**Elementary Analysis Math 140B—Winter 2007**
**Sample Final Examination; March 16, 2007**

1. Which of the following functions $f(x)$ has the property that $f'(x)$ can be calculated for each $x$ on the specified interval by differentiating the series for $f(x)$ term by term?

   **(a)** $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad 0 \le x \le 1.$

   **(b)** $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{n^2 (\log n)^2} \qquad -1 \le x \le 1.$

   **(c)** $f(x) = \sum_{n=1}^{\infty} \left( \frac{1}{x - n\pi} + \frac{1}{n\pi} \right) \qquad 0 < x < \pi.$

2. Suppose $f(x)$ is defined by a power series in $x$ with positive radius of convergence.

   **(a)** Let $k$ be the smallest positive integer such that $f^{(k)}(0) \neq 0$. (We suppose $f$ is not a constant, so that there really is such an integer $k$.) If $k$ is odd, show that $f$ has neither a relative maximum nor a relative minimum at $x = 0$.

   **(b)** If the function $f$ is an even function ($f(-x) = f(x)$), show that all the coefficients of odd powers of $x$ are zero.

3. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on any interval of the form $[p, \infty)$ with $p > 1$.

4. Let $f_n(x) = (x/n)e^{-x/n}$ for $x \ge 0$. Prove that the sequence converges uniformly on $[0, A]$ for any $A > 0$. Is the convergence uniform on the interval $[0, \infty)$?

5. Suppose that $f$ is defined and differentiable in an open interval containing $x = a$. Suppose that $f''(x)$ exists in an interval containing $a$ and that $f''$ is continuous at $a$.

   **(a)** Show that
   $$f''(a) = \lim_{x \to a} \frac{f(x) - f(a) - (x - a) f'(a)}{\frac{(x-a)^2}{2}}.$$

   **(b)** Show that if $f$ has a relative minimum at $x = a$, then $f''(a) \ge 0$ and that if $f$ has a relative maximum at $x = a$, then $f''(a) \le 0$.

6. Find the following limits.

   (a) $$\lim_{x \to 0} \frac{(e^x - 1)\sin x}{\cos x - \cos^2 x}$$

   (b) $$\lim_{x \to \infty} \frac{1}{x^2} \int_1^x \sin^2 t \, dt$$

   (c) $$\lim_{x \to 0+} \left( \frac{1}{x^2} - \frac{1}{x \log x} \right)$$

7. Prove that if the bounded function $f$ is integrable on $[a, b]$ and if $f(x) \ge \delta$ for all $x \in [a, b]$ and some $\delta > 0$, then $1/f$ is integrable on $[a, b]$.

8. Show that
   $\int_x^y \tan t \, dt = \log \frac{\cos x}{\cos y}, \quad -\frac{\pi}{2} < x \le y < \frac{\pi}{2}$

9. Show by an appropriate substitution that $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n y \, dy$

10. Show that $\int_a^b x f''(x) \, dx = b f'(b) - f(b) + f(a) - a f'(a)$.