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On Evolution Algebras*

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Abstract. The structural constants of an evolution algebra are given by a quadratic matrix. In this work we establish an equivalence between nil, right nilpotent evolution algebras and evolution algebras defined by upper triangular matrices. The classification of 2-dimensional complex evolution algebras is obtained. For an evolution algebra with a special form of the matrix, we describe all its isomorphisms and their compositions. We construct an algorithm running under Mathematica which decides if two finite dimensional evolution algebras are isomorphic.

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1 Introduction

In this paper we consider a class of algebras called evolution algebras. The concept of evolution algebra lies between algebras and dynamical systems. Algebraically,

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evolution algebras are non-associative Banach algebras; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc. (see [4] and [5]).

In the book [5], the foundations of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed.

Let $(E, \cdot)$ be an algebra over a field K. If it admits a basis $\{e_1, e_2, \dots\}$ such that $e_i \cdot e_j = 0$ for $i \neq j$ and $e_i \cdot e_i = \sum_k a_{ik} e_k$ for any i, then this algebra is called an *evolution algebra*. We denote by $A = (a_{ij})$ the matrix of the structural constants of the evolution algebra E.

In [2] an evolution algebra $\mathcal{A}$ associated to the free population is introduced, and using this non-associative algebra, many results are obtained in an explicit form, e.g., the explicit description of stationary quadratic operators, and the explicit solutions of a non-linear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of a selection.

In the study of any class of algebras, it is important to describe up to isomorphism at least algebras of lower dimensions because such description gives examples to establish or reject certain conjectures. In this way, in [3] and [6], the classifications of associative and nilpotent Lie algebras of low dimensions were given.

In this paper we study properties of evolution algebras. In Section 2 we establish an equivalence between nil, right nilpotent evolution algebras and evolution algebras defined by upper triangular matrices. In [1] it was proved that these notions are equivalent to the nilpotency of evolution algebras, but right nilindex and nilindex do not coincide in general. Thus, it is natural to study conditions when some power of an evolution algebra is equal to zero. In Section 3 we consider an evolution algebra E with an upper triangular matrix A and obtain solutions of a system of equations (for entries of the matrix A) which give $E^k = 0$ for small values of k. Section 4 is devoted to the classification of 2-dimensional complex evolution algebras. In Section 5 for an evolution algebra with a special form of the matrix A we describe all its isomorphisms and their compositions. Finally, in Appendix, we construct an algorithm running under Mathematica, using Gröbner bases and the star product of two evolution matrices, which decides if two finite dimensional evolution algebras are isomorphic.

2 Nil and Right Nilpotent Evolution Algebras

In this section we prove that notions of nil and right nilpotency are equivalent for evolution algebras. Moreover, the matrix A of structural constants for such algebras has upper (or lower, up to permutation of basis of the algebra) triangular form.

Definition 2.1. An element a of an evolution algebra E is called nil if there exists $n(a) \in \mathbb{N}$ such that $\underbrace{((a \cdot a) \cdot a) \cdots a}_{n(a)} = 0$. An evolution algebra E is called nil if every element of the algebra is nil.

Theorem 2.2. *Let E be a nil evolution algebra with basis $\{e_1, \dots, e_n\}$. Then for* *the elements of the matrix* $A = (a_{ij})$, *the following relation holds*

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1} = 0 \tag{1}$$

for all $i_1, \ldots, i_k \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, n\}$ *with* $i_p \neq i_q$ *for* $p \neq q$.

Proof. Note that $(e_i \cdot e_i) \cdot e_i = a_{ii} e_i^2$, hence $a_{ii} = 0$ (otherwise the element e_i is not nil). We shall prove the equality (1) for right normed terms by induction.

For $1 \leq i, j \leq n$, by induction we can prove $(e_i + e_j)^{2s} = a_{ij}^{s-1} a_{ji}^{s-1} (e_i + e_j)^2$. The nil condition for the element $e_i + e_j$ leads to $a_{ij} a_{ji} = 0$ or $e_i^2 + e_j^2 = 0$. Take in account the fact $a_{ii} = a_{jj} = 0$ for any i, j and comparing the coefficients at the basic elements, from $e_i^2 + e_j^2 = 0$ we obtain $a_{ij} = a_{ji} = 0$. Hence, the equation $a_{ij} a_{ji} = 0$ for all i, j is obtained and therefore the equality (1) is true for $k = 2$.

Let (1) be true for $k - 1$. We shall prove it for k. For this purpose we consider the element $e_{i_1} + e_{i_2} + \cdots + e_{i_k}$. Without loss of generality, instead of this element we can consider the element $e_1 + e_2 + \cdots + e_k$. Using the hypothesis of the induction it is not difficult to note that

$$\Big(\sum_{i=1}^{k} e_i\Big)^{s+1} = \sum_{\substack{i_1, \ldots, i_s = 1 \\ i_p \neq i_q,\, p \neq q}}^{k} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s} e_{i_s}^2.$$

Let us take $s = k + 1$ in the above expression, then $i_{s-1} = i_k$. From induction hypothesis the coefficient $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s}$ is equal to zero if $i_s \in \{i_2, \ldots, i_{s-1}\}$. Therefore, we need to consider the case $i_s = i_1$ and the above expression will have the form

$$\begin{aligned}
\Big(\sum_{i=1}^{k} e_i\Big)^{k+2} &= \sum_{\phi \in S_k} a_{\phi(1)\phi(2)} a_{\phi(2)\phi(3)} \cdots a_{\phi(k)\phi(1)} e_{\phi(1)}^2 \\
&= \sum_{i=1}^{k} \Big(\sum_{\phi \in S_k : \phi(1) = i} a_{i\phi(2)} a_{\phi(2)\phi(3)} \cdots a_{\phi(k)i} \Big) e_i^2,
\end{aligned}$$

where S_k denotes the symmetric group of permutations of k elements.

Denote

$$\mathcal{F}_i = \sum_{\phi \in S_k : \phi(1) = i} a_{i\phi(2)} a_{\phi(2)\phi(3)} \cdots a_{\phi(k)i}.$$

We need the following lemmas:

Lemma 2.3. *For any* $i, j = 1, \ldots, k$, *we have* $\mathcal{F}_i = \mathcal{F}_j$.

Proof. For $\phi \in S_k$ with $\phi(1) = i$, we construct a unique $\bar{\phi} \in S_k$ such that $\bar{\phi}(1) = j$ and $a_{i\phi(2)} a_{\phi(2)\phi(3)} \cdots a_{\phi(k)i} = a_{j\bar{\phi}(2)} a_{\bar{\phi}(2)\bar{\phi}(3)} \cdots a_{\bar{\phi}(k)j}$. Indeed, let s be the number such that $\phi(s) = j$, then $\bar{\phi}$ is defined as

$$\bar{\phi} = \begin{pmatrix} 1 & 2 & \cdots & k-s+1 & k-s+2 & k-s+3 & \cdots & k \\ j & \phi(s+1) & \cdots & \phi(k) & i & \phi(2) & \cdots & \phi(s-1) \end{pmatrix}.$$

Thus, we get $\mathcal{F}_i = \mathcal{F}_j$. $\square$

Put $a = \sum_{i=1}^{k} e_i$.

Lemma 2.4. *If $a^2 = 0$, then $\mathcal{F}_1 = 0$.*

Proof. From $a^2 = 0$ we obtain $\sum\limits_{i=1,\, i\neq j}^{k} a_{ij} = 0$ for $j = 1, \ldots, n$. Using this equality we get

$$\mathcal{F}_1 = \sum_{\phi\in S_k:\phi(1)=1} a_{1\phi(2)}a_{\phi(2)\phi(3)}\cdots a_{\phi(k)1} = -\sum_{\phi\in S_k:\phi(1)=1}\sum_{\substack{i=2\\ i\neq\phi(2)}}^{k} a_{i\phi(2)}a_{\phi(2)\phi(3)}\cdots a_{\phi(k)1}.$$

Since for any $i = 2, \ldots, k$ there exists s_i such that $\phi(s_i) = i$, by the assumption of the induction we get

$$a_{i\phi(2)}a_{\phi(2)\phi(3)}\cdots a_{\phi(k)1} = a_{i\phi(2)}a_{\phi(2)\phi(3)}\cdots a_{\phi(s_i-1)i}a_{i\phi(s_i+1)}\cdots a_{\phi(k)1} = 0. \quad \square$$

Now we continue the proof of theorem. Using Lemma 2.3, we get

$$\Big(\sum_{i=1}^{k} e_i\Big)^{k+2} = \mathcal{F}_1 a^2 = 0.$$

By Lemma 2.4, we obtain $\mathcal{F}_1 = 0$. Fix an arbitrary $\phi_0 \in S_k$ with $\phi_0(1) = 1$ and multiply both sides of $\mathcal{F}_1 = 0$ by $a_{1\phi_0(2)}a_{\phi_0(2)\phi_0(3)}\cdots a_{\phi_0(k)1}$, then (again using the assumption of the induction) we obtain $a^2_{1\phi_0(2)}a^2_{\phi_0(2)\phi_0(3)}\cdots a^2_{\phi_0(k)1} = 0$, that is, $a_{1\phi_0(2)}a_{\phi_0(2)\phi_0(3)}\cdots a_{\phi_0(k)1} = 0$, which completes the induction and the proof of Theorem 2.2. $\square$

For an evolution algebra E, we introduce $E^{<1>} = E$ and $E^{<k+1>} = E^{<k>}E$ for $k \geq 1$.

Definition 2.5. An evolution algebra is called right nilpotent if there exists some $s \in \mathbb{N}$ such that $E^{<s>} = 0$.

Let E be a right nilpotent evolution algebra, then it is evident that E is a nil algebra.

Lemma 2.6. *Let the matrix A satisfy* (1). *Then for any $j \in \{1, \ldots, n\}$, there is a row π_j of A with j zeros. Moreover, $\pi_{j_1} \neq \pi_{j_2}$ if $j_1 \neq j_2$.*

Proof. First we shall prove that there is a row π_n with n zeros, i.e., all entries are zeros. Assume that there is no such a row. Then for any $i \in \{1, \ldots, n\}$, there is a number $\beta(i) \in \{1, \ldots, n\}\backslash\{i\}$ such that $a_{i\beta(i)} \neq 0$. Consider the sequence $i_1 = 1$, $i_2 = \beta(1)$, $\ldots$, $i_{n+1} = \beta(i_n)$. Then by the assumption we have $a_{i_m i_{m+1}} \neq 0$ for all $m = 1, \ldots, n$, hence $a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ni_{n+1}} \neq 0$. Since $i_j \in \{1, \ldots, n\}$, we have $i_p = i_q$ for some $p \neq q \in \{1, \ldots, n+1\}$. Thus, $a_{i_pi_{p+1}}a_{i_{p+1}i_{p+2}}\cdots a_{i_{q-1}i_p} \neq 0$, a contradiction with (1). So there is a row π_n which consists of zeros (n zeros). We also call this as the π_n-th row.

Now we shall prove that there is a row $\pi_{n-1} \neq \pi_n$ of A with $n-1$ zeros. Consider the A_{π_n}-minor of A, which is obtained from A deleting the π_n-th row and π_n-th column. A_{π_n} is an $(n-1)\times(n-1)$ matrix satisfying condition (1). To prove that A has a row with $n-1$ zeros, it is sufficient to prove that A_{π_n} has a row with all zeros. But this problem is the same as above. Iterating this argument, we can show that for any j, there exists a row π_j with j zeros. □

The following theorem is the main result of this section.

Theorem 2.7. *The following statements are equivalent for an n-dimensional evolution algebra E:*

(a) *The matrix corresponding to E can be written as*

$$\widehat{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2}$$

(b) *E is a right nilpotent algebra.*

(c) *E is a nil algebra.*

Proof. (b)⇒(a) Since the equality (1) is true for right nilpotent algebra, we are in the conditions of Lemma 2.6. Consider the permutation of the first indices $\{1,\dots,n\}$ of the matrix A as $\pi(j)=\pi_j$, where π_j is defined in the proof of Lemma 2.6. Note that Lemma 2.6 is also true for columns: for any j, there is a column τ_j with j zeros. Moreover, $\tau_p \neq \tau_q$ if $p \neq q$. Now consider the permutation of the second indexes $\{1,\dots,n\}$ of A as $\tau(j)=\tau_j$. Then $\tau(\pi(A))=\widehat{A}$.

The implication (b)⇒(c) is evident since every right nilpotent evolution algebra is a nil algebra.

The implication (a)⇒(b)&(c) is also true because the table of the multiplication of the evolution algebra is defined by an upper triangular matrix A, which is right nilpotent and nil.

The implication (c)⇒(a) follows from Theorem 2.2. □

3 Conditions for $E^k = 0$

For an evolution algebra E, we define the *lower central series* by $E^1 = E$ and $E^k = \sum_{i=1}^{k-1} E^i E^{k-i}$ for $k \geq 2$, and the *derived series* by $E^{(1)} = E$ and $E^{(k)} = E^{(k-1)}E^{(k-1)}$ for $k \geq 2$. An evolution algebra E is called *nilpotent* if there exists $n \in \mathbb{N}$ such that $E^n = 0$, and it is called *solvable* if there exists $m \in \mathbb{N}$ such that $E^{(m)} = 0$. Any nilpotent evolution algebra is solvable, and in [1], it is proved that the notions of nilpotent and right nilpotent are equivalent.

In this section we consider an n-dimensional evolution algebra E with a triangular matrix A (as $\widehat{A}$ in Theorem 2.7), and for small values of k, we present conditions on entries of A under which $E^k = 0$.

First, for $n=3$ we have $E^2 = 0 \Leftrightarrow a_{ij} = 0$ and $E^3 = 0 \Leftrightarrow a_{12}a_{23} = 0$. For $n = 4$ one easily finds $E^2 = 0 \Leftrightarrow a_{ij} = 0$ and

$$E^3 = 0 \iff a_{12}a_{23} = 0,\ a_{12}a_{24} = 0,\ a_{13}a_{34} = 0,\ a_{23}a_{34} = 0.$$

Now we consider an arbitrary $n \in \mathbb{N}$ and $k = 3, 4, 5$, we shall derive a system of equations (for a_{ij}) whose solutions give $E^k = 0$ for $k = 3, 4, 5$.

Let $E = \langle e_1, \dots, e_n \rangle$ be an evolution algebra with the matrix (2).

Case $k = 3$: We have $e_i^2 e_j = e_j e_i^2$ for all i, j and $e_i e_j e_k = 0$ for $i \neq j$. Thus, $e_i^2 e_j = 0$ for $j \leq i$, and $e_i^2 e_j = \sum_{s=j+1}^n a_{ij} a_{js} e_s$ for $j \geq i+1$. So the system of equations is

$$a_{ij} a_{js} = 0 \quad (i = 1, \dots, n,\ j \geq i+1,\ s \geq j+1). \tag{$n;3$}$$

Case $k = 4$: Since $e_i^2 e_j^2 = e_j^2 e_i^2$, $(e_i^2 e_j) e_s = (e_j e_i^2) e_s = e_s (e_j e_i^2) = e_s (e_i^2 e_j)$, $e_i e_j e_s e_t = 0$ if $i \neq j$; $(e_i e_j e_s) e_t = e_t (e_i e_j e_s), \dots$, it is sufficient to consider $e_i^2 e_j^2$ and $(e_i^2 e_j) e_s$. We have $e_i^2 e_j^2 = \sum_{t=j+2}^n \left(\sum_{u=j+1}^{t-1} a_{iu} a_{ju} a_{ut} \right) e_t$ for $i \leq j$, and $(e_i^2 e_j) e_s = \sum_{t=s+1}^n a_{ij} a_{js} a_{st} e_t$ for $j \geq i+1$ and $s \geq j+1$. So the system of equations is

$$\begin{cases} \sum\limits_{u=j+1}^{t-1} a_{iu} a_{ju} a_{ut} = 0 & (j = 1, \dots, n,\ i \geq j,\ t \geq j+2); \\ a_{ij} a_{js} a_{st} = 0 & (j \geq i+1,\ s \geq j+1,\ t \geq s+1). \end{cases} \tag{$n;4$}$$

Case $k = 5$: We should only use previous non-zero words and multiply them to get a word of length 5:

$$\begin{aligned} e_i^2 e_j^2 e_s &= \sum_{t=s+1}^n \Big(\sum_{u=j+1}^{s-1} a_{iu} a_{ju} a_{us} a_{st} \Big) e_t \quad (i \leq j,\ s \geq j+2); \\ (e_i^2 e_j) e_s^2 &= a_{ij} \sum_{u=s+2}^n \Big(\sum_{t=s+1}^{u-1} a_{jt} a_{st} a_{tu} \Big) e_u \quad (j \leq i+1,\ s \geq j); \\ e_i^2 e_j e_s e_v &= \sum_{u=v+1}^n \big(a_{ij} a_{js} a_{sv} a_{vu} \big) e_u \quad (j \geq i+1,\ s \geq j+1,\ v \geq s+1). \end{aligned}$$

Thus, we get the following system of equations

$$\begin{cases} \sum\limits_{u=j+1}^{s-1} a_{iu} a_{ju} a_{us} a_{st} = 0 & (j \leq i,\ s \geq j+2,\ t \geq s+1); \\ a_{ij} \sum\limits_{t=s+1}^{u-1} a_{jt} a_{st} a_{tu} = 0 & (j \geq i+1,\ s \geq j,\ u \geq s+2); \\ a_{ij} a_{js} a_{sv} a_{vu} = 0. \end{cases} \tag{$n;5$}$$

Therefore, we have proved the following:

Proposition 3.1. *Let E be an evolution algebra with the matrix* (2). *Then $E^k = 0$ if the elements of the matrix* (2) *satisfy the equations* $(n; k)$, *where* $k = 3, 4, 5$.

4 Classification of Complex 2-Dimensional Evolution Algebras

Let E and E' be evolution algebras and $\{e_i\}$ a natural basis of E. A linear map $\varphi \colon E \to E'$ is called a *homomorphism* of evolution algebras if $\varphi(xy) = \varphi(x)\varphi(y)$ and the set $\{\varphi(e_i)\}$ is a subset of a natural basis of E'. Moreover, if φ is bijective, then it is called an *isomorphism*.

Let E be a 2-dimensional complex evolution algebra and $\{e_1, e_2\}$ be a basis of the algebra E. It is evident that if $\dim E^2 = 0$, then E is an abelian algebra, i.e., an algebra with all products equal to zero.

Theorem 4.1. *Any 2-dimensional complex evolution algebra E is isomorphic to one of the following pairwise non-isomorphic algebras:*

(i) $\dim E^2 = 1:$

- $E_1 : e_1e_1 = e_1,\ e_2e_2 = 0;$
- $E_2 : e_1e_1 = e_1,\ e_2e_2 = e_1;$
- $E_3 : e_1e_1 = e_1 + e_2,\ e_2e_2 = -e_1 - e_2;$
- $E_4 : e_1e_1 = e_2,\ e_2e_2 = 0.$

(ii) $\dim E^2 = 2:$

- $E_5 : e_1e_1 = e_1 + a_2e_2,\ e_2e_2 = a_3e_1 + e_2,\ 1 - a_2a_3 \neq 0,$ *where* $E_5(a_2, a_3) \cong E_5'(a_3, a_2)$;
- $E_6 : e_1e_1 = e_2,\ e_2e_2 = e_1 + a_4e_2,$ *where for* $a_4 \neq 0$, $E_6(a_4) \cong E_6(a_4')$ *if and only if* $\frac{a_4'}{a_4} = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ *for some* $k = 0, 1, 2.$

Proof. We have $e_1e_1 = a_1e_1 + a_2e_2$, $e_2e_2 = a_3e_1 + a_4e_2$ and $e_1e_2 = e_2e_1 = 0$.

(i) Since $\dim E^2 = 1$, we have $e_1e_1 = c_1(a_1e_1 + a_2e_2)$, $e_2e_2 = c_2(a_1e_1 + a_2e_2)$ and $e_1e_2 = e_2e_1 = 0$. Evidently, $(c_1, c_2) \neq (0, 0)$, because otherwise our algebra will be abelian. Since e_1 and e_2 are symmetric, we can suppose $c_1 \neq 0$, and by a simple change of basis we can suppose $c_1 = 1$.

Case 1. $a_1 \neq 0$. We take an appropriate change of basis $e_1' = a_1e_1 + a_2e_2$, $e_2' = Ae_1 + Be_2$, where $a_1B - a_2A \neq 0$. Consider the product

$$\begin{aligned} 0 = e_1'e_2' &= (a_1e_1 + a_2e_2)(Ae_1 + Be_2) \\ &= a_1A(a_1e_1 + a_2e_2) + a_2Bc_2(a_1e_1 + a_2e_2) = (a_1A + a_2Bc_2)(a_1e_1 + a_2e_2). \end{aligned}$$

Therefore, $a_1A + a_2Bc = 0$, i.e., $A = -\frac{a_2Bc_2}{a_1}$ and $a_1B - a_2A = a_1B + \frac{a_2^2Bc_2}{a_1} \neq 0$.

It means that in the case when $a_1^2 + a_2^2c_2 \neq 0$ we can take the above change.

Consider the products

$$\begin{aligned} e_1'e_1' &= (a_1e_1 + a_2e_2)(a_1e_1 + a_2e_2) = a_1^2(a_1e_1 + a_2e_2) + a_2^2c_2(a_1e_1 + a_2e_2) \\ &= (a_1^2 + a_2^2c_2)(a_1e_1 + a_2e_2) = (a_1^2 + a_2^2c_2)e_1', \\ e_2'e_2' &= (Ae_1 + Be_2)(Ae_1 + Be_2) = A^2(a_1e_1 + a_2e_2) + B^2c_2(a_1e_1 + a_2e_2) \\ &= (A^2 + B^2c_2)(a_1e_1 + a_2e_2) = \left(\frac{a_2^2B^2c_2^2}{a_1^2} + B^2c_2\right)e_1' = \frac{B^2c_2(a_1^2 + a_2^2c_2)}{a_1^2}e_1'. \end{aligned}$$

Case 1.1. $c_2 = 0$. Then $e_1e_1 = a_1^2e_1$ and $e_2e_2 = e_1e_2 = e_2e_1 = 0$. Taking $e_1' = \frac{e_1}{a_1^2}$, we obtain the algebra E_1.

Case 1.2. $c_2 \neq 0$. Then taking $B = \sqrt{\frac{a_1^2}{c_2}}$, we obtain $e_1e_1 = (a_1^2 + a_2^2c_2)e_1$ and $e_2e_2 = (a_1^2 + a_2^2c_2)e_1$.

If $a_1^2 + a_2^2c_2 \neq 0$, the change of basis $e_1' = \frac{e_1}{a_1^2 + a_2^2c_2}$, $e_2' = \frac{e_2}{a_1^2 + a_2^2c_2}$ brings us to the algebra with multiplication $e_1e_1 = e_1$ and $e_2e_2 = e_1$.

If $a_1^2 + a_2^2c_2 = 0$, then $c_2 = -\frac{a_1^2}{a_2^2}$ and we have $e_1e_1 = a_1e_1 + a_2e_2$ and $e_2e_2 = -\frac{a_1^3}{a_2^2}e_1 - \frac{a_1^2}{a_2}e_2$. The change of basis $e_1' = \frac{e_1}{a_1}$, $e_2' = \frac{a_2}{a_1^2}e_2$ gives the algebra E_3.

Case 2. $a_1 = 0$. Then we have $e_1e_1 = a_2e_2$ and $e_2e_2 = c_2a_2e_2$, where $a_2 \neq 0$.

If $c_2 = 0$, then by the change $e'_1 = \frac{e_1}{\sqrt{a_2}}$ we get the algebra E_4.

If $c_2 \neq 0$, then by $e'_1 = \frac{e_1}{\sqrt{c_2 a_2^2}}$ and $e'_2 = \frac{e_2}{c_2 a_2}$, we get the algebra $e_1e_1 = e_2$, $e_2e_2 = e_2$, which is isomorphic to the algebra E_2.

(ii) Now we consider algebras with $\dim E^2 = 2$. Let us write $e_1e_1 = a_1e_1 + a_2e_2$ and $e_2e_2 = a_3e_1 + a_4e_2$, where $a_1a_4 - a_2a_3 \neq 0$.

Case 1. $a_1 \neq 0$ and $a_4 \neq 0$. Then the change of basis $f_1 = a_1^{-1}e_1$, $f_2 = a_4^{-1}e_2$ makes possible to suppose $a_1 = a_4 = 1$. Therefore, we have the two-parametric family $E_5(a_2, a_3)$: $e_1e_1 = e_1 + a_2e_2$, $e_2e_2 = a_3e_1 + e_2$, $1 - a_2a_3 \neq 0$.

Let us take the general change of basis $e'_1 = A_1e_1 + A_2e_2$, $e'_2 = B_1e_1 + B_2e_2$, where $A_1B_2 - A_2B_1 \neq 0$. Consider the product

$$\begin{aligned}0 &= e'_1e'_2 = (A_1e_1 + A_2e_2)(B_1e_1 + B_2e_2) = A_1B_1(e_1 + a_2e_2) + A_2B_2(a_3e_1 + e_2)\\ &= (A_1B_1 + A_2B_2a_3)e_1 + (A_1B_1a_2 + A_2B_2)e_2.\end{aligned}$$

Since in this new basis the algebra should be also an evolution algebra, we have $A_1B_1 + A_2B_2a_3 = 0$ and $A_1B_1a_2 + A_2B_2 = 0$. From this we have $A_2B_2(1-a_2a_3) = 0$ and $A_1B_1(1 - a_2a_3) = 0$. Since $1 - a_2a_3 \neq 0$, we have $A_1B_1 = A_2B_2 = 0$.

Case 1.1. $A_2 = 0$. Then $B_1 = 0$. Consider the products

$$\begin{aligned}e'_1e'_1 &= A_1^2(e_1 + a_2e_2) = e'_1 + a'_2e'_2 = A_1e_1 + a'_2B_2e_2\\ &\implies A_1^2 = A_1,\ A_1^2a_2 = a'_2B_2 \implies A_1 = 1,\\ e'_2e'_2 &= B_2^2(a_3e_1 + e_2) = a'_3e'_1 + e'_2 = a'_3A_1e_1 + B_2e_2\\ &\implies B_2^2a_3 = a'_3A_1,\ B_2^2 = B_2 \implies B_2 = 1.\end{aligned}$$

Case 1.2. $A_1 = 0$. Then $B_2 = 0$, and from the family of algebras $E_5(a_2, a_3)$ we get the family $E_5(a_3, a_2)$.

Case 2. $a_1 = 0$ or $a_4 = 0$. Since e_1 and e_2 are symmetric, without loss of generality we can suppose $a_1 = 0$, i.e., $e_1e_1 = a_2e_2$ and $e_2e_2 = a_3e_1 + a_4e_2$, where $a_2a_3 \neq 0$.

Taking the change of basis $e'_1 = \sqrt[3]{\frac{1}{a_2^2a_3}}\, e_1$, $e'_2 = \sqrt[3]{\frac{1}{a_2a_3^2}}\, e_2$, we obtain the one-parametric family of algebras $E_6(a_4)$: $e_1e_1 = e_2$, $e_2e_2 = e_1 + a_4e_2$.

Let us take the general change of basis $e'_1 = A_1e_1 + A_2e_2$, $e'_2 = B_1e_1 + B_2e_2$, where $A_1B_2 - A_2B_1 \neq 0$. Consider the product

$$0 = e'_1e'_2 = (A_1e_1 + A_2e_2)(B_1e_1 + B_2e_2) = A_1B_1e_2 + A_2B_2(e_1 + a_4e_2).$$

Therefore, $A_1B_1 + A_2B_2a_4 = 0$ and $A_2B_2 = 0$, implying $A_1B_1 = 0$ and $A_2B_2 = 0$. Without loss of generality we can assume $A_2 = 0$. Then $B_1 = 0$.

Consider the products

$$\begin{aligned}e'_1e'_1 &= A_1^2e_2 = e'_2 = B_2e_2 \implies A_1^2 = B_2,\\ e'_2e'_2 &= B_2^2(e_1 + a_4e_2) = e'_1 + a'_4e'_2 = A_1e_1 + a'_4B_2e_2\\ &\qquad\qquad \implies B_2^2 = A_1,\ B_2^2a_4 = B_2a'_4.\end{aligned}$$

From these equalities we have $B_2^3 = 1$ and $B_2 a_4 = a_4'$.

If $\frac{a_4'}{a_4} = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ for some $k = 0, 1, 2$, putting $B_2 = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ we obtain the isomorphism between the algebras $E_6(a_4)$ and $E_6(a_4')$.

The obtained algebras are pairwise non-isomorphic, which may be checked by comparison of the algebraic properties listed in the following table:

	$\dim E^2$	Right Nilpotency	dim(Center)	Nil Elements	Solvability
E_1	1	No	1	Yes	No
E_2	1	No	0	Yes	No
E_3	1	No	0	Yes	Yes
E_4	1	Yes	1	Yes	Yes
E_5	2	No	0	No	No
E_6	2	No	0	Yes	No

This completes the proof of the theorem. □

5 Isomorphisms of Evolution Algebras

Since the study of isomorphisms for any class of algebras is a crucial task and taking into account the great difficulties of their description, in this section we consider a particular case of evolution algebras.

Let E be an evolution algebra which has a matrix A in the following form

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \oplus \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

The multiplication in the basis $\{e_1, \ldots, e_{2n}\}$ of this evolution algebra is $e_i e_j = 0$ for $i \neq j$, $e_{2k-1}^2 = a_k e_{2k-1} + b_k e_{2k}$ and $e_{2k}^2 = c_k e_{2k-1} + d_k e_{2k}$ for $k = 1, 2, \ldots, n$.

Let φ be an isomorphism of the evolution algebra E onto E with matrix A'. Write $\varphi = (\alpha_{ij})_{2n\times 2n}$ with $\det(\varphi) \neq 0$. For $i = 1, 2, \ldots, 2n$, we have

$$\begin{aligned}(e_i')^2 = (\varphi(e_i))^2 = {} & (\alpha_{i1}^2 a_1 + \alpha_{i2}^2 c_1)e_1 + (\alpha_{i1}^2 b_1 + \alpha_{i2}^2 d_1)e_2 + \cdots \\ & + (\alpha_{i,2n-1}^2 a_n + \alpha_{i,2n}^2 c_n)e_{2n-1} + (\alpha_{i,2n-1}^2 b_n + \alpha_{i,2n}^2 d_n)e_{2n}.\end{aligned}$$

For $i \neq j$ we get

$$\begin{aligned} e_i' e_j' = {} & (\alpha_{i1}\alpha_{j1}a_1 + \alpha_{i2}\alpha_{j2}c_1)\, e_1 + (\alpha_{i1}\alpha_{j1}b_1 + \alpha_{i2}\alpha_{j2}d_1)\, e_2 + \cdots \\ & + (\alpha_{i,2n-1}\alpha_{j,2n-1}a_n + \alpha_{i,2n}\alpha_{j,2n}c_n)e_{2n-1} \\ & + (\alpha_{i,2n-1}\alpha_{j,2n-1}b_n + \alpha_{i,2n}\alpha_{j,2n}d_n)\, e_{2n} = 0.\end{aligned}$$

From this we obtain

$$\begin{cases} \alpha_{i1}\alpha_{j1}a_1 + \alpha_{i2}\alpha_{j2}c_1 = 0, \\ \alpha_{i1}\alpha_{j1}b_1 + \alpha_{i2}\alpha_{j2}d_1 = 0, \\ \quad \cdots \\ \alpha_{i,2n-1}\alpha_{j,2n-1}a_n + \alpha_{i,2n}\alpha_{j,2n}c_n = 0, \\ \alpha_{i,2n-1}\alpha_{j,2n-1}b_n + \alpha_{i,2n}\alpha_{j,2n}d_n = 0. \end{cases} \tag{3}$$

Let S_{2n} be the group of permutations of $\{1, 2, \ldots, 2n\}$.

Theorem 5.1. *Assume* $\det(A) \neq 0$.

(i) *For any isomorphism* $\varphi\colon E \to E$, *there exists a unique* $\pi = \pi(\varphi) \in S_{2n}$ *such that* $\varphi \in \Phi_\pi = \big\{(\alpha_{ij})_{2n\times 2n} : \alpha_{i\pi(i)} \neq 0\ (1 \le i \le 2n),\ \text{other } \alpha_{ij} = 0\big\}$. *Moreover,* $\Phi = \bigcup_{\pi\in S_{2n}} \Phi_\pi$ *is the set of all possible isomorphisms.*

(ii) *For any* $\pi, \tau \in S_{2n}$ *we have* $\Phi_\pi\Phi_\tau = \{\varphi\psi : \varphi \in \Phi_\pi, \psi \in \Phi_\tau\} = \Phi_{\tau\pi}$. *The set* $G = \{\Phi_\pi : \pi \in S_{2n}\}$ *is a multiplicative group.*

Proof. (i) Since $\det(A) \neq 0$, we have $a_i d_i - b_i c_i \neq 0$ for any $i = 1, 2, \ldots, n$. Thus, from (3) we have

$$\alpha_{ik}\alpha_{jk} = 0 \quad (i \neq j). \tag{4}$$

By (4) it is easy to see that each row and each column of the matrix φ must contain exactly one non-zero element. It is not difficult to see that every such matrix φ corresponds to a permutation π. The set of all possible solutions of (4) give all the possible isomorphisms, i.e., we get the set Φ.

(ii) Take $\varphi = (\alpha_{ij}) \in \Phi_\pi$ and $\psi = (\beta_{ij}) \in \Phi_\tau$. Denote $\varphi \circ \psi = (\gamma_{ij})$. It is easy to see that $\gamma_{ij} = 0$ if $j \neq \tau(\pi(i))$, and $\gamma_{ij} = \alpha_{i\pi(i)}\beta_{\pi(i)\tau(\pi(i))}$ if $j = \tau(\pi(i))$. This gives $\Phi_\pi\Phi_\tau = \Phi_{\tau\pi}$ and then one can easily check that G is a group. □

Now for a fixed φ (i.e., π) we shall find the matrix A'. Consider $\pi \in S_{2n}$ and the corresponding $\varphi_\pi = (\alpha_{ij})$, where $\alpha_{ij} = 0$ if $j \neq \pi(i)$, and $\alpha_{ij} = \alpha_{i\pi(i)}$ if $j = \pi(i)$. We have

$$e_i' = \alpha_{i\pi(i)} e_{\pi(i)} \quad (i = 1, \ldots, 2n). \tag{5}$$

Using this equality we get

$$(e_i')^2 = \alpha_{i\pi(i)}^2 e_{\pi(i)}^2 = \begin{cases} \alpha_{i(2k-1)}^2 (a_k e_{2k-1} + b_k e_{2k}) & \text{if } \pi(i) = 2k-1, \\ \alpha_{i(2k)}^2 (c_k e_{2k-1} + d_k e_{2k}) & \text{if } \pi(i) = 2k. \end{cases}$$

By (5), from the last equality we get

$$(e_i')^2 = \begin{cases} (\alpha_{i(2k-1)} a_k) e_i' + \big(\frac{\alpha_{i(2k-1)}^2}{\alpha_{i(2k)}}\, b_k\big) e'_{\pi^{-1}(2k)} & \text{if } \pi(i) = 2k-1, \\ \big(\frac{\alpha_{i(2k)}^2}{\alpha_{i(2k-1)}}\, c_k\big) e'_{\pi^{-1}(2k-1)} + (\alpha_{i(2k)} d_k) e_i' & \text{if } \pi(i) = 2k. \end{cases}$$

Thus, $A' = (a'_{ij})$ is a matrix with

$$a'_{ij} = \begin{cases} \alpha_{i(2k-1)} a_k & \text{if } \pi(i) = 2k-1,\ j = i, \\ \frac{\alpha_{i(2k-1)}^2}{\alpha_{i(2k)}}\, b_k & \text{if } \pi(i) = 2k-1,\ \pi(j) = 2k, \\ \frac{\alpha_{i(2k)}^2}{\alpha_{i(2k-1)}}\, c_k & \text{if } \pi(i) = 2k,\ \pi(j) = 2k-1, \\ \alpha_{i(2k)} d_k & \text{if } \pi(i) = 2k,\ j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Theorem 5.2. *Assume* $\det(A) \neq 0$. *Let* $\varphi\colon E \to E$ $(A \to A')$ *be an isomorphism. Then* A' *has the same form as* A *if and only if* φ *belongs to* Φ_π, *where* $\pi \in S^b_{2n} = \{\pi = (\pi(1), \ldots, \pi(2n)) \in S_{2n} : \pi(i) \in \{\pi(i-1) \pm 1\},\ i = 1, 2, \ldots, 2n\}$.

Proof. Using the above formula (6) for A' and the condition $\det(A) \neq 0$, one can see that A' has form as A if and only if $\pi(i) \in \{\pi(i-1) \pm 1\}$ for all i. □

Properties of the matrix A can be uniquely defined by properties of its non-zero blocks. So if we consider $n = 1$, then for $\det(A) \neq 0$ we have two classes of isomorphisms:

$$\Phi_{12} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha\delta \neq 0 \right\} \quad \text{with } A' = \begin{pmatrix} a\alpha & b\frac{\alpha^2}{\delta} \\ c\frac{\delta^2}{\alpha} & d\delta \end{pmatrix};$$

$$\Phi_{21} = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : \beta\gamma \neq 0 \right\} \quad \text{with } A' = \begin{pmatrix} d\beta & c\frac{\beta^2}{\gamma} \\ b\frac{\gamma^2}{\beta} & a\gamma \end{pmatrix}.$$

It is easy to check $\Phi_{12}\Phi_{21} \subset \Phi_{21}$, $\Phi_{21}\Phi_{12} \subset \Phi_{21}$, $\Phi_{21}\Phi_{21} \subset \Phi_{12}$, and Φ_{12} is a group.

Assuming that the matrices A and A' are symmetric, we get the following classes of isomorphisms:

$$\Phi_{12}^s = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha\delta \neq 0,\ ad - b^2 \neq 0,\ b\alpha^3 = b\delta^3 \right\},$$

$$\Phi_{21}^s = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} : \beta\gamma \neq 0,\ ad - b^2 \neq 0,\ b\beta^3 = b\gamma^3 \right\}.$$

Appendix. The following program written in Mathematica allows to check the existence (or non-existence) of an isomorphism between two evolution algebras of dimension n. It is based on the star product $A * B$ of two evolution matrices (see [4, page 31]) and the computation of Gröbner bases. In particular, one can check that the algebras E_i $(i = 1, \dots, 6)$ are pairwise non-isomorphic.

```
StarProduct[A_List, B_List] := Module[{icont, jcont, kcont, AEB,
ndim, Indices}, ndim = Dimensions[A][[1]];
  Indices = {};
  Do[Indices = Join[Indices, {{icont, jcont}}];
  ,{icont, 1, ndim}, {jcont, icont + 1, ndim}];
  AEB = Table[Table[aux[icont, jcont], {icont, 1, ndim}],
              {jcont,1, (ndim^2 - ndim)/2}];
    Do[AEB[[icont, kcont]] =
    A[[Indices[[icont]][[1]], kcont]]*
    B[[Indices[[icont]][[2]], kcont]];
    ,{icont, 1, Length[Indices]}, {kcont, 1, ndim}];
        Return[AEB];]
  SystemEquations[P_List, Q_List] :=Module[{FirstEquation,
  SecondEquation, ThirdEquation, A, ndim, Result},
    ndim = Dimensions[P][[1]];
    A = Table[Table[aux[icont, jcont], {jcont, 1, ndim}],
    {icont, 1, ndim}];
    FirstEquation = (A*A).Q - P.A;
    SecondEquation = StarProduct[Transpose[A], Transpose[A]].Q;
    ThirdEquation = {Det[A]*Y - 1};
    Result = Join[Flatten[FirstEquation],
```

```
    Flatten[SecondEquation], ThirdEquation]; Return[Result];]
    IsoEvolAlgebrasQ[A1_, A2_] := Module[{Equations, BGrobner},
    Equations = SystemEquations[A1, A2];
    BGrobner = GroebnerBasis[Equations, Variables[Equations]];
    (* Print temporal *)
    Print[BGrobner];
    If[BGrobner == {1},
    Print["Evolution algebras are NOT isomorphic"];
      Print["Evolution algebras are isomorphic"]; ];]
```

Example 5.3. We check that the evolution algebras E_5 and E_6 are not isomorphic.

```
IsoEvolAlgebrasQ[{{1, a2}, {a3, 1}}, {{0, 1}, {1, a4}}]
{1}
Evolution algebras are NOT isomorphic
```

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