On the structure of Bernstein algebras.


A finite-dimensional commutative algebra $A$ over a field $\Phi$, equipped with an algebra morphism $\omega: A \to \Phi$, is called a Bernstein algebra if, for all $x \in A$, $(x^2)^2 = \omega(x^2)x^2$.

Every Bernstein algebra has a nonzero idempotent. In characteristic $\neq 2$, if $e \in A$ is such an idempotent, $A$ has the Peirce decomposition $A = \Phi \oplus U_e \oplus V_e$, where $U_e = \{x \in A | 2ex = x\}$, $V_e = \{x \in A | ex = 0\}$ and $\ker(\omega) = U_e \oplus V_e$. The Peirce components satisfy $U_eV_e \subseteq U_e$, $V_e^2 \subseteq U_e$, $U_e^2 \subseteq V_e$, and $U_eV_e^2 = 0$. The set of idempotents of $A$ is given by $I(A) = \{e + u + u^2 | u \in U_e\}$. For two idempotents $e$ and $f = e + u + u^2$, one has the following relations among the Peirce components: $U_f = \{x + 2xu | x \in U_e\}$ and $V_f = \{x - 2x(u + u^2) | x \in V_e\}$. The numbers $\dim U_e$ and $\dim V_e$ being invariants of $A$, the pair $(1 + \dim U_e, \dim V_e)$ is called the type of $A$.

In the first paper the authors define the direct product of Bernstein algebras and the usual notions of decomposable and indecomposable Bernstein algebra. The classical Krull-Schmidt theorem is then given.

A commutative algebra $A$ is a Jordan algebra if $x^2(yx) = (x^2y)x$ for all $x \in A$. After recalling the characterization theorem for Bernstein-Jordan algebras, the authors define $J(A)$ as the smallest ideal of $A$ such that $A/J(A)$ is a Jordan algebra. In this way they show that the correspondence $F: A \to A/J(A)$ is a functor preserving the direct product. A Bernstein algebra is said to be reduced if the ideal $U_0(A) = U_e \cap \text{Ann} U_e = 0$. It is known that $A/U_0(A)$ is a Bernstein-Jordan algebra. The authors show that $J(A) \subseteq U_0(A)$ and that $U_0(A/J_0(A)) = U_0(A)/U_0(A) = 0$. Then they establish a sharper version of the Krull-Schmidt theorem for reduced nuclear Bernstein algebras, where a Bernstein algebra is said to be nuclear if $A^2 = A$.

In the second paper the authors use these ideas to give a classification of reduced Bernstein algebras of dimension $\leq 5$, and then derive, via several lemmas, a classification of Bernstein-Jordan algebras of dimension $\leq 5$. 

\textcopyright Copyright American Mathematical Society 1996, 2015

\textbf{Moussa Ouattara}