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Maynard, James [Maynard, James Alexander] (4-OXM)
Small gaps between primes. (English summary)
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In the present paper, the author makes significant progress in the bounds of gaps between primes. More explicitly, denote by p_n the n-th prime; then it is proved that

$$\liminf_n (p_{n+1}-p_n) \leqslant 600$$

and

$$\liminf_n (p_{n+m}-p_n) \ll m^3 e^{4m},$$

improving the corresponding results obtained by using Y. T. Zhang's method [see, for example, W. Castryck et al., Algebra Number Theory **8** (2014), no. 9, 2067–2199; MR3294387; Y. T. Zhang, Ann. of Math. (2) **179** (2014), no. 3, 1121–1174; MR3171761]. The author also proves that

$$\liminf_n (p_{n+1}-p_n) \leqslant 12$$

and

$$\liminf_n (p_{n+2}-p_n) \leqslant 600$$

under the Elliott-Halberstam conjecture.

The method used here is a refinement of Goldston, Pintz and Yıldırım's (GPY's) method [Ann. of Math. (2) **170** (2009), no. 2, 819–862; MR2552109 (2011c:11146)]. The basic idea is to consider the sum

$$\sum_{N\leqslant n<2N} \left(\sum_{i=1}^{k} \chi_{\mathbb{P}}(n+h_i) - \rho \right) w_n,$$

where $\mathcal{H}=\{h_1,\dots,h_k\}$ is a fixed admissible set, $\chi_{\mathbb{P}}$ is the characteristic function of the primes and w_n are non-negative weights. In the previous works based on GPY's method the w_n are usually chosen to mimic A. Selberg's sieve weights, while in this paper

$$w_n = \left(\sum_{d_i \mid n+h_i,\ \forall\, i} \lambda_{d_1,\dots,d_k} \right)^2,$$

where

$$\lambda_{d_1,\dots,d_k} = \left(\prod_{i=1}^{k} \mu(d_i) d_i \right) \sum_{\substack{r_1,\dots,r_k \\ d_i \mid r_i \forall\, i \\ (r_i,W)=1,\ \forall\, i}} \frac{\mu(\prod_{i=1}^k r_i)^2}{\prod_{i=1}^k \varphi(r_i)} F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_k}{\log R} \right),$$

with W, R and F being suitably chosen. The idea of this weight can be traced back to Selberg [see *Collected papers. Vol. II*, Springer, Berlin, 1991; MR1295844 (95g:01032)], and the superiority is that Bombieri-Vinogradov theorem is sufficient in the proof (note that in the original paper of Goldston, Pintz and Yıldırım it is an essential barrier). With the help of some numerical calculations, the desired results can be obtained by choosing F as a special symmetric polynomial. *Ya Ming Lu*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.