Classification of three dimensional complex Leibniz algebras

Article · May 2012

DOI: 10.1063/1.4724168

2 authors:

Ikrom Rikhsiboev
University of Kuala Lumpur

Isamiddin Rakhimov
Universiti Putra Malaysia

Classification of three dimensional complex Leibniz algebras

Ikrom M. Rikhsiboev and Isamiddin S. Rakhimov

Citation: AIP Conf. Proc. **1450**, 358 (2012); doi: 10.1063/1.4724168
View online: http://dx.doi.org/10.1063/1.4724168
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=1450&Issue=1
Published by the American Institute of Physics.

Classification of three dimensional complex Leibniz algebras

Ikrom M.Rikhsiboev* and Isamiddin S.Rakhimov†

*Malaysian Institute of Industrial Technology, Universiti Kuala Lumpur, Malaysia.
†Department of Mathematics, Faculty of Science & Institute for Mathematical Research, Universiti Putra Malaysia.
ikromr@gmail.com, risamiddin@gmail.com

Abstract. The aim of this paper is to complete the classification of three-dimensional complex Leibniz algebras. The description of isomorphism classes of three-dimensional complex Leibniz algebras has been given by Ayupov and Omirov in 1999. However, we found that this list has a little redundancy. In this paper we apply a method which is more elegant and it gives the precise list of isomorphism classes of these algebras. We compare our list with that of Ayupov-Omirov and show the corrections which should be made.

Keywords: Leibniz algebra, isomorphism
PACS: 02.10.DE, 02.10.Ud, 02.20.Sv

INTRODUCTION

We start with the following

Definition 1 *An algebra L over field F is said to be a Lie algebra, if the following identities hold:*

- $[x,y] = -[y,x]$ *antisymmericity,*
- $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0,$ *Jacobi identity,*

where $[\cdot,\cdot]$ denotes the multiplication in L.

Lie algebras play an important role in different areas of mathematics and physics, such as: theoretical physics, quantum field theory, and others. Nowadays the structural theory of Lie algebras is one of the most intensively developed part of modern algebra. The study of Lie algebra and its applications has raised the different generalizations of it. One of this kind generalization has been introduced by Loday [4]. He called this generalization Liebniz algebra because of an identity which this algebra satisfies.

Definition 2 *An algebra L over a field F is called a Leibniz algebra if it satisfies the following Leibniz identity:*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]] \qquad \forall x,y,z \in L,$$

where $[\cdot,\cdot]$ is the multiplication in L.

If $d_z(\cdot) = [\cdot,z]$, then the Leibniz identity is exactly the Leibniz rule:

$$d_z([x,y]) = [d_z(x),y] + [x,d_z(y)].$$

It is observed that if the bracket $[\cdot,\cdot]$ in Leibniz algebra has the antisymmetric property then the Leibniz identity easily can be reduced to the Jacobi identity. Therefore, Leibniz algebra is a generalization of Lie algebra.

Definition 3 *The subset $R(L)$ defined*

$$R(L) = \{x \in L : [y,x] = 0, \forall y \in L\}.$$

is said to be the right annihilator of a Leibniz algebra L.

It is observed that $R(L)$ is a two sided ideal of L.

Further all algebras considered are assumed to be over the field of complex numbers $\mathbb{C}$.

Note that there is no one-dimensional non trivial Lie algebra and it is an easy exercise to get the list of possible Lie algebra structures on two-dimensional complex space. There exists only one non trivial Lie algebra with the composition law as follows on a basis $\{x,y\}$: $[x,y] = x$.

Here is the result from [3] on classification of three-dimensional non trivial Lie algebras.

Theorem 1 *Any three-dimensional complex Lie algebra G is included in one of the following isomorphism classes of Lie algebras*

$G_1 : [x,y] = x,$
$G_2 : [x,z] = x+y, \quad [y,z] = y,$
$G_3 : [x,z] = 2x, \quad [y,z] = -y, \quad [x,y] = z,$
$G_4(\alpha) : [x,z] = x, \quad [y,z] = \alpha y, \quad \alpha \in \mathbb{C}\setminus\{0\},$
where $\{x,y,z\}$ is a basis of G.

In the theorem above if $\alpha_1 \neq \alpha_2$ then $G_4(\alpha_1)$ is not isomorphic to $G_4(\alpha_2)$.

It is not to hard to describe the isomorphism classes of two-dimensional non Lie Leibniz algebras as well. There are two non trivial classes with representatives

$L_1 : [x,x] = y$ and $L_2 : [x,x] = y, \quad [y,x] = y.$

The 5th International Conference on Research and Education in Mathematics
AIP Conf. Proc. 1450, 358-362 (2012); doi: 10.1063/1.4724168

Notice that there is no parametric family of isomorphism classes.

The purpose of this paper is the description of all possible Leibniz algebra structures on three-dimensional complex space.

DESCRIPTION OF THREE-DIMENSIONAL COMPLEX LEIBNIZ ALGEBRAS

Theorem 2 *Up to isomorphism, there exist three one parametric families and six explicit representatives of non Lie complex Leibniz algebras of dimension three:*

$RR_1 : [e_1,e_3] = -2e_1,\ [e_2,e_2] = e_1,\ [e_3,e_2] = e_2,\ [e_2,e_3] = -e_2;$
$RR_2 : [e_1,e_3] = \alpha e_1,\ [e_3,e_2] = e_2,\ [e_2,e_3] = -e_2,\ \alpha \in \mathbb{C};$
$RR_3 : [e_3,e_3] = e_1,\ [e_3,e_2] = e_2, [e_2,e_3] = -e_2;$
$RR_4 : [e_2,e_2] = e_1,\ [e_3,e_3] = \alpha e_1,\ [e_2,e_3] = e_1,\ \alpha \in \mathbb{C};$
$RR_5 : [e_2,e_2] = e_1,\ [e_3,e_3] = e_1;$
$RR_6 : [e_1,e_3] = e_2,\ [e_2,e_3] = e_1;$
$RR_7 : [e_1,e_3] = e_2,\ [e_2,e_3] = \alpha e_1 + e_2,\ \alpha \in \mathbb{C};$
$RR_8 : [e_3,e_3] = e_1,\ [e_1,e_3] = e_2;$
$RR_9 : [e_3,e_3] = e_1,\ [e_1,e_3] = e_1 + e_2.$

Proof. Consider a three dimensional non Lie complex Leibniz algebra L. Let $R_x(y) = [y,x]$ be the right multiplication operator on L. We specify several cases.

Case 1. $dimR(L) = 1$. Let e_1 be a basis of $R(L)$. Consider a basis of L including e_1: $\{e_1,e_2,e_3\}$. Since $R(L)$ is an ideal of the Leibniz algebra L, we find the products of basis vectors in L as follows:

$[e_1,e_2] = \alpha_1 e_1,$
$[e_2,e_2] = \alpha_2 e_1,$
$[e_1,e_3] = \alpha_3 e_1,$
$[e_3,e_3] = \alpha_4 e_1,$
$[e_3,e_2] = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3,$
$[e_2,e_3] = \gamma_1 e_1 - \beta_2 e_2 - \beta_3 e_3.$

Case 1.1. Let $dimL^2 = 2$. Then we have $(\beta_1,\beta_2) \neq (0,0)$. We may assume that $\beta_2 \neq 0$, otherwise applying the base change $e_1' = e_1$, $e_2' = e_3$ and $e_3' = -e_2$ we can make $\beta_2 \neq 0$.

If $\beta_2 \neq 0$ then the base change $e_1' = e_1$, $e_2' = \beta_2 e_2 + \beta_3 e_3$ and $e_3' = \frac{1}{\beta_2} e_3$ leads to $\beta_2 = 1$ and $\beta_3 = 0$. Then the multiplication table of L has the form:

$[e_1,e_2] = \alpha_1 e_1,$
$[e_2,e_2] = \alpha_2 e_1,$
$[e_1,e_3] = \alpha_3 e_1,$
$[e_3,e_3] = \alpha_4 e_1,$
$[e_3,e_2] = \beta_1 e_1 + e_2,$
$[e_2,e_3] = \gamma_1 e_1 - e_2.$

Applying the Leibniz identity to L we find the following constraints for the structure constants

$$\alpha_1 = 0,\ \alpha_2(2+\alpha_3) = 0,\ \beta_1 + \alpha_3\beta_1 + \gamma_1 = 0. \qquad (1)$$

Now we consider a few cases again.

Case 1.1.1. $\alpha_2 \neq 0$. Then from (1) we get $\alpha_3 = -2, \gamma_1 = \beta_1$. As a result we have the table of multiplication for L as follows

$[e_2,e_2] = \alpha_2 e_1,$
$[e_1,e_3] = -2e_1,$
$[e_3,e_3] = \alpha_4 e_1,$
$[e_3,e_2] = \beta_1 e_1 + e_2,$
$[e_2,e_3] = \beta_1 e_1 - e_2.$

Taking the base change

$$e_1' = \alpha_2 e_1,\ e_2' = e_2 \text{ and } e_3' = \frac{\alpha_2\alpha_4 - \beta_1^2}{2\alpha_2} e_1 - \frac{\beta_1}{\alpha_2} e_2 + e_3,$$

we obtain the following algebra:

$\mathbf{I}: \quad [e_1,e_3] = -2e_1,$
$[e_2,e_2] = e_1,$
$[e_2,e_3] = -e_2,$
$[e_3,e_2] = e_2.$

Case 1.1.2. Let $\alpha_2 = 0$. Then we get

$$\begin{aligned} &[e_1,e_3] = \alpha_3 e_1,\\ &[e_3,e_3] = \alpha_4 e_1,\\ &[e_3,e_2] = \beta_1 e_1 + e_2,\\ &[e_2,e_3] = -\beta_1(1+\alpha_3)e_1 - e_2. \end{aligned} \qquad (2)$$

Case 1.1.2.1 If we assume that $\alpha_4 = 0$ (in this case $\alpha_3 \neq 0$ otherwise L is Lie algebra)

$$\begin{aligned} &[e_1,e_3] = \alpha_3 e_1,\\ &[e_3,e_2] = \beta_1 e_1 + e_2,\\ &[e_2,e_3] = -\beta_1(1+\alpha_3)e_1 - e_2. \end{aligned} \qquad (3)$$

Case 1.1.2.1.1 Let $\beta_1 = 0$ then we have the following multiplication table:

$\mathbf{II}: \quad [e_1,e_3] = \alpha_3 e_1,$
$[e_3,e_2] = e_2,$
$[e_2,e_3] = -e_2.$

Case 1.1.2.1.2 Let $\beta_1 \neq 0$ then we set $e_1' = e_1$, $e_2' = e_2$, $e_3' = \frac{1}{\alpha_3} e_1 + e_3$, and

$$\begin{aligned} &[e_1,e_3] = \alpha_3 e_1,\\ &[e_3,e_3] = e_1,\\ &[e_3,e_2] = \beta_1 e_1 + e_2,\\ &[e_2,e_3] = -\beta_1(1+\alpha_3)e_1 - e_2. \end{aligned} \qquad (4)$$

The base change $e_1' = e_1$, $e_2' = \beta_1 e_1 + e_2$, $e_3' = e_3$, in (4) yields

$[e_1,e_3] = \alpha_3 e_1,$
$[e_2,e_3] = -e_2,$
$[e_3,e_2] = e_2,$
$[e_3,e_3] = e_1.$

In this case if $\alpha_3 \neq 0$ then by using base change $e_1' = e_1$, $e_2' = e_2$, $e_3' = (-\frac{1}{\alpha_3})e_1 + e_2 + e_3$ we will have algebra **II**. If $\alpha_3 = 0$ then we get the following algebra:

$\mathbf{III}: \quad [e_2,e_3] = -e_2,$

$[e_3,e_2]=e_2,$
$[e_3,e_3]=e_1.$

Case 1.1.2.2 If we assume that $\alpha_4\neq 0$ then we may get $\alpha_4=1$ using the base change $e_1'=\alpha_4 e_1, e_2'=e_2, e_3'=e_3$. Multiplication table will has form (4) considered in case 1.1.2.1.2.

Straightforward calculations show that for different values of α_3 the algebras from **II** are not isomorphic to each other.

Case 1.2. Let now $dimL^2=1$. Then the table of multiplications of L has the form:

$[e_1,e_2]=\alpha_1 e_1,$
$[e_2,e_2]=\alpha_2 e_1,$
$[e_1,e_3]=\alpha_3 e_1,$
$[e_3,e_3]=\alpha_4 e_1,$
$[e_3,e_2]=\beta_1 e_1,$
$[e_2,e_3]=\gamma_1 e_1.$

The Leibniz identity gives constraints for the structure constant $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\gamma_1$:

$$\alpha_1\gamma_1=\alpha_2\alpha_3,\ \alpha_1\alpha_4=\alpha_3\beta_1. \tag{5}$$

Case 1.2.1. let $(\alpha_1,\alpha_3)=(0,0)$. It leads to the following multiplication rules in L:

$$\begin{aligned}&[e_2,e_2]=\alpha_2 e_1,\\ &[e_3,e_3]=\alpha_4 e_1,\\ &[e_3,e_2]=\beta_1 e_1,\\ &[e_2,e_3]=\gamma_1 e_1.\end{aligned} \tag{6}$$

It is easy to see that $(\alpha_2,\alpha_4,\beta_1+\gamma_1)\neq(0,0,0)$, otherwise L is a Lie algebra. Let us consider a basis change of the form $e_1'=e_1, e_2'=Ae_2+Be_3, e_3'=e_3$ then we have $[e_2',e_2']=[A^2\alpha_2+B^2\alpha_4+AB(\beta_1+\gamma_1)]e_1$.
The condition $(\alpha_2,\alpha_4,\beta_1+\gamma_1)\neq(0,0,0)$ implies that there are A and B such that

$$A^2\alpha_2+B^2\alpha_4+AB(\beta_1+\gamma_1)\neq 0.$$

Thus in (6) we may assume that $\alpha_2\neq 0$. Now consider the base change $e_1'=\alpha_2 e_1, e_2'=e_2, e_3'=e_3-\frac{\beta_1}{\alpha_2}e_2$ in (6) and obtain $\alpha_1=1$, $\beta_1=0$:

$$[e_2,e_2]=e_1,\quad [e_3,e_3]=\alpha_4 e_1,\quad [e_2,e_3]=\gamma_1 e_1, \tag{7}$$

where $(\alpha_4,\gamma_1)\neq(0,0)$. Otherwise L is a two-dimensional algebra.

If $\alpha_4=0$ in (7) then $e_2-\frac{1}{\gamma}e_3\in R(L)$ that contradicts to $dimR(L)=1$. Therefore, $\alpha_4\neq 0$. Hence, consider two cases: $\gamma_1\neq 0$ and $\gamma_1=0$. In the first case taking $e_1'=e_1$, $e_2'=e_2, e_3'=\frac{1}{\gamma_1}e_3$ we get $\gamma_1=1$.

$$\textbf{IV}.\quad \begin{aligned}&[e_2,e_2]=e_1,\\ &[e_3,e_3]=\alpha_4 e_1,\\ &[e_2,e_3]=e_1, \text{ with } \alpha_4\neq 0.\end{aligned} \tag{8}$$

If $\gamma_1=0$ then we have

$$[e_2,e_2]=e_1,\ [e_3,e_3]=\alpha_4 e_1, \text{ with } \alpha_4\neq 0. \tag{9}$$

But the base change $e_1'=\alpha_4 e_1$, $e_2'=e_1+e_3$, $e_3'=e_1+\sqrt{\alpha_4}e_2$ brings (9) to

V. $[e_2,e_2]=e_1,$
$[e_3,e_3]=e_1.$

Case 1.2.2. Let $(\alpha_1,\alpha_3)\neq(0,0)$. We can suppose that $\alpha_1\neq 0$, otherwise, we have the following table of multiplication in L:

$[e_1,e_3]=\alpha_3 e_1$, $[e_3,e_3]=\alpha_4 e_1$, $[e_2,e_3]=\gamma_1 e_1$ i.e., $e_2\in R(L)$, which contradicts to $dimR(L)=1$.
Now since $\alpha_1\neq 0$, from (5) we find $\gamma_1=\frac{\alpha_2\alpha_3}{\alpha_1}$ and $\alpha_4=\frac{\alpha_3\beta_1}{\alpha_1}$. This yields

$[e_1,e_2]=\alpha_1 e_1,$
$[e_2,e_2]=\alpha_2 e_1,$
$[e_1,e_3]=\alpha_3 e_1,$
$[e_3,e_3]=\frac{\alpha_3\beta_1}{\alpha_1}e_1,$
$[e_3,e_2]=\beta_1 e_1,$
$[e_2,e_3]=\frac{\alpha_2\alpha_3}{\alpha_1}e_1.$

It is not difficult to see that $e_3-\frac{\alpha_3}{\alpha_1}e_2\in R(L)$, which leads again to a contradiction.

Case 2. Let $dimR(L)=2$ and $\{e_1,\ e_2\}$ be a basis of $R(L)$. Then the table of multiplication of L on the basis $\{e_1,e_2,e_3\}$ can be given as follows

$[e_3,e_3]=\alpha_1 e_1+\alpha_2 e_2,$
$[e_1,e_3]=\beta_1 e_1+\beta_2 e_2,$
$[e_2,e_3]=\gamma_1 e_1+\gamma_2 e_2.$

We specify two cases.

Case 2.1. $dimL^2=1$. Then we get

$[e_3,e_3]=\alpha_1 e_1$, $[e_1,e_3]=\beta_1 e_1$, $[e_2,e_3]=\gamma_1 e_1$.

Clearly, $\gamma_1\neq 0$, otherwise L is two-dimensional. Making the base change $e_1'=\gamma_1 e_1$, $e_2'=e_2$, $e_3'=e_3-\frac{\alpha_1}{\gamma_1}e_2$ we reduce α_1 to zero and γ_1 to one:

$[e_1,e_3]=\beta_1 e_1$, $[e_2,e_3]=e_1$.

If now $\beta_1=0$ then we derive the algebra

VI: $[e_2,e_3]=e_1.$

But if $\beta_1\neq 0$ then the base change $e_1'=e_1$, $e_2'=\beta_1 e_2$, $e_3'=\frac{1}{\beta_1}e_3$ yields $\beta_1=1$, hence we get,

VII: $[e_1,e_3]=e_1,$
$[e_2,e_3]=e_1.$

Case 2.2. Let $dimL^2=2$. Then one has

$[e_3,e_3]=\alpha_1 e_1+\alpha_2 e_2,$
$[e_1,e_3]=\beta_1 e_1+\beta_2 e_2,$
$[e_2,e_3]=\gamma_1 e_1+\gamma_2 e_2,$

where $rank\begin{pmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\\ \gamma_1 & \gamma_2\end{pmatrix}=2.$

Case 2.2.1. Let $det(R_{z|_{R(L)}})\neq 0$. It is the same to say that

$$\beta_1\gamma_2-\gamma_1\beta_2\neq 0. \tag{10}$$

Case 2.2.1.1. $\beta_2\neq 0$. The condition (10) implies that $\gamma_1\neq 0$ and this along with the base change $e_1'=e_1$, $e_2'=e_2$, $e_3'=\frac{\alpha_1\gamma_2}{\gamma_1}e_1+\frac{\alpha_1}{\gamma_1}e_2+e_3$ yields

$$[e_1,e_3]=e_2,\ [e_2,e_3]=\gamma_1 e_1+\gamma_2 e_2, \qquad (\gamma_1\neq 0).$$

Now it is easy to see that if $\gamma_2=0$ then we obtain the algebra

$$[e_1,e_3]=e_2,\ [e_2,e_3]=\gamma_1 e_1, \qquad \gamma_1\neq 0.$$

But the base change $e_1'=\sqrt{\gamma_1}e_1,\ e_2'=e_2,\ e_3'=\frac{1}{\sqrt{\gamma_1}}e_3$ leads it to

VIII: $[e_1,e_3]=e_2,$
$[e_2,e_3]=e_1.$

And if $\gamma_2\neq 0$ then we obtain

IX: $[e_1,e_3]=e_2,$
$[e_2,e_3]=\gamma_1e_1+e_2,\quad \gamma_1\neq 0.$

Case 2.2.1.2. Let $\beta_2=0$. Setting $e_1'=\frac{1}{\beta_1}e_1,\ e_2'=e_2,\ e_3'=e_3$, we get
$[e_3,e_3]=\alpha_1e_1+\alpha_2e_2,$
$[e_1,e_3]=e_1,$
$[e_2,e_3]=\gamma_1e_1+\gamma_2e_2.$

The condition (10) implies that $\gamma_2\neq 0$. Applying the base change

$$e_1'=e_1,\ e_2'=e_2,\ e_3'=(\frac{\alpha_1\gamma_1}{\gamma_2}-\alpha_1)e_1-\frac{\alpha_2}{\gamma_2}+e_3$$

one gets $\alpha_1=\alpha_2=0$ and we come to the table
$[e_1,e_3]=e_1,$
$[e_2,e_3]=\gamma_1e_1+\gamma_2e_2$, with $\gamma_2\neq 0$.

If $(\gamma_1,\gamma_2)\neq(0,1)$, then there are complex numbers A and B such that $AB(\gamma_2-1)-B^2\gamma_1\neq 0$. Therefore first we apply the base change $e_1'=Ae_1+Be_2,\ e_2'=e_2,\ e_3'=e_3$, with $A\neq 0$. And then the base change
$e_1''=e_1',$
$e_2''=(1+\frac{B}{A}\gamma_1)e_1'+[B\gamma_2-(B+\frac{B^2}{A}\gamma_1)e_2',$
$e_3''=e_3',$
reduces it to the Case 2.2.1.1, which has already been considered.

If $(\gamma_1,\gamma_2)=(0,1)$, it gives rise the algebra

X: $[e_1,e_3]=e_1,$
$[e_2,e_3]=e_2.$

Case 2.2.2. Let $det(R_{z|R(L)})=0$. In this case we have the following general table of multiplication:

$$\begin{aligned}&[e_3,e_3]=\alpha_1e_1+\alpha_2e_2,\\ &[e_1,e_3]=k_1(\beta_1e_1+\beta_2e_2), \qquad (11)\\ &[e_2,e_3]=k_2(\gamma_1e_1+\gamma_2e_2).\end{aligned}$$

Since in (11) the vectors e_1 and e_2 are "symmetric", we may suppose that $k_1\neq 0$ and taking the base change $e_1'=e_1,\ e_2'=\frac{k_2}{k_1}e_1-e_2,\ e_3'=e_3$ we reduce (11) to

$$\begin{aligned}&[e_3,e_3]=\alpha_1e_1+\alpha_2e_2,\\ &[e_1,e_3]=\beta_1e_1+\beta_2e_2, \qquad (12)\end{aligned}$$

Then the base change $e_1'=\alpha_1e_1+\alpha_2e_2,\ e_2'=e_2,\ e_3'=e_3$ in (12) gives
$[e_3,e_3]=e_1,$
$[e_1,e_3]=\beta_1e_1+\beta_2e_2.$

Then making the change $e_1'=e_1,\ e_2'=\beta_2e_2,\ e_3'=e_3$ we come to the following table of multiplication:

$$\begin{aligned}&[e_3,e_3]=e_1,\\ &[e_1,e_3]=\beta_1e_1+e_2. \qquad (13)\end{aligned}$$

Consider two cases in (13):

Case 2.2.2.1. Let $\beta_1=0$. In the first case we get the algebra

XI: $[e_3,e_3]=e_1,$
$[e_1,e_3]=e_2.$

Case 2.2.2.2. let now $\beta_1\neq 0$. Then making the base change $e_1'=\frac{1}{\beta_1^2}e_1,\ e_2'=\frac{1}{\beta_1^3}e_2,\ e_3'=\frac{1}{\beta_1}e_3$ we obtain the following algebra

XII: $[e_3,e_3]=e_1,$
$[e_1,e_3]=e_1+e_2.$

Summarizing all the observations above we conclude that any three dimensional non Lie complex Leibniz algebra L is included in one of the following classes of algebras, where the parameters α_3, α_4 and γ_1 for algebras II, IV, and IX are replaced by a parameter α:

1. $dimR(L)=1,\ dimL^2=2$

I: $[e_1,e_3]=-2e_1,$
$[e_2,e_2]=e_1,$
$[e_2,e_3]=-e_2,$
$[e_3,e_2]=e_2.$

II : $[e_1,e_3]=\alpha e_1,$
$[e_3,e_2]=e_2,$
$[e_2,e_3]=-e_2,\qquad \alpha\in\mathbb{C}$

III: $[e_2,e_3]=-e_2,$
$[e_3,e_2]=e_2,$
$[e_3,e_3]=e_1.$

2. $dimR(L)=1,\ dimL^2=1$

IV: $[e_2,e_2]=e_1,$
$[e_3,e_3]=\alpha e_1,$
$[e_2,e_3]=e_1,\qquad \alpha\in\mathbb{C}\setminus\{0\}$

V: $[e_2,e_2]=e_1,\ [e_3,e_3]=e_1.$

3. $dimR(L)=2,\ dimL^2=1$

VI: $[e_2,e_3]=e_1.$

VII: $[e_1,e_3]=e_1,$
$[e_2,e_3]=e_1.$

4. $dimR(L)=2,\ dimL^2=2$

VIII: $[e_1,e_3]=e_2,$
$[e_2,e_3]=e_1.$

IX: $[e_1,e_3]=e_2,$
$[e_2,e_3]=\alpha e_1+e_2,\qquad \alpha\in\mathbb{C}.$

X: $[e_1,e_3]=e_1,$
$[e_2,e_3]=e_2.$

XI: $[e_3,e_3]=e_1,$
$[e_1,e_3]=e_2.$

XII: $[e_3,e_3]=e_1,$
$[e_1,e_3]=e_1+e_2.$

It is easily observed that in the list **I - XII** above the algebra **VI** is included in **IV** at $\alpha=0$. The algebra **VII** is isomorphic to **IX**$(\alpha=0)$, the transition matrix is $\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$. The similar base change leads the algebra **X** to **VIII**. Below we present the comparison result with the list given in [1] in form of a table. Our notation of isomorphism classes is RR_i and Ayupov-Omirov list is denoted by AO_i. According to these observations we conclude that the final list of isomorphism classes for three-dimensional complex non Lie Leibniz algebras is represented as $RR_1 - RR_9$ below.

	List of this paper	List in [1]
1	RR_1	AO_6
2	RR_2	AO_5
3	RR_3	–
4	RR_4	AO_2
5	RR_5	AO_3
6	RR_6	AO_{13}
7	RR_7	AO_1
8	RR_8	AO_{10}
9	RR_9	AO_{11}
10	–	$AO_7, dimAO_7 = 2$
11	–	$AO_4 \cong AO_7$
12	–	$AO_8 \cong AO_2, (\alpha = 0)$
13	–	$AO_9 \cong AO_1$
14	–	$AO_{12} \cong AO_{13}$

Remark. When the Galley Proofs of the paper has been ready the authors were informed by Prof. J.M.Casas on his paper in *Linear Algebra and its Applications* [2] containing an algorithm implemented in Mathematica notebook for the classification of three-dimensional complex Leibniz algebras. We also have compared our list with that of paper [2] and found them identical.

ACKNOWLEDGEMENT.

The research was supported by grant 05-01-09-0733RU UPM.

REFERENCES

1. Ayupov Sh.A., Omirov B.A., *On* 3*-dimensional Leibniz algebras*, Uzbek Math. Jour., No.1,(1999), pp. 9 – 14.
2. Casas J.M., Insua M.A., Landra M., Landra S., *An algorithm for the classification of* 3*-dimensional complex Leibniz algebras*, Linear Algebra and its Applications, 2012, article in press.
3. Jacobson N., *Lie algebras*. Interscience Tracts in Pure and Applied Mathematics, New York, No.10,(1962).
4. Loday J.-L., *Une version non commutative dés algébras de Lie: les algébras de Leibniz*, L'Ens. Math., 39, (1993), pp. 269–293.