Mendel’s laws were formulated in terms of non-associative algebras by I. M. H. Etherington in [Proc. Roy. Soc. Edinburgh 59 (1939), 242–258; MR0000597 (1,99e)] and [Proc. Roy. Soc. Edinburgh. Sect. B. 61 (1941), 24–42; MR003557 (2,237e)]. In these works, algebraic patterns outlined by Gregor J. Mendel in his famous papers of 1865 and 1866 were developed, by improving the mathematical approach for a formulation of Mendel’s laws given by Jennings (1917), Serebrovskii (1934) and Glivenko (1936).

Since then, many authors have made relevant contributions to genetics in the context of non-associative algebras: Schafer, Gonshor, Haldane, Hoigate, Heuch, Reiersøl, Abraham, Lyubich and Wörz-Busekros, among others. That is why non-associative algebras are considered an adequate theoretical framework to address important topics in genetics. In fact, there are so many non-associative algebras that have attracted the interest of geneticists that it would be difficult to make an exhaustive list of them. Let us mention, as an example, the following classes of algebras: Mendelian algebras, gametic algebras, zygotic algebras, baric algebras, train algebras, copular algebras, Bernstein algebras and evolution algebras. These algebras, and generally all the algebras used to model inheritance in genetics, are referred to as genetic algebras. For surveys about genetic algebras, see [M. Bertrand, Algèbres non associatives et algèbres génétiques, Mémorial des Sciences Mathématiques, Fasc. 162, Gauthier-Villars Éditeur, Paris, 1966; MR0215885 (35 #6720); A. Wörz-Busekros, Algebras in genetics, Lecture Notes in Biomathematics, 36, Springer, Berlin, 1980; MR0599179 (82e:92033); Yu. I. Lyubich, Mathematical structures in population genetics, English translation, Biomathematics, 22, Springer, Berlin, 1992; MR1224676 (95f:92018); M. L. Reed, Bull. Amer. Math. Soc. (N.S.) 34 (1997), no. 2, 107–130; MR1414973 (98e:17043)]. Usually these algebras are non-associative algebras that, nevertheless, do not belong to any of the standard classes of non-associative algebras, such as Lie, Jordan, alternative or power associative algebras. In the words of M. L. Reed, “In addition, many of the algebraic properties of these structures have genetic significance. Indeed, it is the interplay between the purely mathematical structure and the corresponding genetic properties that makes this subject so fascinating.”

The evolution algebras are a particular class of genetic algebras, relevant because of its close connection with other mathematical fields like dynamical systems, stochastic processes, graph theory, group theory, or mathematical physics, for instance. Evolution algebras are non-associative Banach algebras which are not even power-associative in general.

An algebra is a linear space $E$ provided with a product, i.e. a bilinear map $(a, b) \mapsto ab$, $E \times E \to E$. If an algebra $E$ admits a basis $B = \{e_1, \ldots, e_n\}$ such that $e_i e_j = 0$, for $i \neq j$, then it is said that $E$ is an evolution algebra, and that $B = \{e_1, \ldots, e_n\}$ is a natural basis of $E$. Given a natural basis $B = \{e_1, \ldots, e_n\}$ in an evolution algebra $E$, if

$$e_i^2 = e_i e_i = \sum_{k=1}^{n} a_{ik} e_k,$$
then it is said that \( A = (a_{ij}) \) is the matrix of the structural constants of the evolution algebra \( E \). This matrix encodes somehow the laws of inheritance as algebraic properties of the algebra [see J. P. Tian, *Evolution algebras and their applications*, Lecture Notes in Math., 1921, Springer, Berlin, 2008; MR2361578 (2008m:17052)].

Let \( E \) and \( E' \) be evolution algebras and let \( B = \{e_1, \ldots, e_n\} \) be a natural basis on \( E \). A linear map \( \varphi: E \to E' \) is said to be a homomorphism of evolution algebras if \( \varphi(ab) = \varphi(a) \varphi(b) \), for every \( a, b \in E \), and the set \( \{\varphi(e_i)\} \) is a natural basis of \( E' \). Moreover, if \( \varphi \) is bijective then it is said to be an isomorphism of evolution algebras.

In the paper reviewed here, the classification of 2-dimensional complex evolution algebras is obtained (Theorem 4.1). Indeed, it is proved that if \( E \) is any 2-dimensional complex evolution algebra, with matrix of structural constants \( A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \) with \( \det A \neq 0 \), then \( E \) is isomorphic to one of the following pairwise non-isomorphic algebras:

(i) \( \dim E^2 = 1 \):
- \( E_1: \begin{pmatrix} e_1^2 = 1 \\ e_2^2 = 0 \end{pmatrix} \)
- \( E_2: \begin{pmatrix} e_1^2 = e_1 \\ e_2^2 = e_1 \end{pmatrix} \)
- \( E_3: \begin{pmatrix} e_1^2 = e_1 + e_2 \\ e_2^2 = -(e_1 + e_2) \end{pmatrix} \)
- \( E_4: \begin{pmatrix} e_1^2 = e_2 \\ e_2^2 = 0 \end{pmatrix} \)

(ii) \( \dim E^2 = 2 \):
- \( E_5: \begin{pmatrix} e_1^2 = 1 + a_2 e_2, e_2^2 = a_3 e_1 + e_2, 1 - a_2 a_3 \neq 0 \), where \( E_5(a_2, a_3) \cong E_5(a_3, a_2) \);
- \( E_6: \begin{pmatrix} e_1^2 = e_2, e_2^2 = e_1 + a_4 e_2 \), where for \( a_4 \neq 0 \), \( E_6(a_4) \cong E_6(a'_4) \) if and only if \( a_4^2 = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3} \) for some \( k = 0, 1, 2 \).

An element \( a \) of an evolution algebra \( E \) is called nil if there exists \( n(a) \in \mathbb{N} \) such that \( (aa) \cdots a = 0 \). An evolution algebra \( E \) is called nil if every element of \( E \) is nil. On the other hand, for an evolution algebra \( E \) let us define \( E^{(1)} = E \), and \( E^{(k+1)} = \text{Center}(E^{(k)}) \) for \( k \geq 1 \). It is said that the evolution algebra \( E \) is right nilpotent if \( E^{(s)} = 0 \), for some \( s \in \mathbb{N} \).

If \( E \) is a right nilpotent evolution algebra, then clearly \( E \) is a nil algebra. In fact, as proved in Theorem 2.7 of this paper, the property of being nil and right nilpotence are equivalent for evolution algebras, and are equivalent to the matrix \( A \) of structural constants being upper triangular after a suitable permutation of the basis.

By investigating the six families of two-dimensional complex evolution algebras described above, the authors get the following table summarizing their properties:

<table>
<thead>
<tr>
<th>( E_i )</th>
<th>( \dim E^2 )</th>
<th>Right nilpotency</th>
<th>( \dim(\text{Center}) )</th>
<th>Nil elements</th>
<th>Solvability</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_1 )</td>
<td>1</td>
<td>No</td>
<td>1</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>1</td>
<td>No</td>
<td>0</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>( E_4 )</td>
<td>1</td>
<td>No</td>
<td>0</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( E_5 )</td>
<td>2</td>
<td>Yes</td>
<td>1</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>2</td>
<td>No</td>
<td>0</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

In the appendix to this paper, the authors provide an algorithm (running under Mathematica) which decides whether two finite-dimensional evolution algebras are isomorphic.

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