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The bounded gap conjecture and bounds between consecutive Goldbach numbers. **(English summary)**

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It is not surprising that there are connections between the distribution of primes and the Goldbach conjecture. The Goldbach conjecture can be formulated as a result about consecutive Goldbach numbers (even numbers which can be written as the sum of two primes), namely, $g_{n+1} - g_n = 2$ for all $n \geq 1$. It is interesting to note that it is exactly the problem of estimating the bounds between consecutive Goldbach numbers. Then it is natural to expect the following weaker conjecture.

Conjecture 1. $g_{n+1} - g_n \ll g_n^\varepsilon$ for any $\varepsilon > 0$.

In this paper the author shows that some unexpected distribution of primes, namely, the non-existence of infinitely many bounded gaps between primes, implies Conjecture 1.

Conjecture 2. (Bounded Gap Conjecture). We have

$$\liminf_{n\to\infty}(p_{n+1} - p_n) = \infty.$$

Conjecture 2 trivially follows from the following conjecture.

Conjecture 3. If k is sufficiently large ($k > k_0$, an absolute constant) then for an arbitrary admissible k-tuple $\mathfrak{H}$, the set $n + \mathfrak{H} := \{n + h_i\}_{i=1}^k$ contains at least two primes for infinitely many values of n.

In this paper it is proved that at least one of Conjectures 1 and 2 is true. In fact the author proves the result in a stronger form: At least one of Conjectures 1 and 3 is true.

It should be remarked that recently Conjecture 3 has been proved by Y. T. Zhang [Ann. of Math. (2) **179** (2014), no. 3, 1121–1174; MR3171761]. The breakthrough theorem of Zhang states that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) < 7 \times 10^7,$$

where p_n is the n-th prime. *Guangshi Lü*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.