A Bernstein algebra $A$ over a commutative ring $K$ is one that admits a homomorphism $w: A \to K$ and satisfies the identity $(x^2)^2 = \{w(x)\}^2 x^2$, $x \in A$. Relative to each idempotent $e$, Ker $w$ has a Pierce decomposition $U \oplus V$, such that $ex = \frac{1}{2}x$ for $x \in U$, $ex = 0$ for $x \in V$. Jacobi’s identity holds in $U$, $x(yz) + y(xz) = 0$ for $x,y \in U$, $z \in V$, $(xy)(zt) + (xz)(yt) + (xt)(yz) = 0$ for $x,y,z \in$ Ker $w$, $t \in A$, and $U^2 \subset V$, $UV \subset U$, $V^2 \subset U$, $UV^2 = 0$.

The authors begin with the study of derivations of Bernstein algebras. Theorem 3.1 gives the necessary and sufficient conditions for a linear mapping $d$ to be a derivation when char $K \neq 2$. They include $d(e) \in U$ and the fact that $d$ has the representations $d(x) = f_d(x) + 2erd(e)$, $d(x) = -2erd(e) + gd(x)$ where $f_d,g_d$ are endomorphisms of $U,V$, respectively, corresponding to $d$ in a morphism of Lie algebras $f: \text{Der}_K(A) \to \text{End}_K(U)$, $g: \text{Der}_K(A) \to \text{End}_K(V)$. The remaining conditions relate to the effects of $f$ and $g$ on $U,V$. As a corollary we have $w \cdot d = 0$. The type of a Bernstein algebra is $(\dim U + 1, \dim V)$. The theorem is applied to Bernstein algebras of the extreme types. If $\dim A = n + 1$, type $A = (n + 1,0)$, then $\text{Der}_K(A)$ is isomorphic to $K^n \times M_n(K)$, the product being semidirect and $M_n$ the full matrix algebra over $K$. If type $A = (1,n)$, then $\text{Der}_K(A)$ is isomorphic to $M_n(K)$.

The next section deals with derivations in the case char $K = 2$. Here, the conditions include $d(e) = 0$, a decomposition $d(x) = (w \cdot d)(x)e + f_d(x)$, and an appropriately modified set of detailed identities. In order that $w \cdot d = 0$, it is necessary and sufficient that $f$ should be injective. Further interesting results are obtained for char $K = 2$, the first time that this case has been extensively studied in the context of genetic algebras. The next sections deal with the automorphism group of $A$ for char $K \neq 2$, = 2, respectively. The general theorems are, mutatis mutandis, related to those on derivations, but the exposition is illustrated by a wide range of examples. An important role is played by the abelian group $K(\theta)$, $\theta \in K$. It comprises those elements $\lambda \in K$ such that $1 - 4\lambda \theta \in U(K)$, $U$ the group of invertible elements of $K$ (an unfortunate clash of notation), with addition defined by $\lambda + \lambda' = \lambda + \lambda' - 4\lambda \lambda'$. In some cases Aut $K(A)$ is isomorphic to $K(\theta)$. If type $A = (n + 1,0)$, then Aut $K(A)$ is isomorphic to $I_p(A) \times \text{GL}_K(U)$, while if type $A = (1,n)$, Aut $K(A)$ is isomorphic to $\text{GL}_K(V)$. The final section examines in detail the cases of all Bernstein algebras of dimension 3.

(See also the following review.)

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