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Bounded gaps between primes. (English summary)
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Historically, much of the progress in sieve methods has been driven by the Twin Prime Conjecture, which states that there are infinitely many primes p such that $p+2$ is also prime. This could also be stated by saying that $p_{n+1}-p_n=2$ infinitely often. More generally, let $\mathcal{H}=\{h_1<\cdots<h_k\}$ and $\nu_p(\mathcal{H})$ be the number of distinct residue classes modulo p in $\mathcal{H}$. If $\nu_p(\mathcal{H})<p$ for all primes p, we say that $\mathcal{H}$ is admissible. The prime k-tuple conjecture is that if $\mathcal{H}$ is admissible, then there are infinitely many integers n such that $n+h_1,\dots,n+h_k$ are all prime. The Twin Prime Conjecture is the special case $\mathcal{H}=\{0,2\}$.

The primary result of this paper is the following:

Theorem. Suppose that $\mathcal{H}$ is admissible with $k\geq 3.5\times 10^6$. Then there are infinitely many positive integers n such that the set

$$\{n+h_1,n+h_2,\dots,n+h_k\}$$

contains at least two primes. Consequently,

$$\liminf_{n\to\infty}(p_{n+1}-p_n)<7\times 10^7.$$

This is a very significant result because it is the first proof that there are infinitely many prime gaps smaller than some given constant. The starting point is a conditional result due to D. A. Goldston, J. Pintz and C. Y. Yıldırım [Ann. of Math. (2) **170** (2009), no. 2, 819–862; MR2552109 (2011c:11146)].

We say that the primes have level of distribution θ if for any $\varepsilon>0$,

$$\sum_{q\leq N^{\theta-\varepsilon}}\max_{\substack{a\\(a,q)=1}}\left|\sum_{\substack{p\leq N\\p\equiv a\pmod q}}\log p-\frac{N}{\varphi(q)}\right|\ll\frac{N}{(\log N)^{-A}}.$$

Unconditionally, we know that the primes have level of distribution $1/2$; this is the Bombieri-Vinogradov Theorem. P. D. T. A. Elliott and H. Halberstam [in *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, 59–72, Academic Press, London, 1970; MR0276195 (43 #1943)] conjectured that the primes have level of distribution 1. Goldston, Pintz, and Yıldırım proved that if the primes have a level of distribution $\theta>1/2$, then there exists an explicitly calculable constant $C(\theta)$ depending only on θ such that for any admissible k-tuple $\mathcal{H}$ with $k\geq C(\theta)$, the set $\{n+h_1,\dots,n+h_k\}$ contains at least two primes infinitely often. From this, one sees immediately that

$$\liminf_{n\to\infty}(p_{n+1}-p_n)\leq h_k-h_1.$$

For example, they proved that if θ is sufficiently close to 1, then one may take $k=6$. From the admissible set $\{7,11,13,17,19,23\}$, one obtains $\liminf_{n\to\infty}(p_{n+1}-p_n)\leq 16$ provided the Elliott-Halberstam Conjecture is true.

It has long been known that it is possible to get a level of distribution larger than $1/2$ provided certain restrictions are placed on the residue classes [see, for example, É. Fouvry and H. Iwaniec, Acta Arith. **42** (1983), no. 2, 197–218; MR0719249 (84k:10035); E. Bombieri, J. B. Friedlander and H. Iwaniec, J. Amer. Math. Soc. **2** (1989), no. 2, 215–

224; MR0976723 (89m:11087)]. However, these theorems require that the residue class $a \pmod q$ be fixed, and this requirement is incompatible with the Goldston-Pintz-Yıldırım method.

The central idea behind the proof of Goldston, Pintz, and Yıldırım is to consider the sum

$$\sum_{N<n\leq 2N} \left(\sum_{i=1}^{k} \theta(n+h_i) - \log 3N \right) \Lambda_R(n;\mathcal{H})^2,$$

where $\theta(n) = \log n$ if n is a prime and 0 otherwise, and the choice of $\Lambda_R(n;\mathcal{H})$ is motivated by the weights in the usual Selberg sieve. More specifically, $\Lambda_R(n;\mathcal{H})$ is a sum of the form

$$\sum_{d|(n+h_1)(n+h_2)\cdots(n+h_k)} \lambda_d.$$

Y. Motohashi and Pintz [Bull. Lond. Math. Soc. **40** (2008), no. 2, 298–310; MR2414788 (2009d:11132)] observed that the results are not materially changed if one restricts this sum to smooth d, that is, values of d with no large prime divisors. Zhang makes critical use of the same idea in this paper. Here, the residue classes $a \pmod q$ run over the roots of the polynomial

$$\prod_{h,h'\in\mathcal{H};h\neq h'} (x+h'-h).$$

There is no issue when q is prime because the number of residue classes is bounded. However, when q is composite, the residue classes a can become quite large. Zhang's approach is to apply the Chinese Remainder Theorem before opening the dispersion. The estimates are adequate only for smooth moduli, but this is sufficient because of the above-mentioned observation of Motohashi and Pintz.

This paper has already motivated considerable other research on prime gaps. Zhang states that his "result is, of course, not optimal". Several authors, including Trudigan and Pintz, found improvements in Zhang's arguments that lead to smaller bounds for the gaps. The most successful of these efforts was the Polymath8a project led by T. Tao [D. H. J. Polymath, "New equidistribution estimates of Zhang type, and bounded gaps between primes", preprint, arXiv:1402.0811]. This group employed the same basic approach as Zhang, but they made numerous technical improvements, and they were able to show that $p_{n+1} - p_n \leq 4680$ infinitely often. J. B. Friedlander and H. Iwaniec ["Close encounters among the primes", preprint, arXiv:1312.2926] gave an alternate exposition of Zhang's work. J. Maynard ["Small gaps between primes", preprint, arXiv:1311.4600] found an alternate approach by using

$$\Lambda_R(n;\mathcal{H}) = \sum_{d_1|(n+h_1),\dots,d|(n+h_k)} \lambda_{d_1,d_2,\dots,d_k}.$$

With this approach, he proved that

$$p_{n+1} - p_n \leq 600$$

infinitely often. The only distributional hypothesis necessary for this work is the Bombieri-Vinogradov Theorem. Further work by the Polymath group (unpublished) has reduced the bound to 246. *S. W. Graham*

References

1. Bombieri, J. B. Friedlander, and H. Iwaniec, Primes in arithmetic progressions to large moduli, *Acta Math.* **156** (1986), 203–251. MR 0834613. Zbl 0588.10042.

http://dx.doi.org/10.1007/BF02399204. MR0834613 (88b:11058)

2. E. Bombieri, J. B. Friedlander, and H. Iwaniec, Primes in arithmetic progressions to large moduli. II, *Math. Ann.* **277** (1987), 361–393. MR 0891581. Zbl 0625.10036. http://dx.doi.org/10.1007/BF01458321. MR0891581 (88f:11085)
3. E. Bombieri, J. B. Friedlander, and H. Iwaniec, Primes in arithmetic progressions to large moduli. III, *J. Amer. Math. Soc.* **2** (1989), 215–224. MR 0976723. Zbl 0674.10036. http://dx.doi.org/10.2307/1990976. MR0976723 (89m:11087)
4. P. Deligne, La conjecture de Weil. I, *Inst. Hautes Études Sci. Publ. Math.* **43** (1974), 273–307. MR 0340258. Zbl 0287.14001. http://dx.doi.org/10.1007/BF02684373. MR0340258 (49 #5013)
5. E. Fouvry and H. Iwaniec, Primes in arithmetic progressions, *Acta Arith.* **42** (1983), 197–218. MR 0719249. Zbl 0517.10045. Available at http://pldml.icm.edu.pl/pldml/element/bwmetal.element. bwnjournal-article-aav42i2pl97bwm?q=bwmetal.element. bwnjournal-number-aa-1982–1983-42–2;5&qt=CHILDREN-STATELESS. MR0719249 (84k:10035)
6. J. B. Friedlander and H. Iwaniec, Incomplete Kloosterman sums and a divisor problem, *Ann. of Math.* **121** (1985), 319–350. MR 0786351. Zbl 0572.10029. http://dx.doi.org/10.2307/1971175. MR0786351 (86i:11050)
7. D. A. Goldston, J. Pintz, and C. Y. Yildirim, Primes in tuples. I, *Ann. of Math.* **170** (2009), 819–862. MR 2552109. Zbl 1207.11096. http://dx.doi.org/10.4007/annals.2009.170.819. MR2552109 (2011c:11146)
8. D. A. Goldston, J. Pintz, and C. Y. Yildirm, Primes in tuples. II, *Acta Math.* **204** (2010), 1–47. MR 2600432. Zbl 1207.11097. http://dx.doi.org/10.1007/s11511-010-0044-9. MR2600432 (2011f:11121)
9. D. R. Heath-Brown, Prime numbers in short intervals and a generalized Vaughan identity, *Canad. J. Math.* **34** (1982), 1365–1377. MR 0678676. Zbl 0478.10024. http://dx.doi.org/10.4153/CJM-1982-095-9. MR0678676 (84g:10075)
10. D. R. Heath-Brown, The divisor function $d_3(n)$ in arithmetic progressions, *Acta. Arith.* **47** (1986), 29–56. MR 0866901. Zbl 0549.10034. Available at http://pldml.icm.edu.pl/pldml/element/bwmetal.element. bwnjournal-article-aav47ilp29bwm?q=bwmetal.element. bwnjournal-number-aa-1986–47-1;2&qt=CHILDREN-STATELESS. MR0866901 (88a:11088)
11. H. Iwaniec, A new form of the error term in the linear sieve, *Acta Arith.* **37** (1980), 307–320. MR 0598883. Zbl 0444.10038. Available at http://pldml.icm.edu.pl/pldml/element/bwmetal.element. bwnjournal-article-aav37i1p307bwm?q=bwmetal.element. bwnjournal-number-aa-1980–37-1;26&qt=CHILDREN-STATELESS. MR0598883 (82d:10069)
12. P. Shiu, A Brun-Titchmarsh theorem for multiplicative functions, *J. Reine Angew. Math.* **313** (1980), 161–170. MR 0552470. Zbl 0412.10030. http://dx.doi.org/10.1515/crll.1980.313.161. MR0552470 (81h:10065)
13. K. Soundararajan, Small gaps between prime numbers: the work of Goldston-Pintz-Yildirim, *Bull. Amer. Math. Soc.* **44** (2007), 1–18. MR 2265008. Zbl 1193. 11086. http://dx.doi.org/10.1090/S0273-0979-06-01142-6. MR2265008 (2007k:11150)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.