.

Now we apply these considerations to the case of a Leibniz algebra $(\mathcal{E},\cdot)$ with enveloping Lie algebra $(\mathfrak{g},\mathfrak{h},f)$. The map $\Delta \equiv \Delta_{1/2}\colon \mathcal{E}\times\mathcal{E} \to \mathfrak{h}$ is given by (2.16); we repeat it here for convenience:

$$\Delta(x,y) = -\frac{1}{4} f\,([\![x,y]\!])$$

for $x,y \in \mathcal{E}$. Therefore the ternary product is given in terms of the Leibniz and skew-symmetrized Leibniz products by

$$\{x,y,z\} = -\frac{1}{4}[\![x,y]\!]\cdot z \tag{5.3}$$

for $x,y,z \in \mathcal{E}$. Since $\lambda(x\cdot y + y\cdot x) = 0$, we can also write the ternary product purely in terms of the Leibniz product:

$$\{x,y,z\} = -\frac{1}{4}(x\cdot y)\cdot z. \tag{5.4}$$

Summarizing, we have the following.

Proposition 5.2. *Let $(\mathcal{E},\cdot)$ be a Leibniz algebra. Then $(\mathcal{E},[\![\cdot,\cdot]\!],\{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti algebra, where $[\![\cdot,\cdot]\!]$ is the skew-symmetrization of $\cdot$, and $\{\cdot,\cdot,\cdot\}$ is defined by (5.3) or (5.4).*

Example 5.3. For the hemisemidirect product $\mathcal{E} = \mathfrak{h} \ltimes_H V$ of Example 2.2, a short calculation using (5.3) and (2.5) shows that the trilinear product in the associated Lie-Yamaguti algebra $(\mathcal{E},[\![\cdot,\cdot]\!],\{\cdot,\cdot,\cdot\})$ is given by

$$\{(\xi,x),(\eta,y),(\zeta,z)\} = -\frac{1}{4}\,([[\xi,\eta],\zeta],[\xi,\eta]z)$$

for $\xi,\eta,\zeta \in \mathfrak{h}$, $x,y,z \in V$.

Example 5.4. For the Courant bracket (1.1), the associated trilinear product is

$$\{(\xi_1,\theta_1),(\xi_2,\theta_2),(\xi_3,\theta_3)\} = -\frac{1}{4}([[\xi_1,\xi_2],\xi_3],\mathcal{L}_{[\xi_1,\xi_2]}\theta_3 - i_{\xi_3}d(\mathcal{L}_{\xi_1}\theta_2 - \mathcal{L}_{\xi_2}\theta_1)).$$

Another approach to this bracket, and to spaces of sections of more general Courant algebroids, is to consider them as *strongly homotopy Lie algebras* [19, 20]. Here, the not-quite-Lie algebra also carries a differential d, and the Jacobiator $\sum_{(x,y,z)} [\![[\![x,y]\!],z]\!]$ is expressed as the differential applied to a completely antisymmetric trilinear product. A higher-order Jacobiator of the trilinear product is again a differential, and so on. In the Lie-Yamaguti approach, it is essential that the trilinear operation *not* be completely antisymmetric.

We showed that a Lie algebra with a reductive decomposition naturally induces the structure of a Lie-Yamaguti algebra on the reductive complement to the subalgebra. Now we show the converse: for any Lie-Yamaguti algebra $\mathfrak{m}$, there exists a Lie algebra $\mathfrak{g}$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that the induced Lie-Yamaguti structure on $\mathfrak{m}$ agrees with the original one.

Let $(\mathfrak{m}, [\![\cdot,\cdot]\!], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra, and let $\mathrm{Der}(\mathfrak{m})$ denote the Lie subalgebra of $\mathfrak{gl}(\mathfrak{m})$ consisting of derivations of both the bilinear and trilinear products. Let $\mathfrak{h}$ be a Lie algebra with a derivation action $\mathfrak{h} \to \mathrm{Der}(\mathfrak{m})$, and let $\Delta\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$ be a skew-symmetric bilinear mapping satisfying the following properties:

$$\Delta(x,y)z = \{x,y,z\} \tag{5.5}$$

$$\left[\xi, \Delta(x,y)\right] = \Delta(\xi x, y) + \Delta(x, \xi y) \tag{5.6}$$

$$\Delta\left([\![x,y]\!],z\right) + \Delta\left([\![y,z]\!],x\right) + \Delta\left([\![z,x]\!],y\right) = 0 \tag{5.7}$$

for all $x,y,z \in \mathfrak{m}$, $\xi \in \mathfrak{h}$. On the vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, define a skew-symmetric bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\left[\xi + x, \eta + y\right] = \left([\xi,\eta] + \Delta(x,y)\right) + \left(\xi y - \eta x + [\![x,y]\!]\right). \tag{5.8}$$

PROPOSITION 5.5. *$\mathfrak{g}$ is a Lie algebra, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition. The Lie-Yamaguti bilinear and trilinear products on $\mathfrak{m}$ induced by the decomposition agree with $[\![\cdot,\cdot]\!]$ and $\{\cdot,\cdot,\cdot\}$, respectively.*

Proof. It is straightforward to check that (5.5)–(5.7) and (LY3) give the Jacobi identity for the bracket (5.8). That $[\mathfrak{h},\mathfrak{m}] \subseteq \mathfrak{m}$ is immediate from (5.8), and so the decomposition is reductive. From (5.8) we have $\pi_{\mathfrak{m}}([x,y]) = [\![x,y]\!]$ and $[\pi_{\mathfrak{h}}([x,y]),z] = [\Delta(x,y),z] = \{x,y,z\}$ for $x,y,z \in \mathfrak{m}$, which proves the remaining assertion. □

Definition 5.6. Let $(\mathfrak{m}, [\![\cdot,\cdot]\!], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra, let $\mathfrak{h}$ be a Lie algebra with a derivation action $\mathfrak{h} \to \mathrm{Der}(\mathfrak{m})$, and let $\Delta\colon \mathfrak{m} \times \mathfrak{m} \to \mathfrak{h}$ be a skew-symmetric bilinear mapping satisfying (5.5)–(5.7). The Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\cdot,\cdot]$ is given by (5.8), is called an *enveloping Lie algebra* of $(\mathfrak{m}, [\![\cdot,\cdot]\!], \{\cdot,\cdot,\cdot\})$.

Since a Leibniz algebra has a natural Lie-Yamaguti structure, the coincidence of their notions of enveloping Lie algebra is both clear and expected.

PROPOSITION 5.7. *Let* $(\mathcal{E},\cdot)$ *be a Leibniz algebra with enveloping Lie algebra* $(\mathfrak{g},\mathfrak{h},f)$*. Then* $\mathfrak{g}\cong\mathfrak{h}\oplus\mathcal{E}_{1/2}$ *is an enveloping Lie algebra of the induced Lie-Yamaguti algebra* $(\mathcal{E},[\![\cdot,\cdot]\!],\{\cdot,\cdot,\cdot\})$.

To conclude this section, we follow Yamaguti [22] to show that *every* Lie-Yamaguti algebra has an enveloping Lie algebra.

Let $(\mathfrak{m},[\![\cdot,\cdot]\!],\{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti algebra. Define a bilinear mapping $\delta\colon\ \mathfrak{m}\times\mathfrak{m}\to\mathfrak{gl}(\mathfrak{m})$ by

$$\delta(x,y)z=\{x,y,z\} \tag{5.9}$$

for $x,y,z\in\mathfrak{m}$. By (LY2), δ is skew-symmetric. By (LY5) and (LY6), $\delta(x,y)\in\mathrm{Der}(\mathfrak{m})$ for all $x,y\in\mathfrak{m}$. We call $\delta(x,y)$ an *inner derivation* of $\mathfrak{m}$. Let $\mathrm{IDer}(\mathfrak{m})$ denote the subspace of $\mathrm{Der}(\mathfrak{m})$ spanned by the inner derivations. By (LY6), $\mathrm{IDer}(\mathfrak{m})$ is a Lie algebra. Now let $\mathfrak{h}$ be any Lie algebra satisfying $\mathrm{IDer}(\mathfrak{m})\subseteq\mathfrak{h}\subseteq\mathrm{Der}(\mathfrak{m})$. Then (5.9) implies (5.5) and (5.6) directly, while (5.7) follows from both (5.9) and (LY4).

Summarizing, we have established the following [22].

PROPOSITION 5.8. *Every Lie-Yamaguti algebra has an enveloping Lie algebra.*

Remark 5.9. It was essentially by this route, after guessing the ternary product (5.3), that we first discovered the enveloping Lie algebras of Leibniz algebras.

Remark 5.10. It would be interesting to find conditions on a reductive decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ (equivalently, on a Lie-Yamaguti algebra $\mathfrak{m}$) which would insure that $\mathfrak{m}$ is the skew-symmetrization of a Leibniz algebra. What is necessary, of course, is that $\Delta\colon\ \mathfrak{m}\times\mathfrak{m}\to\mathfrak{h}$ factors into an $\mathfrak{h}$-equivariant linear map $-\frac{1}{4}f\colon\ \mathfrak{m}\to\mathfrak{h}$ and the binary product $[\![\cdot,\cdot]\!]\colon\ \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$. The question then becomes: what conditions on the decomposition guarantee the existence of such a factorization?

6. Reductive homogeneous spaces and loops. Let G be a Lie group with Lie algebra $\mathfrak{g}$, let $H\subseteq G$ be a closed subgroup with Lie algebra $\mathfrak{h}\subseteq\mathfrak{g}$, and let $M=G/H$. The homogeneous space M is said to be *reductive* if there exists a reductive complement $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$, i.e.,

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m} \tag{6.1}$$

$$\mathrm{Ad}_G(H)\mathfrak{m}\subseteq\mathfrak{m}. \tag{6.2}$$

Condition (6.2) implies that

$$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})\mathfrak{m}\subseteq\mathfrak{m}. \tag{6.3}$$

When H is connected, (6.3) implies (6.2). We may identify $\mathfrak{m}$ with the tangent space $T_{\pi(e)}(M)$ where $e \in G$ is the identity element and $\pi\colon\ G \to M$ is the canonical projection.

Now let $(\mathcal{E}, \cdot)$ be a Leibniz algebra with enveloping Lie algebra $(\mathfrak{g}, \mathfrak{h}, f)$, and let H be a Lie group with Lie algebra $\mathfrak{h}$. Suppose that the derivation action of $\mathfrak{h}$ on $\mathcal{E}$ lifts to an automorphism action $H \to \mathrm{Aut}(\mathcal{E})$, i.e. $h(x \cdot y) = hx \cdot hy$ for $h \in H$, $x, y \in \mathcal{E}$. Let $G = H \ltimes \mathcal{E}$ be the semidirect product group, $\mathcal{E}$ considered as usual as an abelian Lie group. Then G is a Lie group with Lie algebra $\mathfrak{g} = \mathfrak{h} \ltimes \mathcal{E}$.

For $s \neq 0$, we already know that (6.3) is satisfied with $\mathfrak{m} = \sigma_s(\mathcal{E})$. Now since H acts by automorphisms, the mapping $f\colon\ \mathcal{E} \to \mathfrak{h}$ is H-equivariant: for all $h \in H$, $x \in \mathcal{E}$,

$$\mathrm{Ad}(h) f(x) = f(hx). \tag{6.4}$$

It follows that the sections $\sigma_s\colon\ \mathcal{E} \to \mathfrak{g}$ defined by $x \mapsto (sf(x), x)$ are also H-equivariant, i.e.,

$$h\sigma_s(x) = h(sf(x), x) = (sf(hx), hx) = \sigma_s(hx).$$

Thus (6.2) holds. Therefore G/H is a reductive homogeneous space.

The vector space projection $\pi_{\mathcal{E}}\colon\ \mathfrak{g} \to \mathcal{E}$ defined by $(\xi, x) \mapsto x$ exponentiates to the group projection $\hat{\pi}_{\mathcal{E}}\colon\ G \to \mathcal{E}; (h, x) \mapsto x$. The sections σ_s exponentiate to the group sections $\hat{\sigma}_s\colon\ \mathcal{E} \to G$ defined by

$$\hat{\sigma}_s(x) = (\exp(sf(x)), x).$$

For $(h, x) \in G = H \ltimes \mathcal{E}$, we have

$$(h, x) = \hat{\sigma}_s(x)(\exp(-sf(x))h, 0).$$

This is clearly a unique factorization of (h, x) into an element of the image $\hat{\mathcal{E}}_s = \hat{\sigma}_s(\mathcal{E})$ and an element of $H \cong H \times \{0\}$. This implies that $\hat{\mathcal{E}}_s$ is a *left transversal* of H in G, i.e., a subset of G consisting of one representative of each left coset in G/H.

Summarizing, we use the semidirect product structure of G to identify the homogeneous space $(H \ltimes \mathcal{E})/H$ with $\mathcal{E}$ itself, and we have a distinguished family of sections $\hat{\sigma}_s\colon\ \mathcal{E} \to G$ whose images are transversals of the subgroup H.

In order to motivate our next construction for $\mathcal{E}$, we will temporarily forget about differentiable structure. Let G be a group acting transitively on a set X. Fix a distinguished element $e \in X$, and let H be the isotropy subgroup of e. Associated to e is the canonical projection $\pi_X\colon\ G \to X; g \mapsto g(e)$. Assume that $\phi\colon\ X \to G$ is a section of π_X, i.e., $\pi_X(\phi(x)) = \phi(x)(e) = x$ for all $x \in X$, and assume that $\phi(e) = 1 \in G$. Let $\pi_H\colon\ G \to H$ be the corresponding projection onto H defined by $\pi_H(g) = (\phi(\pi_X(g)))^{-1} g$. We define a binary operation $\diamond\colon\ X \times X \to X$

and a mapping $l\colon X \times X \to H$ by

$$x \diamond y = \pi_X(\phi(x)\phi(y)) \tag{6.5}$$

$$l(x,y) = \pi_H(\phi(x)\phi(y)) \tag{6.6}$$

for $x, y \in X$. Then $(X, \diamond)$ is a *left loop* [9], i.e., given $a, b \in X$, the equation $a \diamond x = b$ has a unique solution $x \in X$, and $e \diamond a = a \diamond e = a$ for all $a \in X$. (Conversely, every left loop can be realized in this way, specifically, as a left transversal in a group.) For $a, b \in X$, the action of $l(a,b) \in H$ on X defines a permutation $L(a,b)\colon X \to X$, called a *left inner mapping*, by $L(a,b)(c) = l(a,b)(c)$ for all $c \in X$. Left inner mappings measure the nonassociativity of $(X, \diamond)$; they are equivalently defined by the equation $a \diamond (b \diamond c) = (a \diamond b) \diamond L(a,b)(c)$ for $a, b, c \in X$.

Two useful conditions on sections $\phi\colon X \to G$ are the following:

(H1) For each $x \in X$, there exists (a necessarily unique) $x' \in X$ such that

$$\phi(x)^{-1} = \phi(x').$$

(H2) For all $x \in X$, $h \in H$,

$$h\phi(x)h^{-1} = \phi(h(x)).$$

If (H1) holds, then the left loop $(X, \diamond)$ satisfies the *left inverse property*, which means that $a' \diamond (a \diamond b) = a \diamond (a' \diamond b) = b$ for all $a, b \in X$. If (H2) holds, then the action of H on X is by automorphisms of $(X, \diamond)$, i.e.

$$h(x \diamond y) = h(x) \diamond h(y)$$

for all $x, y \in X$, $h \in H$. In particular, every left inner mapping $L(a,b)$ is an automorphism, and in this case, $(X, \diamond)$ is said to have the A_l (or *left A* or *left special*) *property*. A left loop with both the left inverse and A_l properties is said to be *homogeneous*. For more on these matters, see [9] and the references therein.

Remark 6.1. Let G be a Lie group with closed (Lie) subgroup $H \subseteq G$ and a smooth section $\phi\colon G/H \to G$ of the natural projection $G \to G/H$. Kikkawa [7] showed that if (H1) and (H2) hold, then G/H is a reductive homogeneous space. The converse problem, which is to characterize reductive homogeneous spaces G/H such that there exists a smooth section $\phi\colon G/H \to G$ satisfying (H1) and (H2), is still open.

We apply the preceding notions to the sections $\hat{\sigma}_s\colon \mathcal{E} \to G$ in the group $G = H \ltimes \mathcal{E}$. For $x, y \in \mathcal{E}$, we have

$$\begin{aligned}\hat{\sigma}_s(x)\hat{\sigma}_s(y) &= (\exp(sf(x)), x)(\exp(sf(y)), y)\\ &= (\exp(sf(x))\exp(sf(y)), x + \exp(sf(x))y).\end{aligned}$$

Following (6.5), we take the projection of this product onto $\mathcal{E}$ to define a family of left loop structures $(\mathcal{E}, \diamond_s)$ by

$$x \diamond_s y = x + \exp(sf(x))y \tag{6.7}$$

for $x, y \in \mathcal{E}$. Also, following (6.6), we define a corresponding family of maps $l_s\colon \mathcal{E} \times \mathcal{E} \to H$ by

$$l_s(x, y) = \exp(s(-x - \exp(sf(x))y)) \exp(sf(x)) \exp(sf(y))$$

for $x, y \in \mathcal{E}$.

Before considering homogeneity, we first note the following important property.

PROPOSITION 6.2. *The left loops $(\mathcal{E}, \diamond_s)$ defined by (6.7) depend only on the Leibniz algebra structure $(\mathcal{E}, \cdot)$, and not on the choice of enveloping algebra $(\mathfrak{g}, \mathfrak{h}, f)$. In particular,*

$$x \diamond_s y = x + \exp(s\lambda(x))y \tag{6.8}$$

for all $x, y \in \mathcal{E}$.

Proof. Indeed, for $x, y \in \mathcal{E}$, we have

$$\exp(sf(x))y = \exp(s\lambda(x))y$$

using (2.13). This establishes the equivalence of (6.7) and (6.8). □

Next we turn to homogeneity. For each $x \in \mathcal{E}$, we have

$$\begin{aligned} \hat{\sigma}_s(x)^{-1} &= (\exp(sf(x)), x)^{-1} \\ &= (\exp(-sf(x)), -\exp(-sf(x))x). \end{aligned} \tag{6.9}$$

Now

$$\begin{aligned} \exp(-sf(x)) &= \exp(-s\,\mathrm{Ad}(\exp(-sf(x)))f(x)) \\ &= \exp(sf(-\exp(-sf(x))x)), \end{aligned}$$

using (6.4). Applying this to (6.9), we have

$$\hat{\sigma}_s(x)^{-1} = \hat{\sigma}_s(-\exp(-sf(x))x).$$

Thus (H1) holds, and the left loops $(\mathcal{E}, \diamond_s)$ satisfy the left inverse property. In particular, the inverse of x in $(\mathcal{E}, \diamond_s)$ is given by

$$x' = -\exp(-s\lambda(x))x.$$

The $\mathrm{Ad}_G(H)$-invariance of $\mathcal{E}$ implies that H normalizes $\hat{\sigma}_s(\mathcal{E})$. More precisely, for $h \in H$, $x \in \mathcal{E}$,

$$\begin{aligned}(h, 0)\hat{\sigma}_s(x)(h^{-1}, 0) &= (h \exp(sf(x))h^{-1}, hx) \\ &= (\exp(s\mathrm{Ad}(h)f(x)), hx) \\ &= (\exp(sf(hx), hx) \\ &= \hat{\sigma}_s(hx).\end{aligned}$$

Thus (H2) holds, and so $(\mathcal{E}, \diamond_s)$ has the A_l property. Summarizing, we have the following.

PROPOSITION 6.3. *Let $(\mathcal{E}, \cdot)$ be a Leibniz algebra, and for $s \neq 0$, let $\diamond_s$: $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$ be defined by*

$$x \diamond_s y = x + \exp(s\lambda(x))y$$

for $x, y \in \mathcal{E}$. Then $(\mathcal{E}, \diamond_s)$ is a homogeneous left loop for which the binary bracket in the associated Lie-Yamaguti algebra is $2s$ times the skew-symmetrized Leibniz product. In particular, the skew-symmetrized Leibniz product itself is recovered when $s = 1/2$.

In addition, $(\mathcal{E}, \diamond_s)$ is a *geodesic* left loop, which means that it agrees with the natural local left loop structure defined in a neighborhood of $0 \in \mathcal{E}$ by parallel transport of geodesics along geodesics [6, 7]. We note that the loop structure $\diamond_1$ was described in the case of Lie algebras by Kikkawa in Proposition 4 of [8].

Example 6.4. For the hemisemidirect product Leibniz algebra $\mathfrak{h} \ltimes_H V$ of Example 2.2, built from the representation ρ: $\mathfrak{h} \to \mathfrak{gl}(V)$, the left loop structures coming from Proposition 6.3 are

$$(\xi, x) \diamond_s (\eta, y) = (\xi + \exp(\mathrm{ad}(s\xi))(\eta), x + \exp(\rho(s\xi))y).$$

Example 6.5. For the Courant bracket (1.1) on $\mathcal{X}(P) \oplus \Omega^1(P)$, the left loop structures coming from Proposition 6.3 are

$$(\xi_1, \theta_1) \diamond_s (\xi_2, \theta_2) = (\xi, \theta),$$

where

$$\xi = \xi_1 + (\exp s\xi_1)^* \xi_2,$$

and

$$\theta = \theta_1 + (\exp s\xi_1)^*\theta_2 - (\exp s\xi_1)^*(\xi_2) \lrcorner d \int_0^s (\exp t\xi_1)^*\theta_1 dt.$$

7. Connections. The following classification theorem for G-invariant connections on reductive homogeneous spaces is due to Nomizu ([18], Thm. 8.1).

PROPOSITION 7.1. *Let $M = G/H$ be a reductive homogeneous space with fixed decomposition (6.1)–(6.2). There exists a one-to-one correspondence between the set of all G-invariant connections on M and the set of all $\mathrm{Ad}_G(H)$-equivariant bilinear mappings α: $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$.*

The $\mathrm{Ad}_G(H)$-equivariance of α: $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ means that $\mathrm{Ad}(h)\alpha(X,Y) = \alpha(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y)$ for all $h \in H$, $X, Y \in \mathfrak{m}$. Equivalently, if one thinks of $(\mathfrak{m}, \alpha)$ as a nonassociative algebra, $\mathrm{Ad}_G(H)$ is a subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{m}, \alpha)$.

Let $(\mathfrak{m}, [\![\cdot,\cdot]\!], \{\cdot,\cdot,\cdot\})$ be the Lie-Yamaguti algebra structure on $\mathfrak{m}$ determined by the decomposition (6.1)–(6.2); thus $[\![\cdot,\cdot]\!] = [\cdot,\cdot]_{\mathfrak{m}}$ and $\{\cdot,\cdot,\cdot\} = [[\cdot,\cdot]_{\mathfrak{h}},]$, where, as before, the subscripts indicate projections. The *zero* bilinear mapping on $\mathfrak{m}$ corresponds to the *canonical connection* (of the 2nd kind) on M. This connection is characterized by the following property: for each $X \in \mathfrak{m}$, parallel displacement of tangent vectors along the curve $\pi(\exp tX)$ is the same as the translation of tangent vectors by the natural action of $\exp tX$ on M. The torsion and curvature are

$$T(X,Y) = -[\![X,Y]\!] \tag{7.1}$$

$$R(X,Y)Z = -\{X,Y,Z\} \tag{7.2}$$

for $X, Y, Z \in \mathfrak{m}$.

If $M = G/H$ is a homogeneous left loop, Kikkawa showed that the canonical connection can be constructed directly from the loop multiplication ([7], Thm. 3.7). The torsion and curvature tensors then define a Lie-Yamaguti algebra structure on $\mathfrak{m}$ by (7.1)–(7.2), which is considered to be the tangent algebra structure of the homogeneous loop. In case $H = \{1\}$, the canonical connection is just Cartan's $(-)$-connection on G, and the corresponding Lie-Yamaguti algebra is the Lie algebra of G ([7], Ex. 3.3).

Now let $(\mathcal{E}, \cdot)$ be a Leibniz algebra with enveloping Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathcal{E}_s$, and homogeneous left loop structure $(\mathcal{E}, \diamond_s)$. We identify $\mathcal{E}$ with the reductive homogeneous space G/H. Following Kikkawa [7] (see also Miheev and Sabinin [15]), one finds that the canonical connection is given by

$$(\nabla_X Y)(x) = DY(x)X(x) - sX(x) \cdot Y(x) \tag{7.3}$$

for $x \in \mathcal{E}$. Here X and Y are vector fields on $\mathcal{E}$ which we are identifying with mappings $X, Y\colon\ \mathcal{E} \to \mathcal{E}$.

We observe that: (1) the reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathcal{E}_s$ give a one-parameter deformation of the semidirect product $\mathfrak{g} = \mathfrak{h} \ltimes \mathcal{E}$; (2) the connection (7.3) is a deformation of the standard flat, torsion-free connection on $\mathcal{E}$; (3) the loop structures $\diamond_s$ form a deformation of the addition operation on $\mathcal{E}$.

8. Further questions. We recall from §4 that Courant's algebra $\mathcal{E} = \mathcal{X}(P) \oplus \Omega^1(P)$ is a Courant algebroid; i.e. it is also the C^∞-module of sections of a vector bundle over P, and the bracket satisfies identities which relate it to the module structure. A simpler version of these identities defines *Lie algebroids* [14], which are the infinitesimal objects associated to Lie groupoids. The sections of a Lie algebroid form a Lie algebra which acts by derivations of $C^\infty(P)$, and the corresponding group of "bisections" of the groupoid acts by automorphisms, i.e., by diffeomorphisms of P. It has been our hope to find a group-like object associated to $\mathcal{E} = \mathcal{X}(P) \oplus \Omega^1(P)$ which has something like an action on P, and which can be considered as the sections of some kind of nonassociative generalization of a groupoid (a loopoid?). So far, we have not succeeded. The difficulty might be related to the absence of a natural adjoint representation of a Lie algebroid on itself (as opposed to the adjoint representation of the Lie algebra of sections). Perhaps a weak version of the adjoint representation, such as is described in the appendices of [4], could be a model for what we seek in the case of Courant algebroids.

Finally, we are left with the problem of constructing a group-like object attached to a Leibniz algebra in such a way that the object is a group when the Leibniz algebra is a Lie algebra. A possible approach to this problem is via path spaces. At the end of chapter 1 in [3], Duistermaat and Kolk prove "Lie's third theorem" by beginning with a Lie algebra $\mathcal{E}$ and defining a Banach Lie group structure on the space $\mathcal{P}(\mathcal{E})$ of continuous paths $\gamma\colon\ [0,1] \to \mathcal{E}$. When the Lie algebra of this group is identified with $\mathcal{P}(\mathcal{E})$ itself, the integration map $I\colon\ \gamma \mapsto \int_0^1 \gamma(t)dt$ is found to be a homomorphism from $\mathcal{P}(\mathcal{E})$ to $\mathcal{E}$. The closed ideal $\ker I$ integrates to a normal Lie subgroup $\mathcal{P}_0(\mathcal{E}) \subset \mathcal{P}(\mathcal{E})$ which is shown to be closed. The quotient $\mathcal{P}(\mathcal{E})/\mathcal{P}_0(\mathcal{E})$ is then a Lie group whose Lie algebra is isomorphic to $\mathcal{E}$. (Nothing comes for free. The proof that $\mathcal{P}_0(\mathcal{E})$ is closed relies on the same vanishing theorem for the second cohomology of a finite dimensional simply connected Lie group which goes into other proofs of Lie III. Of course, the result holds only when $\mathcal{E}$ is finite dimensional, as it should.)

Cattaneo and Felder [1] have used a similar path space construction to construct symplectic groupoids from Poisson manifolds. Their construction is a variation of the Duistermaat-Kolk construction applied to the cotangent bundle Lie algebroid. Rather than using an associative product corresponding to pointwise multiplication of group(oid) paths, they use concatenation of Lie algebroid paths, which becomes a groupoid structure only after an equivalence relation is applied;

the idea should be extendible to arbitrary Lie algebroids. Here, the resulting groupoid may have singularities; only the local groupoid is smooth.

We have begun to investigate the Duistermaat-Kolk construction when $(\mathcal{E}, \cdot)$ is a Leibniz algebra. If we use the same formula as in [3], the multiplication on $\mathcal{P}(\mathcal{E})$ is, remarkably, still associative, so we still have a Banach Lie group. On the other hand, the kernel $\mathcal{P}_0(\mathcal{E})$ of the integration map is no longer an ideal; in fact it is not even a subalgebra unless $\mathcal{E}$ is a Lie algebra. We can still recover the skew-symmetrized Leibniz bracket by identifying $\mathcal{E}$ with the constant paths, and projecting the path space Lie bracket along $\mathcal{P}_0(\mathcal{E})$, but since the latter is not a Lie subalgebra, it is not clear how to pass the group product to a quotient space.

Although the kernel $\mathcal{P}_0(\mathcal{E})$ is not a subalgebra, there is a different complement of the constant paths which *is* a subalgebra, namely the kernel of the evaluation map $E_{1/2}\colon \gamma \mapsto \gamma(1/2)$ (a crude "midpoint approximation" to I). The kernel $\ker E_{1/2}$ is also the corresponding Lie subgroup of $\mathcal{P}(\mathcal{E})$. Although the adjoint action of $\ker E_{1/2}$ does not leave the constant paths invariant, we can still use projection along $\ker E_{1/2}$ to construct Lie algebra and loop structures on $\mathcal{E}$. The projected bracket turns out to be the antisymmetrized Leibniz product $[\![\,.\,]\!]$, while the projected product is the same multiplication $\diamond_{1/2}$ which was obtained in §6 by a completely different method.

In a sense, then, we are back where we started. What we still need is a construction which incorporates the best properties of I (which produces the right group when $\mathcal{E}$ is a Lie algebra) and $E_{1/2}$ (whose kernel is a subalgebra for any Leibniz algebra $\mathcal{E}$), to reduce $\mathcal{P}(\mathcal{E})$ to manageable size. Our search will continue.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, INDIANA UNIVERSITY, SOUTH BEND, IN 46634
E-mail: mkinyon@iusb.edu
URL: http://www.iusb.edu/˜mkinyon

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720
E-mail: alanw@math.berkeley.edu
URL: http://www.math.berkeley.edu/˜alanw

REFERENCES

[1] A. S. Cattaneo and G. Felder, Poisson sigma models and symplectic groupoids, preprint, math.SG/0003023.
[2] T. J. Courant, Dirac manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 631–661.
[3] J. J. Duistermaat and J. A. C. Kolk, *Lie Groups*, Springer-Verlag, New York, 1999.
[4] S. Evens, J.-H. Lu, and A. Weinstein, Transverse measures, the modular class, and a cohomology pairing for Lie algebroids, *Quart. J. Math Oxford Ser.* **50** (1999), 417–436.

[5] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math.* **78** (1963), 267–288.
[6] M. Kikkawa, On local loops in affine manifolds, *J. Sci. Hiroshima Univ. Ser. A-I Math.* **28** (1964), 199–207.
[7] ———, Geometry of homogeneous Lie loops, *Hiroshima Math. J.* **5** (1975), 141–179.
[8] ———, Projectivity of left loops on R^n, *Mem. Fac. Sci. Shimane Univ.* **22** (1988), 33–41.
[9] M. K. Kinyon and O. Jones, Loops and semidirect products, *Comm. Algebra* **28** (2000), 4173–4164.
[10] Y. Kosmann-Schwarzbach, From Poisson algebras to Gerstenhaber algebras, *Ann. Inst. Fourier (Grenoble)* **46** (1996), 1243–1274.
[11] Z.-J. Liu, A. Weinstein, and P. Xu, Manin triples for Lie bialgebroids, *J. Differential Geom.* **45** (1997), 547–574.
[12] J. L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Enseign. Math.* **39** (1993), 269–293.
[13] J. L. Loday and T. Pirashvili, The tensor category of linear maps and Leibniz algebras, *Georgian Math. J.* **5** (1998), 263–276.
[14] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry, London Math. Soc. Lecture Note Ser.*, vol. 124, Cambridge Univ. Press, 1987.
[15] P. O. Miheev and L. V. Sabinin, Quasigroups and differential geometry, *Quasigroups and Loops: Theory and Applications* (O. Chein, H. O. Pflugfelder, and J. D. H. Smith, eds.), Heldermann-Verlag, Berlin, 1990, pp. 357–430.
[16] H. C. Myung, *Malcev-Admissible Algebras*, Birkhäuser, Boston, 1986.
[17] A. Nijenhuis, On a class of common properties of some different types of algebras, I, II, *Nieuw Arch. Wisk. III* **17** (1969) 17–46, 87–108.
[18] K. Nomizu, Invariant affine connections on homogeneous spaces, *Amer. J. Math.* **76** (1954), 33–65.
[19] D. Roytenberg, Courant algebroids, derived brackets and even symplectic supermanifolds, Ph.D. thesis, University of California, Berkeley, 1999, preprint math.DG/9910078.
[20] D. Roytenberg and A. Weinstein, Courant algebroids and strongly homotopy Lie algebras, *Lett. Math. Phys.* **46** (1998), 81–93.
[21] A. Weinstein, Omni-Lie algebras, preprint math.RT/9912190, *RIMS Kôkyûroku* **1176** (2000), 95–102.
[22] K. Yamaguti, On the Lie triple system and its generalization, *J. Sci. Hiroshima Univ. Ser. A-I Math.* **21** (1957/1958), 155–160.