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On some "minimal" Leibniz algebras

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The aim of this paper is to describe some "minimal" Leibniz algebras, that are the Leibniz algebras whose proper subalgebras are Lie algebras, and the Leibniz algebras whose proper subalgebras are abelian.

Keywords: Leibniz algebra; Lie algebra; cyclic subalgebra; left center; right center; center of Leibniz algebra; nilpotent subalgebras; abelian subalgebras.

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1. Introduction

An algebra L over a field F is said to be a Leibniz algebra (more precisely a left Leibniz algebra) if its second binary operation (commutation, [,]), if it satisfies the Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]], \quad \text{for all } a,b,c \in L. \tag{1.1}$$

Leibniz algebras are generalizations of Lie algebras. Indeed, a Leibniz algebra L is a Lie algebra if and only if $[a,a] = 0$ for every element $a \in L$. By this reason, we may consider Leibniz algebras as "nonanticommutative" analogs of Lie algebras.

Leibniz algebra first appeared in the papers of Bloh [1–3], in which he called them D-algebras. However, real interest to the subject arose after the work of Loday [12], who introduced the term Leibniz algebras. The term was chosen since it was Leibniz, who discovered and proved the "Leibniz rule" for differentiation of functions. The Leibniz algebras appeared to be linked to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. They found some applications in physics (see, for example, [4, 11, 14]). Some papers concerning Leibniz algebras are devoted to the study of homological problems [5, 7, 13]. The theory of Leibniz algebras has been developing quite intensively but very uneven. On one hand, some deep structural theorems were obtained as analogues of the corresponding results about Lie algebras. On the other hand, there are some issues that seemingly should have been considered first which were not even touched. Thus, we were not able to find any description of the cyclic subalgebras in Leibniz algebras. More precisely, we found descriptions for some various partial cases, but not for a general case. So, we decided to fill this gap. The first main result of our work is the following theorem. But first, let us recall some of the important concepts.

Let L be a Leibniz algebra over a field F. If M is a nonempty subset of L then $\langle M\rangle$ denote the generated by M subalgebra of L.

If A and B are subspaces of L, then denote by $[A, B]$ the subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$.

A Leibniz algebra L has one specific ideal. Denote by $\mathrm{Leib}(L)$ the subspace generated by the elements $[a, a]$, $a \in L$.

We note that $\mathrm{Leib}(L)$ is an ideal of L. Indeed,

$$[a, [a, x]] = [[a, a], x] + [a, [a, x]], \quad \text{so } [[a, a], x] = 0.$$

Furthermore,

$$\begin{aligned}[x + [a, a], x + [a, a]] &= [x, x] + [x, [a, a]] + [[a, a], x] + [[a, a], [a, a]]\\ &= [x, x] + [x, [a, a]].\end{aligned}$$

It follows that $[x, [a, a]] = [x + [a, a], x + [a, a]] - [x, x] \in \mathrm{Leib}(L)$.

We also remark that if H is an ideal of L such that L/H is a Lie algebra, then $\mathrm{Leib}(L) \le H$.

The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel of algebra L*.

We also note the following important property of the Leibniz kernel:

$$[[a, a], x] = 0 \quad \text{for arbitrary elements } a, x \in L.$$

Let L be a Leibniz algebra. Define the lower central series of L

$$L = \gamma_1(L) \ge \gamma_2(L) \ge \cdots \gamma_\alpha(L) \ge \gamma_{\alpha+1}(L) \ge \cdots \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for limit ordinals λ. The last term $\gamma_\delta(L)$ is called the *lower hypocenter of L*.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, \ldots [L, L] \ldots]]$ is a *left normed commutator* of k copies of L.

A Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $\mathrm{ncl}(L)$.

The *left* (respectively *right*) *center* $\zeta^{\mathrm{left}}(L)$ (respectively $\zeta^{\mathrm{right}}(L)$) of a Leibniz algebra L is defined by the rule

$$\zeta^{\mathrm{left}}(L) = \{x \in L \mid [x, y] = 0, \text{ for each element } y \in L\}$$

respectively

$$\zeta^{\mathrm{right}}(L) = \{x \in L \mid [y, x] = 0, \text{ for each element } y \in L\}.$$

It is not hard to prove that the left center of L is an ideal. Moreover, $\mathrm{Leib}(L) \leq \zeta^{\mathrm{left}}(L)$, so that $L/\zeta^{\mathrm{left}}(L)$ is a Lie algebra. In general, the left and the right centers are different, moreover, the left center is an ideal, but it is not true for the right center. One can find the corresponding counterexample in [10].

The *center* $\zeta(L)$ *of* L is defined by the rule

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x], \text{ for each element } y \in L\}.$$

The center is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

As usual, a Leibniz algebra L is called *abelian* if $[x, y] = 0$ for all elements $x, y \in L$. We note that the left and right centers are abelian subalgebras.

We now define the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \cdots \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \cdots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter of* L. A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

Let L be a Leibniz algebra over a field F and d be an element of L. Put

$$\mathrm{ln}_1(d) = d, \quad \mathrm{ln}_2(d) = [d, d], \quad \mathrm{ln}_{k+1}(d) = [d, \mathrm{ln}_k(d)], \quad k \in \mathbb{N}.$$

These elements are called the *left normed commutators of the element* d.

We will further prove that a cyclic subalgebra $\langle d \rangle$ is generated as a subspace by the elements $\mathrm{ln}_k(d)$, $k \in \mathbb{N}$. The following two natural cases appear here.

The elements $d_j = \mathrm{ln}_j(d)$, $j \in \mathbb{N}$ are linearly independent. In this case, a subalgebra $D = \langle d \rangle$ has the lower central series

$$D = \gamma_1(D) \geq \gamma_2(D) \geq \cdots \geq \gamma_j(D) \geq \gamma_{j+1}(D) \geq \cdots \langle 0 \rangle$$

of the length ω, and $\gamma_j(D) = \bigoplus_{t \geq j} F d_t$, $j \in \mathbb{N}$. In this case, we will say that an *element* d *has infinite depth*.

Now consider the second possibility, i.e. the case when elements $d_j = \mathrm{ln}_j(d)$, $j \in \mathbb{N}$, are not linearly independent. In this case, we will show that the subalgebra

$D = \langle d \rangle$ has finite dimension over F. In this setting, we will say that an *element d has finite depth*. Let k be the least positive integer such that $\ln_1(d), \dots, \ln_k(d)$ are linearly independent, but $\ln_1(d), \dots, \ln_k(d), \ln_{k+1}(d)$ are not linearly independent. It is possible to show that in this case, $D = F\ln_1(d) + \cdots + F\ln_k(d)$. In particular, the subset $\{\ln_1(d), \dots, \ln_k(d)\}$ is a basis of D and $\dim_F(D) = k$. In this case, we can say that an *element d has depth k*.

The case when an element d has finite depth turned out to be much more diverse. The following theorem has described this case.

Theorem 1.1. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Suppose that an element a has finite depth. Then D is an algebra of one of the following types*:

(i) $D = Fa$ *is abelian*, $[a, a] = 0$.
(ii) *There exists a positive integer k such that* $\ln_k(a) \neq 0$, *but* $\ln_{k+1}(a) = 0$, *that is D is a nilpotent cyclic algebra.*
(iii) $D = V \oplus U$, *where V is an abelian ideal*, $V \leq \zeta^{\mathrm{left}}(D)$, *$U$ is a nilpotent cyclic subalgebra*, $[D, D] = V \oplus [U, U]$ *is an abelian ideal.*
(iv) $D = \zeta^{\mathrm{left}}(D) \oplus \zeta^{\mathrm{right}}(D)$, *where* $[D, D] = \zeta^{\mathrm{left}}(D) = F\ln_2(a) + \cdots + F\ln_k(a)$, $\zeta^{\mathrm{right}}(D) = Fc$ *for some element $c \in D$ and* $[c, y] = [a, y]$ *for each element* $y \in \zeta^{\mathrm{left}}(D)$.

Another natural subject we want to consider is the minimal Leibniz algebras that are the Leibniz algebras, whose proper subalgebras are Lie algebras.

Theorem 1.2. *Let L be a Leibniz algebra over a field F. Suppose that every proper subalgebra of L is a Lie algebra. Then L is an algebra of one of the following types*:

(i) *L is a Lie algebra.*
(ii) *There exists a positive integer k such that* $\ln_k(a) \neq 0$, *but* $\ln_{k+1}(a) = 0$, *that is L is nilpotent.*
(iii) $L = V \oplus U$, *where V is an abelian ideal*, $V \leq \zeta^{\mathrm{left}}(L)$, $U = Fu$ *and* $[u, u] = 0$, $V = Fv + Fv_1$ *and* $[u, v] = v_1$, $[u, v_1] = 0$.

Since every abelian Leibniz algebra is a Lie algebra, we obtain

Corollary 1.1. *Let L be a Leibniz algebra over a field F. Suppose that every proper subalgebra of L is abelian. Then L is an algebra of one of the following types*:

(i) *L is a Lie algebra whose proper subalgebras are abelian.*
(ii) *There exists a positive integer k such that* $\ln_k(a) \neq 0$, *but* $\ln_{k+1}(a) = 0$, *that is L is nilpotent.*
(iii) $L = V \oplus U$, *where V is an abelian ideal*, $V \leq \zeta^{\mathrm{left}}(L)$, $U = Fu$ *and* $[u, u] = 0$, $V = Fv + Fv_1$ *and* $[u, v] = v_1$, $[u, v_1] = 0$.

This result implies that a description of Leibniz algebras, whose proper subalgebras are abelian, can be deduced to the case of Lie algebras, whose proper subalgebras are abelian. Such Lie algebras are either simple, or solvable. Soluble minimal nonabelian Lie algebras (even soluble minimal non-nilpotent Lie algebras) were described in [9, 15, 16]. Simple minimal nonabelian Lie algebras were studied in [6, 8], but their complete description remains an open problem.

2. The Structure of Cyclic Subalgebras

Lemma 2.1. *Let L be a Leibniz algebra over a field F, $a \in L$. Then* $[\ln_k(a), \ln_j(a)] = 0$, *whenever* $k > 1$.

Proof. Using the Leibniz identity (1.1) and assuming that $a = b$, we obtain $[[a, a], c] = 0$, in particular $[\ln_2(a), \ln_j(a)] = 0$. Let $k > 2$, and suppose that we have already proved that $[\ln_m(a), \ln_j(a)] = 0$ for all $m < k$. We have

$$[\ln_k(a), \ln_j(a)] = [[a, \ln_{k-1}(a)], \ln_j(a)] = [a, [\ln_{k-1}(a), \ln_j(a)]] - [\ln_{k-1}(a), [a, \ln_j(a)]].$$

By the induction hypothesis, $[\ln_{k-1}(a), \ln_j(a)] = 0$. Furthermore,

$$[\ln_{k-1}(a), [a, \ln_j(a)]] = [\ln_{k-1}(a), \ln_{j+1}(a)] = 0$$

so that $[\ln_k(a), \ln_j(a)] = 0$. $\square$

Corollary 2.1. *Let L be a Leibniz algebra over a field F, $a \in L$. Then* $[\ln_k(a), a] = 0$, *whenever* $k > 1$.

Let L be a Leibniz algebra over a field F and $a_1, \dots, a_n$ be elements of L. An arbitrary commutator formed of elements $a_1, \dots, a_n$ we call a commutator of weight n of the elements $a_1, \dots, a_n$.

Lemma 2.2. *Let L be a Leibniz algebra over a field F, $a \in L$. Then every nonzero product of k copies of an element a with any bracketing coincides with* $\ln_k(a)$.

Proof. We will proceed by induction on k. If $k = 4$, we have the following commutators of weight 4: $\ln_4(a)$, $[[[a, a], a], a]$, $[[a, a], [a, a]]$, $[a, [[a, a], a]]$, $[[a, [a, a]], a]$. Since the left center contains $[a, a]$,

$$[[[a, a], a], a] = [[a, a], [a, a]] = [a, [[a, a], a]] = 0.$$

Further, $[[a, [a, a]], a] = [a, [[a, a], a]] - [[a, a], [a, a]] = 0$.

Let $k > 4$ and suppose that, we already proved our assertion for all commutators of weight m where $m < k$. We have the following possibilities: $[a, x]$, $[y, a]$, $[u, v]$, where x, y are commutators of weight $k - 1$, u, v are commutators of weight m, n respectively, and $m + n = k$. If $x \neq 0$, then by the induction hypothesis it coincides with $\ln_{k-1}(a)$, and then $[a, x] = [a, \ln_{k-1}(a)] = \ln_k(a)$. If $y \neq 0$, then by the induction hypothesis it coincides with $\ln_{k-1}(a)$, and then $[y, a] = [\ln_{k-1}(a), a] = 0$ by Corollary 2.1. Further, if $m = 2$, then $u = [a, a]$ and in this case, $[u, v] = [[a, a], v] = 0$. If $n = 2$, then $v = [a, a]$ and in this case,

$[u, v] = [u, [a, a]] = 0$. Of course, we assume that $u \neq 0$. Then by the induction hypothesis it coincides with $\ln_{k-2}(a)$, and by Lemma 2.1 because $k - 2 > 2$, we obtain $[u, [a, a]] = [\ln_{k-2}(a), \ln_2(a)] = 0$. Finally, suppose that $m > 2$ and $n > 2$. Again, we must consider the case when $u \neq 0$ and $v \neq 0$. In this case, the induction hypothesis implies that $u = \ln_m(a)$ and $v = \ln_n(a)$. Then, we have

$$[u, v] = [\ln_m(a), \ln_n(a)] = [\ln_m(a), [a, \ln_{n-1}(a)]]$$
$$= [[\ln_m(a), a], \ln_{n-1}(a)] + [a, [\ln_m(a), \ln_{n-1}(a)]].$$

By Corollary 2.1 $[\ln_m(a), a] = 0$, and since $m > 1$, by Lemma 2.1 $[\ln_m(a), \ln_{n-1}(a)] = 0$, which proves this result. □

Corollary 2.2. *Let L be a Leibniz algebra over a field F, $a \in L$. Then a cyclic subalgebra $\langle a \rangle$ is generated as a subspace by the elements $\ln_k(a)$, $k \in \mathbb{N}$.*

Proof. The subalgebra $\langle a \rangle$ is generated as a subspace by the commutators of elements $a_1, \ldots, a_k$ where $a_j = a$, $1 \leq j \leq k$, and $k \in \mathbb{N}$. Now we must apply Lemma 2.2. □

Corollary 2.3. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Then $[D, D]$ generates as a subspace by the elements $\ln_k(a)$, where $k \geq 2$.*

Proof. Indeed, let x, y be arbitrary elements of D. By Corollary 2.2, $x = \sum_{1 \leq j \leq m} \alpha_j \ln_j(a)$, $y = \sum_{1 \leq t \leq n} \beta_t \ln_t(a)$, where $\alpha_j, \beta_t \in F$, $1 \leq j \leq m$, $1 \leq t \leq n$. Then

$$[x, y] = \left[\sum_{1 \leq j \leq m} \alpha_j \ln_j(a), \sum_{1 \leq t \leq n} \beta_t \ln_t(a)\right] = \sum_{1 \leq j \leq m, 1 \leq t \leq n} \alpha_j \beta_t [\ln_j(a), \ln_t(a)].$$

Taking into account Lemma 2.1, we obtain that

$$[x, y] = \sum_{1 \leq t \leq n} \alpha_1 \beta_t [\ln_1(a), \ln_t(a)] = \sum_{2 \leq t \leq n+1} \alpha_1 \beta_t \ln_t(a).$$ □

Corollary 2.4. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Then $[D, D] = \mathrm{Leib}(D)$.*

Proof. Indeed, we have already shown that $\mathrm{Leib}(L)$ is an ideal. Then the fact that $\ln_2(a) = [a, a] \in \mathrm{Leib}(L)$ implies that $\ln_3(a) = [a, \ln_2(a)] \in \mathrm{Leib}(L)$, and $\ln_{j+1}(a) = [a, \ln_j(a)] \in \mathrm{Leib}(L)$ for all $j \in \mathbb{N}$. It follows that $D \leq \mathrm{Leib}(L)$. The reverse inclusion is obvious. □

Corollary 2.5. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Then $\gamma_k(D)$ is a span of the elements $\ln_t(a)$, where $t \geq k$.*

Proof. We will proceed by induction on k. If $k = 2$, then result {follows from Corollary 2.3. Let $k > 2$ and suppose} that we have already proved that for all $m <$

k $\gamma_m(D)$ is generated as a subspace by the elements $\ln_t(a)$, $t < m$. We have $\gamma_k(D) = [D, \gamma_{k-1}(D)]$. Let $x \in D$, $y \in \gamma_{k-1}(D)$. By Corollary 2.2 $x = \sum_{1 \le j \le m} \alpha_j \ln_j(a)$, by the induction hypothesis, $y = \Sigma_{k-1 \le t \le n} \beta_t \ln_t(a)$, where $\alpha_j, \beta_t \in F$, $1 \le j \le m$, $k - 1 \le t \le n$. Then

$$[x, y] = \left[\sum_{1 \le j \le m} \alpha_j \ln_j(a), \sum_{k-1 \le t \le n} \beta_t \ln_t(a)\right] = \sum_{1 \le j \le m, k-1 \le t \le n} \alpha_j \beta_t [\ln_j(a), \ln_t(a)].$$

Taking into account Lemma 2.1, we obtain that

$$[x, y] = \sum_{k-1 \le t \le n} \alpha_1 \beta_t [\ln_1(a), \ln_t(a)] = \sum_{k \le t \le n+1} \alpha_1 \beta_t \ln_t(a).$$

□

Corollary 2.6. *Let L be a Leibniz algebra over a field F and let a be an element of L of infinite depth, $D = \langle a \rangle$. Then $\gamma_\omega(D) = \bigcap_{n \in \mathbb{N}} \gamma_n(D) = \langle 0 \rangle$.*

Corollary 2.7. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Then $[D, D] \le \zeta^{\text{left}}(L)$, in particular, $[D, D]$ is an abelian subalgebra. Moreover, $[[x, y], z] = 0$ for all elements $x, y \in D$ and an arbitrary element $z \in L$.*

Proof. Indeed, by Corollary 2.3 $[D, D]$ generates as a subspace by elements $\ln_k(a)$, where $k \ge 2$. By above noted $\ln_2(a) = [a, a] \in \zeta^{\text{left}}(L)$. Since $\zeta^{\text{left}}(L)$ is a subalgebra, $\ln_k(a) = [a, \ln_{k-1}(a)] \in \zeta^{\text{left}}(L)$, so that $[D, D] \le \zeta^{\text{left}}(L)$. □

Corollary 2.8. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. Then D is a metabelian subalgebra.*

Lemma 2.3. *Let L be a Leibniz algebra over a field F, $a \in L$, $D = \langle a \rangle$. If there exists a positive integer k such that $\ln_{k+1}(a) \in F \ln_1(a) + \cdots + F \ln_k(a)$, then $D = F \ln_1(a) + \cdots + F \ln_k(a)$.*

Proof. Put $a_1 = a$, $a_j = \ln_j(a)$, $j \in \mathbb{N}$. We have $a_{k+1} = \alpha_1 a_1 + \cdots + \alpha_k a_k$, $\alpha_j \in F$, $1 \le j \le k$. Then

$$\begin{aligned} a_{k+2} = [a_1, a_{k+1}] &= [a_1, \alpha_1 a_1 + \cdots + \alpha_k a_k] = \alpha_1 a_2 + \cdots + \alpha_k a_{k+1} \\ &= \alpha_1 a_1 + \cdots + \alpha_{k-1} a_k + \alpha_k \alpha_1 a_1 + \cdots + \alpha_k^2 a_k \\ &\in F a_1 + \cdots + F a_k. \end{aligned}$$

Using the same arguments as we used above and ordinary induction, we obtain that $a_n \in F a_1 + \cdots + F a_k$ for all $j \in \mathbb{N}$. Corollary 2.2 shows that $D = F a_1 + \cdots + F a_k$. □

Recall some needed concepts. Let L be a Leibniz algebra over a field F, M be nonempty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H^{\text{left}}(M) = \{a \in H \mid [a, M] = 0\}, \quad \mathrm{Ann}_H^{\text{right}}(M) = \{a \in H \mid [M, a] = 0\}.$$

The subset $\mathrm{Ann}_H^{\text{left}}(M)$ is called the *left annihilator* or *left centralizer* of M in the subalgebra H. The subset $\mathrm{Ann}_H^{\text{right}}(M)$ is called the *right annihilator* or *right* *centralizer* of M in the subalgebra H. The intersection

$$\mathrm{Ann}_H(M) = \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \mid [a, M] = \langle 0 \rangle = [M, a]\}$$

is called the *annihilator* or *centralizer* of M in a subalgebra H.

It is not hard to see that all these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then it is not hard to check that $\mathrm{Ann}_L^{\mathrm{left}}(M)$ is an ideal of L.

If M is an ideal of L, then also it is not hard to check that $\mathrm{Ann}_L(M)$ is an ideal of L.

3. Proof Of Theorem 1.1.

Put again $a_1 = a$, $a_j = \mathrm{ln}_j(a)$, $j \in N$. By our condition,

$$Fa_1 + \cdots + Fa_k = Fa_1 \oplus \cdots \oplus Fa_k.$$

If $k = 1$, then $[a, a] = \alpha a$ for some $\alpha \in F$. Suppose that $\alpha \neq 0$, then $0 = [[a, a], a] = [\alpha a, a] = \alpha[a, a]$, which implies that $[a, a] = 0$ and $\alpha a = 0$ which implies that $a = 0$. This contradiction shows that $\alpha = 0$ and we obtain that $[a, a] = 0$ so in this case, $\langle a \rangle$ is abelian.

Let $k > 1$, then $\dim_F(D) > 1$. If we suppose that $D = \mathrm{Leib}(D)$, then D is abelian, and we obtain contradiction. Hence $D \neq \mathrm{Leib}(D)$ and therefore $a \notin \mathrm{Leib}(D)$. Put $\mathrm{Leib}(D) = K$. By Corollaries 2.3 and 2.4, K as a subspace is generated by the elements a_j for $j > 1$.

Suppose, first that $\dim_F(K) = 1$. Then K is an abelian subalgebra and $K = Fa_2$. Since K is an ideal of D, $a_3 = [a, a_2] \in K$, that is $a_3 = \alpha a_2$ for some $\alpha \in F$. If $\alpha = 0$, then $a_2 \in \zeta^{\mathrm{right}}(D)$. Since $K \leq \zeta^{\mathrm{left}}(D)$ and $\zeta^{\mathrm{left}}(D) \neq D$, we obtain that $K = \zeta^{\mathrm{right}}(D) = \zeta^{\mathrm{left}}(D)$, which follows that $K = \zeta(D)$. By Corollary 2.3, $K = \gamma_2(D)$, so that $\gamma_3(D) = [D, \gamma_2(D)] = \langle 0 \rangle$. In other words, D is nilpotent.

Assume now that $\alpha \neq 0$. Put $b = \alpha a - a_2$, then

$$[a, b] = [a, \alpha a - a_2] = \alpha[a, a] - [a, a_2] = \alpha a_2 - \alpha a_2 = 0.$$

Furthermore, $[b, b] = [\alpha a - a_2, b] = \alpha[a, b] - [a_2, b] = 0$. It follows that a subalgebra $\langle b \rangle$ is abelian and $\langle b \rangle = Fb$. Since $\alpha \neq 0$, $b \notin K$, so that $D = \langle b \rangle + K$. The fact that $\dim_F(\langle b \rangle) = 1$ implies that $D = \langle b \rangle \oplus K$. Furthermore, the equalities $D = Fa + K$ and $[a, b] = 0$, $[a_2, b] = 0$ prove that $\langle b \rangle = \zeta^{\mathrm{right}}(D)$. The last commutator

$$[b, a_2] = [\alpha a - a_2, a_2] = \alpha[a, a_2] = \alpha(\alpha a_2) = \alpha^2 a_2.$$

Suppose now that $\dim_F(K) = k - 1 > 1$. Since K is an ideal of D, $a_{j+1} = [a, a_j] \in K$ for all $j \geq 1$, that is $a_{k+1} = \alpha_2 a_2 + \cdots + \alpha_k a_k$ for some $\alpha_j \in F$, $1 \leq j \leq k$. If $a_{k+1} = 0$, then $a_k \in \mathrm{Ann}_K^{\mathrm{right}}(a_1)$, in particular, $\mathrm{Ann}_K^{\mathrm{right}}(a_1) \neq \langle 0 \rangle$. Since $D = Fa_1 + K$ and $K \leq \zeta^{\mathrm{left}}(D)$, $\mathrm{Ann}_K^{\mathrm{right}}(a_1) = \zeta(D)$, and we obtain that $\gamma_{k+1}(D) = \langle 0 \rangle$. In other words, D is nilpotent.

Assume that $a_{k+1} \neq 0$. Then not all of coefficients α_j are zeros. Let t be the first index such that $\alpha_t \neq 0$. In other words, $a_{k+1} = \alpha_t a_t + \cdots + \alpha_k a_k$. Consider first the case when $t > 2$. Then

$$[a, a_k] = \alpha_t[a, a_{t-1}] + \cdots + \alpha_k[a, a_{k-1}] = [a, \alpha_t a_{t-1} + \cdots + \alpha_k a_{k-1}]$$

which implies that $\alpha_t a_{t-1} + \cdots + \alpha_k a_{k-1} - a_k \in \mathrm{Ann}_K^{\mathrm{right}}(a_1)$. The fact that $\alpha_t \neq 0$ implies that $\alpha_t^{-1} \neq 0$, and then

$$\begin{aligned} d_{t-1} &= \alpha_t^{-1}(\alpha_t a_{t-1} + \cdots + \alpha_k a_{k-1} - a_k) \\ &= a_{t-1} + \beta_t a_t + \cdots + \beta_k a_k \in \mathrm{Ann}_K^{\mathrm{right}}(a_1). \end{aligned}$$

Put

$$\begin{aligned} d_{t-2} &= a_{t-2} + \beta_t a_{t-1} + \cdots + \beta_k a_{k-1}, \\ d_{t-3} &= a_{t-3} + \beta_t a_{t-2} + \cdots + \beta_k a_{k-2}, \\ &\ldots, \\ d_1 &= a_1 + \beta_t a_2 + \cdots + \beta_k a_{k-t+1}. \end{aligned}$$

Then

$$\begin{aligned} [d_1, d_1] &= [a_1 + \beta_t a_2 + \cdots + \beta_k a_{k-t+1}, a_1 + \beta_t a_2 + \cdots + \beta_k a_{k-t+1}] \\ &= a_2 + \beta_t a_3 + \ldots + \beta_k a_{k-t+2} = d_2, \\ [d_1, d_1] &= [a_1 + \beta_t a_2 + \cdots + \beta_k a_{k-t+1}, a_2 + \beta_t a_3 + \cdots + \beta_k a_{k-t+2}] \\ &= a_3 + \beta_t a_4 + \cdots + \beta_k a_{k-t+3} = d_3, \\ &\ldots \\ [d_1, d_{t-2}] &= d_{t-1}, \\ [d_1, d_{t-1}] &= [a_1 + \beta_t a_2 + \cdots + \beta_k a_{k-t+1}, d_{t-1}] = [a_1, d_{t-1}] = 0. \end{aligned}$$

It follows that the subspace $U = Fd_1 \oplus Fd_2 \oplus \cdots \oplus Fd_{t-1}$ is a subalgebra, and moreover, this subalgebra is nilpotent. Put further $d_t = a_t$, $d_{t+1} = a_{t+1}, \ldots, d_k = a_k$. The following matrix corresponds to this transaction:

$$\begin{pmatrix} 1 & \beta_t & \beta_{t+1} & \ldots & \beta_k & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \beta_t & \ldots & \beta_{k-1} & \beta_k & 0 & \ldots & 0 & 0 \\ \multicolumn{10}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & 0 & 1 & \beta_t & \ldots & \beta_{k-1} & \beta_k \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\ \multicolumn{10}{c}{\dotfill} \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

This matrix is nonsingular, which proves that the elements $\{d_1, \ldots, d_k\}$ present a new basis. Since $\zeta^{\mathrm{left}}(D)$ contains elements $d_t, \ldots, d_k$, the subspace $V = Fd_t \oplus \cdots \oplus Fd_k$ is a subalgebra. Moreover, V is an ideal of D, because

$$[a_1, d_t] = d_{t+1}, \ldots, [a_1, d_{k-1}] = d_k, \quad [a_1, d_k] = \alpha_t d_t + \cdots + \alpha_k d_k.$$

Thus $D = U \oplus V$, where U is a nilpotent subalgebra, V is an abelian ideal. Furthermore, $\zeta^{\text{left}}(D) = [D, D] = Fd_2 \oplus \cdots \oplus Fd_k$, $Fd_k \leq \zeta(D)$.

Consider now the case, when $t = 2$, that is $\alpha_2 \neq 0$. Then

$$[a, a_k] = \alpha_2[a, a_1] + \alpha_3[a, a_2] + \cdots + \alpha_k[a, a_{k-1}] = [a, \alpha_2 a_1 + \alpha_3 a_2 + \cdots + \alpha_k a_{k-1}]$$

which follows that $\alpha_2 a_1 + \cdots + \alpha_k a_{k-1} - a_k \in \mathrm{Ann}_K^{\text{right}}(a_1)$. The fact that $\alpha_2 \neq 0$ implies the existing of α_2^{-1}. Put

$$c = \alpha_2^{-1}(\alpha_2 a_1 + \cdots + \alpha_k a_{k-1} - a_k) = a_1 + \gamma_2 a_2 + \cdots + \gamma_{k-1} a_{k-1} + \gamma_k a_k.$$

Then $c \in \mathrm{Ann}_K^{\text{right}}(a_1)$, $c \notin K$, so that $D = Fc + K$. We have

$$[c, c] = [a_1 + \gamma_2 a_2 + \cdots + \gamma_k a_k, c] = [a_1, c] + [\gamma_2 a_2 + \cdots + \gamma_k a_k, c] = 0.$$

It follows that the subalgebra $\langle c \rangle$ is abelian and $\langle c \rangle = Fc$. The fact that $\dim_F(\langle c \rangle) = 1$ implies that $D = \langle c \rangle \oplus K$. Furthermore, the equalities $D = Fa + K$ and $[a, c] = 0$, $[a_j, c] = 0$ for $j > 1$ prove that $\langle c \rangle = \zeta^{\text{right}}(D)$. And, finally, we obtain the last commutators:

$$[c, a_2] = [a_1 + \gamma_2 a_2 + \cdots + \gamma_k a_k, a_2] = [a_1, a_2] + [\gamma_2 a_2 + \cdots + \gamma_k a_k, a_2] = a_3,$$

$$[c, a_3] = [a_1, a_3] = a_4, \ldots, [c, a_{k-1}] = [a_1, a_{k-1}] = a_k, [c, a_k] = \alpha_2 a_2 + \cdots + \alpha_k a_k.$$

4. On the Minimal Leibniz Algebras

Let L be a Leibniz algebra, B, C be ideals of L such that $B \leq C$. The factor C/B is called *L-central*, if $[C, L], [L, C] \leq B$.

The factor C/B is called *L-eccentric* if the center of L/B does not include C/B. An ideal C of L is said to be *L-hypereccentric*, if it has an ascending series

$$\langle 0 \rangle \leq C_0 \leq C_1 \leq \cdots C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = C$$

of ideals of L such that each factor $C_{\alpha+1}/C_\alpha$ is an L-eccentric and L-chief for every $\alpha < \gamma$.

Lemma 4.1. *Let L be a Leibniz algebra over a field F, and A, B be ideals of L such that $B \leq A$. Then $\mathrm{Ann}_L^{\text{left}}((A+B)/B) = \mathrm{Ann}_L^{\text{left}}(A/(A \cap B))$, $\mathrm{Ann}_L^{\text{right}}((A+B)/B) = \mathrm{Ann}_L^{\text{right}}(A/(A \cap B))$, $\mathrm{Ann}_L((A+B)/B) = \mathrm{Ann}_L(A/(A \cap B))$.*

Proof. Indeed, if $x \in \mathrm{Ann}_L^{\text{left}}((A+B)/B)$, $a \in A$, then $[x, a] \in B$. On the other hand, since A is an ideal, then $[x, a] \in A$, so that $[x, a] \in A \cap B$. Conversely, if $y \in \mathrm{Ann}_L^{\text{left}}(A/(A \cap B))$, $c \in A + B$, then $c = a + b$ for some elements $a \in A$, $b \in B$. We have $[y, c] = [y, a + b] = [y, a] + [y, b] \in B$, which proves the first equality. The proofs of the rest of the equalities are similar. □

Lemma 4.2. *Let L be a Leibniz algebra over a field F and A be an ideal of L. Suppose that A satisfies the following conditions*:

(i) *A is abelian.*
(ii) *$L/\mathrm{Ann}_L(A)$ is hypercentral.*

(iii) *A includes an ideal C of L such that $C \le \zeta(L)$ and A/C is L-chief factor.*
(iv) $\mathrm{Ann_L(A/C)} \neq \mathrm{L}$.

Then A includes an ideal D of L such that $A = C \oplus D$. In particular, D is L-eccentric and L-chief.

Proof. Let $Y = \mathrm{Ann}_L(A)$ and $Y \neq z + Y \in \zeta(L/Y)$. Consider the mapping $\xi_z : A \mapsto A$, defined by the rule $\xi_z(a) = [z, a]$ for each $a \in A$. Clearly this mapping is linear, $\mathrm{Ker}(\xi_z) = \mathrm{Ann}_A^{\mathrm{right}}(z)$, $\mathrm{Im}(\xi_z) = [z, A]$, and we have the following F-isomorphism $A/\mathrm{Ker}(\xi_z) \cong_F \mathrm{Im}(\xi_z) = D$.

Let $x \in L$ and $c \in \mathrm{Ann}_A^{\mathrm{right}}(z)$. We have

$$[z, [c, x]] = [[z, c], x] + [c, [z, x]].$$

From the choice of z, we obtain that $[z, x] \in \mathrm{Ann}_L(A)$, so that $[c, [z, x]] = 0$. The choice of c yields that $[z, c] = 0$. Thus $[[z, c], x]] = 0$, which shows that $[c, x] \in \mathrm{Ann}_A^{\mathrm{right}}(z)$. Further,

$$[z, [x, c]] = [[z, x], c] + [x, [z, c]].$$

Again $[z, x] \in \mathrm{Ann}_L(A)$ implies $[[z, x], c] = 0$ and $[x, [z, c]] = [x, 0] = 0$, so that $[x, c] \in \mathrm{Ann}_A^{\mathrm{right}}(z)$. This proves that $\mathrm{Ann}_A^{\mathrm{right}}(z)$ is an ideal of L.

Since A is abelian, $D = [z, A]$ is a subalgebra. Let again $x \in L$, $a \in A$. We have

$$[[z, a], x] = [z, [a, x]] - [a, [z, x]].$$

Since A is an ideal, $[a, x] \in A$. Therefore, $[z, [a, x]] \in [z, A]$. The choice of z implies that $[z, x] \in \mathrm{Ann}_L(A)$. Hence $0 = [a, [z, x]]$. Furthermore,

$$[x, [z, a]] = [[x, z], a] + [z, [x, a]].$$

Since A is an ideal, $[x, a] \in A$, so that $[z, [x, a]] \in [z, A]$. The choice of z implies that $[x, z] \in \mathrm{Ann}_L(A)$, hence $[[x, z], a] = 0$. It follows that D is an ideal of L.

An inclusion $C \le \zeta(L)$ implies that $C \le \mathrm{Ker}(\xi_z)$. Since A/C is L-chief factor, then for ideal $\mathrm{Ann}_A^{\mathrm{right}}(z)$ there exists only two possibility: $C = \mathrm{Ann}_A^{\mathrm{right}}(z)$ or $A = \mathrm{Ann}_A^{\mathrm{right}}(z)$.

First, suppose that $\mathrm{Ann}_A^{\mathrm{right}}(z) = C$. Let K be an ideal of L such that $K \le D$, and K_1 be the preimage of K by the mapping ξ_z. If $a \in K_1$, $x \in L$, then

$$\xi_z([a, x]) = [z, [a, x]] = [[z, a], x] + [a, [z, x]].$$

The choice of z implies that $[z, x] \in \mathrm{Ann}_L(A)$, therefore $[a, [z, x]] = 0$. Since $a \in K_1$, $[z, a] \in K$ and the fact that K is an ideal implies that $\xi_z([a, x]) = [[z, a], x] \in K$. In turn out, it implies that $[a, x] \in K_1$. Furthermore,

$$\xi_z([x, a]) = [z, [x, a]] = [[z, x], a] + [x, [z, a]].$$

Again, $[[z, x], a] = 0$. Since $a \in K_1$, $[z, a] \in K$ and the fact that K is an ideal implies that $\xi_z([x, a]) = [x, [z, a]] \in K$. In turn out, it implies that $[x, a] \in K_1$. Hence K_1 is an ideal of L. If now $K \neq \langle 0 \rangle$, then $K_1 \neq \mathrm{Ker}(\xi_z) = C$. If $K \neq D$, then

$K_1 \neq A$. Hence if we assume that D is not L-chief, then A/C also is not L-chief, and we obtain a contradiction with condition (iii). This contradiction shows that D is L-chief.

Now suppose that D is L-central. For arbitrary elements $a \in A$ and $x \in L$, we obtain $[[z,a],x] = [\xi_z(a), x] = 0$. On the other hand,

$$[[z,a],x] = [z,[a,x]] - [a,[z,x]] = [z,[a,x]]$$

because $[a,[z,x]] = 0$. Thus $[z,[a,x]] = 0$, which implies that $[a,x] \in \mathrm{Ker}(\xi_z) = C$. Furthermore, $[x,[z,a]] = [x, \xi_z(a)] = 0$, and

$$[x,[z,a]] = [[x,z],a] + [z,[x,a]] = [z,[x,a]]$$

because $[[x,z],a] = 0$. Thus $[z,[x,a]] = 0$, which implies that $[x,a] \in \mathrm{Ker}(\xi_z) = C$. But this means that the factor A/C is L-central, which contradicts condition (iv). This contradiction shows that D is L-eccentric.

Since D is L-chief, either $D \cap C = D$ or $D \cap C = \langle 0 \rangle$. In the first case, $D \leq C$, which is impossible because D is L-eccentric. Hence $D \cap C = \langle 0 \rangle$. In this case, $(D+C)/C \cong D$ is a nonzero ideal of L/C, and condition (iii) implies that $(D+C)/C = A/C$. Thus $D + C = A$.

Suppose now that $\mathrm{Ann}_A^{\mathrm{right}}(z) = A$. The choice of z implies that $z \notin \mathrm{Ann}_L(A) = \mathrm{Ann}_L^{\mathrm{right}}(A) \cap \mathrm{Ann}_L^{\mathrm{left}}(A)$. Then our assumption implies that $z \notin \mathrm{Ann}_L^{\mathrm{left}}(A)$. Consider the mapping $\eta_z : A \mapsto A$ defined by the rule $\eta_z(a) = [a,z]$ for each $a \in A$. Clearly, this mapping is linear, $\mathrm{Ker}(\eta_z) = \mathrm{Ann}_A^{\mathrm{left}}(z)$, $\mathrm{Im}(\eta_z) = [A,z]$, and we have the following F-isomorphism $A/\mathrm{Ker}(\eta_z) \cong_F \mathrm{Im}(\eta_z)$.

Let $x \in L$ and $c \in \mathrm{Ann}_A^{\mathrm{left}}(z)$. We have

$$[[c,x],z] = [c,[x,z]] - [x,[c,z]].$$

From the choice of z, we obtain that $[x,z] \in \mathrm{Ann}_L(A)$, so that $[c,[x,z]] = 0$. The choice of c yields that $[c,z] = 0$. Thus $[x,[c,z]] = 0$, which shows that $[c,x] \in \mathrm{Ann}_A^{\mathrm{left}}(z)$. Further,

$$[[x,c],z] = [x,[c,z]] - [c,[x,z]].$$

As we have seen above, $[x,[c,z]] = [c,[x,z]] = 0$, so that $[x,c] \in C$. This proves that $\mathrm{Ann}_A^{\mathrm{left}}(z)$ is an ideal of L.

Since A is abelian, $V = [A,z]$ is a subalgebra. Let again $x \in L$, $a \in A$. We have

$$[[a,z],x] = [a,[z,x]] - [z,[a,x]].$$

Since A is an ideal, $[a,x] \in A$, so that $[z,[a,x]] = 0$ because $\mathrm{Ann}_A^{\mathrm{right}}(z) = A$. As above, $[a,[z,x]] = 0$. Thus $[[a,z],x] = 0$. Furthermore,

$$[x,[a,z]] = [[x,a],z] + [a,[x,z]] = [[x,a],z] \in [A,z].$$

It follows that V is an ideal of L.

The inclusion $C \leq \zeta(L)$ implies that $C \leq \mathrm{Ker}(\eta_z)$. Since L/C is L-chief factor, for ideal $\mathrm{Ann}_A^{\mathrm{left}}(z)$ we have only two possibility: $C = \mathrm{Ann}_A^{\mathrm{left}}(z)$ or $A = \mathrm{Ann}_A^{\mathrm{left}}(z)$. If we suppose that $A = \mathrm{Ann}_A^{\mathrm{left}}(z)$, then $z \in \mathrm{Ann}_L^{\mathrm{left}}(A)$, and we obtain contradiction. This contradiction proves that $z \notin \mathrm{Ann}_L^{\mathrm{left}}(A)$.

Let K be an ideal of L such that $K \leq V$, and K_1 be the preimage of K by the mapping η_z. If $a \in K_1$, $x \in L$, then

$$\eta_z([a,x]) = [[a,x],z] = [a,[x,z] + [x,[a,z]] = [x,[a,z]].$$

Since $a \in K_1$, $[a,z] \in K$, and the fact that K is an ideal implies that $\eta_z([a,x]) = [x,[a,z]] \in K$. In turn out, it implies that $[x,a] \in K_1$. Furthermore,

$$\eta_z([x,a]) = [[x,a],z] = [x,[a,z]] - [a,[x,z]] = [x,[a,z]].$$

Since $a \in K_1$, $[z,a] \in K$ and the fact that K is an ideal implies that $\eta_z([x,a]) = [x,[a,z]] \in K$. In turn out, it implies that $[x,a] \in K_1$. Hence K_1 is an ideal of L. If now $K \neq \langle 0 \rangle$, then $K_1 \neq \mathrm{Ker}(\xi_z) = C$. If $K \neq D$, then $K_1 \neq A$. Hence, if we assume that D is not L-chief, then A/C also is not L-chief, and we obtain contradiction with condition (iii). This contradiction shows that V is L-chief.

Suppose now that V is L-central. For arbitrary elements $a \in A$ and $x \in L$, we obtain $[[a,z],x] = [\eta_z(a),x] = 0$. On the other hand,

$$[[a,z],x] = [a,[z,x]] - [z,[a,x]] = -[z,[a,x]]$$

because $[a,[z,x]] = 0$. Thus $[z,[a,x]] = 0$, which implies that $[a,x] \in \mathrm{Ker}(\xi_z) = C$. Furthermore, $[x,[a,z]] = [x,\eta_z(a)] = 0$, and

$$[x,[a,z]] = [[x,a],z] + [a,[x,z]] = [[x,a],z]$$

because $[a,[x,z]] = 0$. Thus $[[x,a],z] = 0$, which implies that $[x,a] \in \mathrm{Ker}(\xi_z) = C$. But this means that the factor A/C is L-central, which contradicts condition (iv). This contradiction shows that V is L-eccentric.

Since V is L-chief, either $V \cap C = V$ or $V \cap C = \langle 0 \rangle$. In the first case, $V \leq C$, which is impossible because V is L-eccentric. Hence $V \cap C = \langle 0 \rangle$. In this case $(V + C)/C \cong V$ is a nonzero ideal of L/C, and condition (iii) implies that $(V + C)/C = A/C$, thus $V + C = A$. □

Corollary 4.1. *Let L be a Leibniz algebra over a field F and A be an ideal of L. Suppose that A satisfies the following conditions*:

(i) *A is abelian.*
(ii) *$L/\mathrm{Ann}_L(A)$ is hypercentral.*
(iii) *A has a series.*

$$\langle 0 \rangle = C_0 \leq C_1 \leq \cdots \leq C_n = C \leq A$$

of ideals of L such that the factors C_j/C_{j-1} are L-central, $1 \leq j \leq n$ *and A/C is L-eccentric, and L-chief factor. Then A includes an ideal D of L such that $A = C \oplus D$. In particular, D is L-eccentric and L-chief.*

5. Proof of Theorem 1.2

We have already seen above that a Leibniz algebra L is a Lie algebra if and only if $[a, a] = 0$ for each element $a \in L$. Thus, if we suppose now that an algebra L is not cyclic, then for every element $a \in L$ the subalgebra $\langle a \rangle$ is proper. In this case, $\langle a \rangle$ is a Lie algebra, which implies that $[a, a] = 0$. Since it is true for every element $a \in L$, L is a Lie algebra.

Suppose now that $L = \langle d \rangle$ is a cyclic algebra. Put $d_1 = d$, $d_j = \ln_j(d)$, $j \in \mathbb{N}$. Consider first the case, when an element d has infinite depth. Then the elements d_j, $j \in \mathbb{N}$ are linearly independent and $\langle d \rangle = \bigoplus_{j\in\mathbb{N}} F d_j$. Consider a subspace K, generated by the elements $d_j - d_{j+1}$, $j \in \mathbb{N}$. We have

$$[d_1 - d_2, d_t - d_{t+1}] = d_{t+1} - d_{t+2} \quad \text{and} \quad [d_j - d_{j+1},\ d_t - d_{t+1}] = 0 \quad \text{if } j > 1.$$

It follows that K is a subalgebra. Furthermore K is a proper subalgebra because $d \notin K$. But in this case, it must be $0 = [d_1 - d_2, d_1 - d_2] = [d_1, d_1]$, and we obtain contradiction. This contradiction shows that the element d has a finite depth.

Assume first that, L is an algebra of type (ii) of Theorem 1.1. Then there exists a positive integer k such that $d_k \neq 0$, but $d_{k+1} = 0$. Let B be a subalgebra of L and suppose that B contains an element $b = \alpha_1 d_1 + \cdots + \alpha_k d_k$, where $\alpha_1 \neq 0$. We have

$$\begin{aligned}
[b, b] &= [\alpha_1 d_1 + \cdots + \alpha_k d_k, \alpha_1 d_1 + \cdots + \alpha_k d_k] \\
&= \alpha_1^2 d_2 + \alpha_1\alpha_2 d_3 + \cdots + \alpha_1\alpha_{k-1} d_k, \\
[b, [b, b]] &= [\alpha_1 d_1 + \cdots + \alpha_k d_k, \alpha_1^2 d_2 + \alpha_1\alpha_2 d_3 + \cdots + \alpha_1\alpha_{k-1} d_k] \\
&= \alpha_1^3 d_3 + \alpha_1^2\alpha_2 d_4 + \cdots + \alpha_1^2\alpha_{k-2} d_k, \cdots, \ln_k(b) = \alpha_1^k d_k.
\end{aligned}$$

Being a subalgebra, B contains $\ln_k(b)$. Since $\alpha_1 \neq 0$, B contains (together with $\alpha_1^k d_k$) also the element d_k. Then B contains an element $b_1 = b - \alpha_k d_k = \alpha_1 d_1 + \cdots + \alpha_{k-1} d_{k-1}$. Using the similar arguments and ordinary induction, we obtain that B contains an element d_1. It follows that $B = L$. This shows that every proper subalgebra lies in $[L, L] = F d_2 + \cdots + F d_k$. The subalgebra $[L, L]$ is abelian, in particular, it is a Lie algebra.

Now, we are in a position to consider the case when L is an algebra of type (iii) of Theorem 1.1. Then $L = V \oplus U$, where V is an abelian ideal, $V \leq \zeta^{\text{left}}(L)$, U is a nilpotent cyclic subalgebra and $[L, L] = V \oplus [U, U]$. Here U is a proper subalgebra, so that U is abelian and has dimension 1, i.e. $U = Fu$ for some element u and $[u, u] = 0$. Suppose that V includes a nonzero ideal $W \neq V$. Then the subalgebra $W + U$ is proper, so that it must be a Lie algebra. Let v be an arbitrary element of W and put $x = v + u$. We have

$$0 = [x, x] = [v + u, v + u] = [v, v] + [u, v] + [v, u] + [u, u] = [u, v].$$

It follows that $W \leq \text{Ann}_A^{\text{right}}(u)$. The inclusion $V \leq \zeta^{\text{left}}(L)$ shows that $W \leq \zeta(L)$. The fact that $\dim_F(V)$ is finite implies that V includes a maximal (in V) ideal M

of L. Then the factor V/M is L-chief. If V/M is L-eccentric, applying Lemma 4.2, we obtain the direct decomposition $V = M \oplus S$, where S is an ideal of L. Since $M \neq V$, $S \neq V$. By above proved, it follows that $S \leq \zeta(L)$, and we obtain contradiction. This contradiction shows that the factor V/M is L-central. By Theorem 1.1, V as an ideal is generated by one element, say v. Then $V/M = F(v + M)$. Furthermore $[u, v] \in M$ and $[u, [u, v]] = 0$. It follows that $V = Fv + Fv_1$, where $v_1 = [u, v]$ and $[u, v_1] = 0$.

Finally, let L be an algebra of type (iii) of Theorem 1.1. This situation repeats the previous one, so that we do not get a new type.

6. Proof of Corollary 1.1

Suppose that L is not a Lie algebra. Then L is an algebra of types (ii) or (iii) of Theorem 1.2. Suppose first that, L is an algebra of type (ii). In the proof of Theorem 1.2, we proved that $[L, L] = F\ln_1(a) + \cdots + F\ln_k(a)$ includes every proper subalgebra of L. The fact that $[L, L] = \zeta^{\mathrm{left}}(L)$ shows that every subalgebra of L is abelian.

Let now L be an algebra of type (iii) of Theorem 1.2. Suppose that B is a subalgebra of L. If $B \leq V$, then B is abelian. We note that the subspace $Fu + Fv_1$ is a subalgebra. Moreover, because $[u, u] = [u, v_1] = [v_1, v_1] = 0$, we can conclude that this subalgebra is abelian. Hence if $Fu + Fv_1$ includes B, then B is abelian. So assume that V and $Fu + Fv_1$ do not include B. Then B contains an element $x = \alpha u + \beta v + \gamma v_1$, where $\alpha, \beta, \gamma \in F$ and $\alpha, \beta \neq 0$. We have

$$[x, x] = [\alpha u + \beta v + \gamma v_1, \alpha u + \beta v + \gamma v_1] = \alpha^2 v + \alpha\beta v_1.$$

Further, $[x, [x, x]] = [\alpha u + \beta v + \gamma v_1, \alpha^2 v + \alpha\beta v_1] = \alpha^3 v_1 \in B$. Since $\alpha \neq 0$, $v_1 \in B$, so that $\gamma v_1 \in B$, $\alpha\beta v_1 \in B$ and $x - \alpha\beta v_1 = \alpha^2 v \in B$. The fact that $\alpha \neq 0$ implies that $v \in B$. Since $v_1 \in B$ and $v \in B$, we obtain that $\alpha u \in B$ and $u \in B$. It follows that $B = L$. Hence every proper subalgebra of L is abelian.

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