PULSE120520pt1_5PP

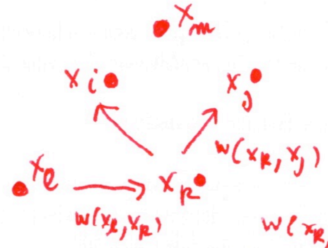
12/2/20 (1)

continued 12/3/20 on p. (6)

pulse.pdf

§2 Pulse Processes

D weighted digraph



vertices $x_1, .., x_n$ $\quad$ $w(x_k, x_\ell) = 0 \quad w(x_\ell, x_i) = 0$

value of vertex at time $t = 0, 1, 2..$ $\quad v_i(t)$

model: $\quad v_i(t+1) = v_i(t) + p_i^0(t+1) + \sum_j w(x_j, x_i)\, p_j(t) \qquad (1)$

vertex x_i

- $v_i(t)$: value at time t
- $p_i^0(t+1)$: value of external pulse introduced at vertex x_i at time $t+1$
- $w(x_j, x_i)$: weight of the arc $x_j x_i$ ($x_j \longrightarrow x_i$)
- $p_j(t)$: pulse at vertex x_j at time t

$p_j(t)$ defined by

$$p_j(t) = \begin{cases} v_j(t) - v_j(t-1) & t > 0 \\ p_j^0(0) & t = 0 \end{cases} \qquad (2)$$

initial vector of values $\quad V(0) = (v_1(0), .., v_n(0))$

external pulse vector $\quad P^0(t) = (p_1^0(t), .., p_n^0(t))$

pulse vector (eq (2)) $\quad P(t) = (p_1(t), .., p_n(t))$

$P(t) = V(t) - V(t-1) \quad t > 0$ (from (2))

(1) can be rewritten as $\quad P(0) = (p_1^0(0), .., p_n^0(0)) = P^0(0)$ (from (2))

$$p_i(t+1) = p_i^0(t+1) + \sum_j w(x_j, x_i)\, p_j(t) \qquad (3)$$

k units in vertex x_j at any time t leads to $k\,w(x_j, x_i)$ units on vertex x_i

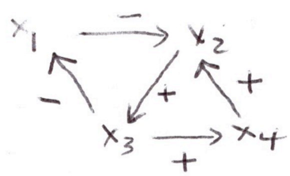
Def 1 autonomous pulse process (no external pulses at any vertex or time $t > 0$)

$P^0(t) = (p_1^0(t), .., p_n^0(t)) = (0, .., 0) \quad \forall t > 0$

simple pulse process starting at vertex x_i = autonomous and

$P^0(0) = (p_1^0(0), .., p_n^0(0)) = (0, .., \underset{i}{1}, .. 0)$ for some i

Example 1



$\omega(x_1, x_2) = -1$
$\omega(x_4, x_2) = 1$
$\omega(x_2, x_3) = 1$
$\omega(x_3, x_4) = 1$
$\omega(x_3, x_1) = -1$

simple pulse process starting at x_3 :

$P(0) = (p_1^0(0), \ldots, p_n^0(0))$

$t = 0 \quad V(0) = (0,0,1,0) = P(0)$

$$v_i(1) = v_i(0) + p_i^0(1) + \sum_j \omega(x_j, x_i)\, p_j(0)$$

$$v_1(1) = 0 + 0 + \overbrace{\omega(x_3, x_1)}^{-1} \cdot \overbrace{p_3(0)}^{1} = -1$$

$$v_2(1) = 0 + 0 + \overbrace{\omega(x_4, x_2)}^{1} \cdot \overbrace{p_4(0)}^{0} + \overbrace{\omega(x_1, x_2)}^{-1}\, \overbrace{p_1(0)}^{0} = 0$$

$$v_3(1) = 1 + 0 + \overbrace{\omega(x_2, x_3)}^{1} \cdot \overbrace{p_2(0)}^{0} = 1$$

$$v_4(1) = 0 + 0 + \overbrace{\omega(x_3, x_4)}^{1} \cdot \overbrace{p_3(0)}^{1} = 1$$

so if $t = 1 \quad V(1) = (-1, 0, 1, 1)$

$$P(1) = V(1) - V(0) = (-1,0,1,1) - (0,0,1,0) = (-1,0,0,1)$$

if $t = 2$

$$v_1(2) = \underbrace{v_1(1)}_{-1} + \underbrace{p_1^0(2)}_{0} + \underset{-1}{\omega(x_3, x_1)} \cdot \underset{0}{p_3(1)} = -1$$

$$v_2(2) = \underbrace{v_2(1)}_{0} + \underbrace{p_2^0(2)}_{0} + \underset{1}{\omega(x_4, x_2)}\, \underset{1}{p_4(1)} + \underset{-1}{\omega(x_1, x_2)}\, \underset{-1}{p_1(1)} = 2$$

$$v_3(2) = \underset{1}{v_3(1)} + \underset{0}{p_3^0(2)} + \underset{1}{\omega(x_2, x_3)}\, \underset{0}{p_2(1)} = 1$$

$$v_4(2) = \underset{1}{v_4(1)} + \underset{0}{p_4^0(2)} + \underset{1}{\omega(x_3, x_4)}\, \underset{0}{p_3(1)} = 1$$

$\therefore \quad V(2) = (-1, 2, 1, 1)$

$$P(2) = V(2) - V(1) = (-1,2,1,1) - (-1,0,1,1) = (0,2,0,0) \ldots \; \blacksquare$$

For autonomous pulse processes

$$p_i(t+1) = \sum_j \omega(x_j, x_i)\, p_j(t)$$

adjacency matrix

in example 1

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \overset{0}{\overset{\shortparallel}{\omega(x_1,x_1)}} & \cdots & \omega(x_1,x_n) \\ \vdots & 0 & \\ \omega(x_n,x_1) & & \overset{0}{\overset{\shortparallel}{\omega(x_n,x_n)}} \end{pmatrix}$$

<u>Theorem 1</u> Autonomous pulse process $\Rightarrow$ $P(t) = P(0)A^t$ $(t \geq 0)$
([2] Roberts AMM 1975 Th. 3)

<u>Def</u> 2 vertex x_j is <u>pulse stable</u> under a pulse process if

$\{|p_j(t)| : t = 0,1,2,\ldots\}$ is bounded

vertex x_j is <u>value stable</u> under a pulse process if

$\{|v_j(t)| : t = 0,1,2,\ldots\}$ is bounded.

If each vertex is <u>pulse (resp value) stable</u> we say the digraph is <u>pulse (resp. value) stable</u>

<u>Remark</u> $|p_j(t)| \leq |v_j(t)| + |v_j(t-1)|$ so value stable $\Rightarrow$ pulse stable.

Theorem 2 ([2: Th 4]) If A has eigenvalue λ with $|\lambda| > 1$ then D is pulse unstable under some simple pulse process.

Theorem 3 ([2: Th 5]) (characterization of pulse stability)

(i) D is pulse stable under all autonomous pulse processes

(ii) " " " " " " simple (autonomous) " "

(iii) $\sigma(A) \subseteq$ closed unit disk, and if $\lambda \in \sigma(A)$ has different algebraic & geometric multiplicities then $|\lambda| < 1$. (DEF by Jordan Canonical form)

Theorem 4 ([2: Th. 6]) characterization of value stability

(i) D is value stable under all autonomous pulse processes

(ii) " " " " " " (") simple pulse processes

(iii) D is pulse stable under all simple pulse processes and $\lambda = 1$ is not an eigenvalue of A.

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§3 Evolution Algebras

<u>Def 3</u> evolution algebra A has a natural basis $B = \{e_1, \dots, e_n\}$, $e_i e_j = 0 \ \forall i \neq j$,

$$e_1^2 = \sum_k \omega_{k1} e_k, \ \dots \quad e_n^2 = \sum_k \omega_{kn} e_k$$

$M_B(A) = (\omega_{ij})$ structure matrix of A relative to B

$$= \begin{array}{c} \begin{array}{ccc} e_1^2 & & e_n^2 \end{array} \\ \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \end{array}$$

evolution operator of A associated to B $\quad L_B : A \to A$

$$L_B(e_i) = e_i^2 = \sum_k \omega_{ki} e_k$$

$$a = \sum \alpha_i e_i \qquad L_B(a) = \sum_i \alpha_i L_B(e_i) = \sum_i \alpha_i e_i^2$$

$$= \sum_i \alpha_i \sum_k \omega_{ki} e_k$$

Let $e = e_1 + \dots + e_n$ $\quad$ Then $ea = \left(\sum_j e_j\right)\left(\sum_i \alpha_i e_i\right) = \sum_{i,j} \alpha_i e_j e_i$

$$= \sum \alpha_i e_i^2 \quad \text{so} \quad \boxed{L_B(a) = ea}$$

<u>Def</u> A <u>non-zero trivial evolution algebra</u> is an evolution algebra with natural basis B s.t. $e_i^2 = \omega_{ii} e_i$ with $\omega_{ii} \neq 0$ $\ 1 \leq i \leq n$

Th 5 (Tian[28]) A has a unit $\Leftrightarrow$ it is non-zero trivial evol. algebra.

Def 5 m-spectrum of $a \in A$ complex algebra with unit e

$$\sigma_m^A(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not m-invertible}\}$$

$b \in A$ is m-invertible if L_b and R_b are bijective

If A has no unit $\sigma_m^A(a) := \sigma_m^{A_1}(a)$, A_1 = unitization of A

If A is real $\sigma_m^A(a) := \sigma_m^{A_{\mathbb{C}}}(a)$, $A_{\mathbb{C}}$ = complexification of A

[29: Prop 2.5] $(A, \| \|)$ nonassoc B alg $\Rightarrow \sigma_m^A(a) \subseteq \{|\lambda| \le \|a\|\}$

[29: Prop 2.2] A complex algebra, $a \in A \Rightarrow$

- if A has no unit: $\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\}$
- if A has a unit: $\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a)$

$$\sigma^{\mathcal{L}(A)}(T) = \{\lambda \in \mathbb{C} : T - \lambda I : A \to A \text{ is not bijective}\}$$

Th 6 $\theta : A \to \tilde{A}$ homo of algebras

(i) θ isomorphism $\Rightarrow \sigma_m^{\tilde{A}}(\theta(a)) = \sigma_m^{a}(A) \quad \forall a \in A$

(ii) θ epimorphism $\Rightarrow \sigma_m^{\tilde{A}}(\theta(a)) \subseteq \sigma_m^{a}(A)$

[proof is given − 1 full page]

Th. 7 [Tian [28], Prop 5.1, 5.3] (with help of) A evol alg, B natural basis

$B = \{e_1, \dots, e_n\}$ $\quad M_B(A) = (\omega_{ij})$ $\quad a = \sum \alpha_i e_i \in A$, $\lambda \in \mathbb{C} \setminus \{0\}$

$$\Rightarrow \quad \lambda \in \sigma_m^A(a) \Leftrightarrow \lambda \text{ is an eigenvalue of } \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & & \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$$

(Hence if A is non-zero trivial evolution algebra, $M_B(A) = \operatorname{diag}(\omega_{11}, \dots, \omega_{nn})$

so $\sigma_m^A(a) = \{\alpha_i \omega_{ii} : i = 1, \dots, n\}$)

Otherwise $0 \in \sigma_m^A(a) \quad \forall a \in A$. ▨