SIGNED DIGRAPHS AND THE ENERGY CRISIS

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1. Introduction. Recently, the seriousness of the "energy crisis" has been increasingly revealed. Attempts to understand the patterns of energy use and the effects of various energy conservation strategies require the understanding of an extremely complex system. Such a system involves many variables interacting with each other, reacting to changes in each other, and so on. In attempting to model such a complex system, one faces a tradeoff between the accuracy of the model's predictions and the ability to obtain the detailed information needed to build the model. In this paper, we describe one way to model complex systems, such as those underlying the energy crisis, which is based on a minimal amount of information about the system.

It is useful to divide methodologies for analyzing complex systems into two types, the arithmetic and the geometric. (Cf. Kane [3], Kane, Vertinsky, and Thompson [5], and Coady, Johnson and Johnson [1].) Arithmetic methodologies tend to be numerical and precise, and usually aim at the optimization of a few parameters. They tend to be present-oriented and relatively sensitive to change or modification of the basic parameters. Geometric methodologies tend to be rather non-numerical, and take account of variables which are not readily quantifiable. Their aim is an analysis of structure and shape, and especially of changing patterns of structure which may have different ramifications for the future. The models we shall discuss are geometric in nature and make use of signed and weighted directed graphs.

In Sections 2 and 3 we describe the models in general terms. Section 4 introduces a specific rule for change of value of the variables. Sections 5 and 6 give criteria for a system to be stable under external pressures. Finally, the results are applied to specific energy questions in Section 7.

2. Signed digraphs. We begin by recalling some elementary definitions of graph theory. A directed graph or **digraph** D consists of a set V called the **vertices** and a subset A of $V \times V$ called the set of **arcs**. Our digraphs may have **loops**, that is, arcs of the form (x, x). A **signed digraph** consists of a digraph together with an assignment of a sign $+$ or $-$ to each arc. Following Harary, Norman, and Cartwright [2] we say that a **sequence** in a digraph D is a sequence of vertices $x_1, x_2, \cdots, x_t$ so that for all i, (x_i, x_{i+1}) is an arc. The **sign** of a sequence is the product of the signs of its arcs, and the **length** of a sequence is the number of arcs in it. Finally, a **cycle** is a sequence $x_1, x_2, \cdots, x_{t-1}, x_t$ with $x_1, x_2, \cdots, x_{t-1}$ distinct and $x_t = x_1$.

The idea of studying energy demand and other environmental problems by means of signed digraphs was introduced in Roberts [9]. In such applications, the vertices of a digraph are taken to be variables relevant to the problem being studied (e.g., population, energy capacity in a given region, energy price, etc.). There is an arc from vertex x to vertex y if a change in x has a significant effect on y. This arc is assigned a + sign if the effect is augmenting, i.e., if, all other things being equal, an increase (decrease) in x leads to an increase (decrease) in y; and a − sign if the effect is inhibiting, i.e., if, all other things being equal, an increase (decrease) in x leads to a decrease (increase) in y.

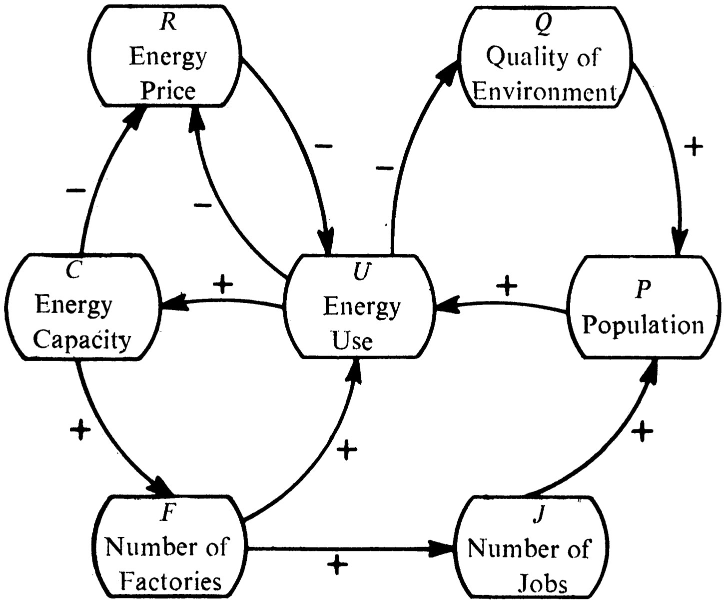

FIG. 1. Signed digraph for energy demand in electrical power. From Roberts [9].

Fig. 1 shows a sample signed digraph for energy demand in electrical power usage in a given area, constructed for talking purposes. (As is conventional, the vertices are represented by ovals, and we draw an arrow from vertex x to vertex y if and only if (x, y) is an arc.) For example, there is an arc from population P to energy use U with a + sign because as population goes up, energy use goes up. There is a negative arc from energy use U to quality of (physical) environment Q because as energy use goes up, the quality of the (physical) environment goes down, as the result of increased smog, thermal pollution, etc.

The change of sign of an arc has an interesting interpretation. Consider for example the arc from energy use U to energy price R. According to the present system, this arc is −, because according to the present rate structure, the more you use, the less you pay (per kilowatt hour). It has been suggested that the rate structure should be inverted, and that large users should pay more rather than less. This strategy, known as **inverting the rate structure,** corresponds to changing the sign of arc (U, R) from − to +. In the same way, other changes in the signed digraph, in particular other changes of sign, might correspond to potential strategies for modifying the energy use system. We shall be interested in evaluating these strategies.

Often, a signed digraph is the most detailed mathematical model of a complex system attainable. This is true in particular if some of the variables cannot be measured, as for example the variable "environmental quality" in the signed digraph of Fig. 1. Difficulty of measurement is a property of many of the variables arising in societal problems. Even with an oversimplified model such as a signed digraph, there are still some precise conclusions which can be reached. For example, if Fig. 1 is an accurate model of the signs of effects in an energy demand system, then one can pinpoint certain augmenting subsystems. The cycle C, F, U, C corresponds to such a system. An increase in energy capacity C leads, through this subsystem, to an increase in the number of factories F, which in turn leads to more energy use U, which finally leads to a further increase in energy capacity C. An augmenting or positive feedback subsystem often contributes to instability, especially if there are many such subsystems present. (Sometimes inhibiting or negative feedback subsystems can create instability of another type, by contributing increasing oscillations.) The directed cycle C, R, U, C is another augmenting subsystem. For an increase in energy capacity leads, via this subsystem, to a decrease in price, which leads to an increase in use and hence to a further increase in capacity. It is easy to see that, in this figure, all subsystems containing the energy capacity vertex C and corresponding to (simple) cycles such as C, F, U, C are augmenting or unstable. This observation already makes precise, from a structural point of view, why the energy capacity system is so unstable. The signed digraph model is especially good for making such structural observations, for digraph theory has concerned itself over the years with just such notions of structure. (Indeed, one book about digraph theory, Harary, Norman, and Cartwright [2], is called *Structural Models.*)

3. Weighted digraphs. The signed digraph model has in it many oversimplifications. For example, some effects of variables on others are stronger than other effects. Thus, in Fig. 1, it seems clear that the effect of an increase in population P on energy use U is very strong compared to the effect of a decrease in quality of the environment Q on population P. The signed digraph model, however, assumes that all effects are equally strong, by placing unit $(+1$ or $-1)$ weights on each arc. It might be more reasonable to place a different weight $w(x, y)$ on each arc (x, y) of a given digraph, thus yielding a **weighted digraph.** The weight is interpreted as the relative strength of the effect, and can be positive or negative. If each weight is an integer, we shall call the weighted digraph **integer-weighted.** Even more realistic than assigning a weight to each arc is to assume that the strength of an effect corresponding to the arc (x, y) changes depending on the levels of the variables x and y. But even weights are hard to estimate in practice, especially if the variables themselves are hard to measure or define.

Another omission in the signed digraph model is the time lag involved before a change in x has an effect on y. For example, an increase in population P will lead almost immediately to an increase in energy use U, while there is a time lag after an increase in the number of jobs J before that attracts more population P to an area. The signed digraph model assumes that all effects take place in one unit time. Thus, a more realistic model would introduce a time lag corresponding to each effect. As with weights, time lags are hard to estimate, and there is a tradeoff between the generality of the model and the possibility of estimating its parameters (weights, time lags) in a realistic way. Also, there is a mathematical problem with the introduction of time lags. Specifically, analysis of the dynamic models — which we shall introduce later — becomes quite difficult with the introduction of time lags. For this reason, we shall discuss weighted digraphs, but we shall disregard time lags.

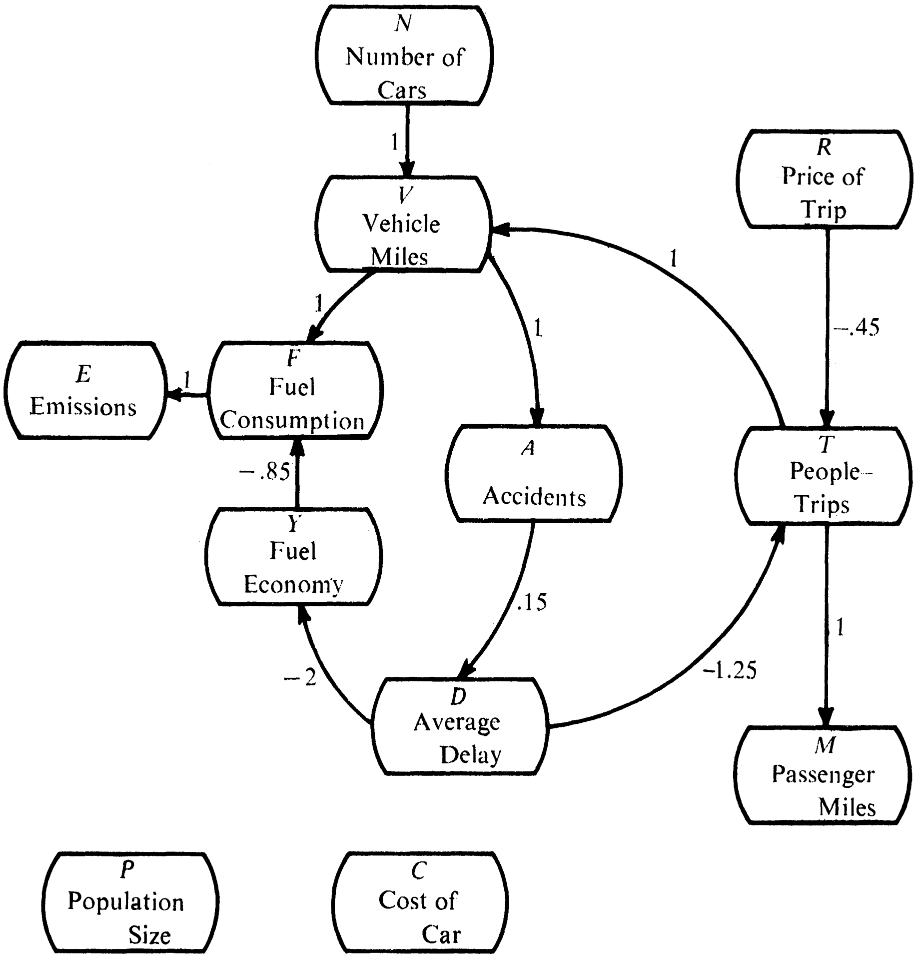

FIG. 2. Weighted digraph for energy use and air pollution produced by the transportation system of San Diego. Short term effects shown only. From Roberts [10].

Fig. 2 shows an example of a weighted digraph, which we shall use to discuss energy use in transportation. This digraph was constructed by a panel of experts to represent the variables relating to energy use and air pollution resulting from the transportation system of San Diego County, California. Only short-term effects are shown. The experts who built this model had been studying San Diego, under an Environmental Protection Agency contract, to assess strategies for meeting requirements of the Clean Air Act, and used their knowledge to construct the weighted digraph. (We shall discuss below the precise meaning of the weights.) In San Diego, 97% of all trips are made by automobile. In recent years, San Diego has exhibited steadily increasing levels of fuel consumption and air pollution. We shall return to these observations later, and see if they are reflected in the properties of the digraph. (For details on how this and related digraphs were constructed, see Roberts [**10**].)

4. A dynamic model. Some rather interesting conclusions can be reached if we introduce a simple dynamic model for the propagation of changes through the vertices of a signed or weighted digraph. Let us begin with a signed digraph, and list its vertices as $x_1, x_2, \cdots, x_n$. We suppose that each vertex x_i attains a value $v_i(t)$ at each discrete time $t = 0, 1, 2, \cdots$. The succeeding value $v_i(t+1)$ is determined from $v_i(t)$, from an outside pulse $p_i^0(t+1)$ introduced at vertex x_i at time $t+1$, and from information about whether other vertices x_j adjacent to x_i went up or down at the last time period. Specifically, we assume that if there is an arc from x_j to x_i and x_j goes up by a units at time t, then as a result x_i goes up at time $t+1$ by an amount equal to a times the sign of arc (x_j, x_i). Moreover, x_i must increase by an amount equal to any external change $p_i^0(t+1)$ introduced at x_i at time $t+1$. To make all this precise, we define

$$v_i(t+1) = v_i(t) + p_i^0(t+1) + \textstyle\sum_j \operatorname{sgn}(x_j, x_i) p_j(t), \tag{1}$$

where

$$\operatorname{sgn}(x_j, x_i) = \begin{cases} +1 & \text{if } (x_j, x_i) \text{ is } + \\ -1 & \text{if } (x_j, x_i) \text{ is } - \\ 0 & \text{if there is no arc } (x_j, x_i), \end{cases}$$

and

$$p_j(t) = \begin{cases} v_j(t) - v_j(t-1) & \text{if } t > 0 \\ p_j^0(0) & \text{if } t = 0. \end{cases}$$

The quantity $p_j(t)$ will be called the **pulse** at vertex x_j at time t. A **pulse process*** on a signed digraph D is defined by the rule (1), by an initial vector of values

$$V(0) = (v_1(0), v_2(0), \cdots, v_n(0)),$$

and by vectors giving the outside pulse introduced at each vertex at each time period.

* Pulse processes were introduced in Roberts [**9**].

We shall denote these vectors by

$$P^0(t) = (p_1^0(t), p_2^0(t), \cdots, p_n^0(t)).$$

We shall also use the **pulse vector** $P(t) = (p_1(t), \cdots, p_n(t))$.

In applications, one usually determines $V(0)$ as follows. Suppose we know the starting value $v_i(\text{start})$ at each vertex x_i. Then $v_i(0)$ is defined by

$$v_i(0) = v_i(\text{start}) + p_i^0(0),$$

i.e., $v_i(0)$ is the starting value at vertex x_i plus the initial pulse introduced at vertex x_i. Thus, we usually define a pulse process by giving the vector

$$V(\text{start}) = (v_1(\text{start}), v_2(\text{start}), \cdots, v_n(\text{start}))$$

rather than the vector $V(0)$.

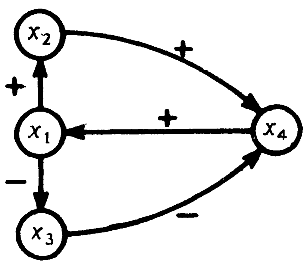

FIG. 3. Signed digraph.

To give an example of how a pulse process works, let us consider a very simple signed digraph, that of Fig. 3. We assume that $P^0(0) = (1,0,0,0)$, that $P^0(t) = 0$, all $t > 0$, and that $V(\text{start}) = (0,0,0,0)$. Thus, $V(0) = (1,0,0,0)$. At time 0, vertex x_1 increased by 1 unit, so at time 1, vertices x_2 and x_3 change, vertex x_2 going up by 1, vertex x_3 going down by 1. Thus, $V(1) = (1,1,-1,0)$ and so $P(1) = (0,1,-1,0)$. Since at time 1, vertex x_2 went up 1, this leads to an increase of 1 unit in vertex x_4 at time 2. But vertex x_3 went down 1 at time 1, so this leads (since $\text{arc}(x_3, x_4)$ is $-$) to a further increase in x_4 by 1 unit at time 2. We conclude that $V(2) = (1,1,-1,2)$, and $P(2) = (0,0,0,2)$. The increase in x_4 of 2 units at time 2 leads in turn to an increase in 2 units in x_1 at time 3. Thus, $V(3) = (3,1,-1,2)$, and so on.

The rule (1) for a pulse process on a signed digraph generalizes, in an obvious way, to a rule for a pulse process on a weighted digraph:

$$v_i(t+1) = v_i(t) + p_i^0(t+1) + \Sigma_j w(x_j, x_i) p_j(t). \tag{2}$$

Eq. (2) is really a system of finite difference equations, with parameters $w(x_j, x_i)$. For it can be rewritten as follows:

$$p_i(t+1) = p_i^0(t+1) + \Sigma_j w(x_j, x_i) p_j(t).$$

With this notion of pulse process, the weights in a weighted digraph have a specific interpretation. For example, in Fig. 2, the weight $-.45$ on the arc from price of trip to people-trips suggests that as the price of a trip goes up by 1 unit, the annual number of people-trips will go down by .45 units. (Here, a unit was taken to be 10% of a base case level.)

If in a pulse process we have $P^0(t) = 0$ for $t > 0$, the process is called **autonomous.** An autonomous pulse process for which $P^0(0)$ is the vector $(0, 0, \cdots, 1, 0, \cdots, 0)$ with a 1 in the ith place, is called a **simple pulse process starting at vertex** x_i. In a simple pulse process starting at vertex x_i of a signed digraph, the quantities $p_j(t)$ and $v_j(t)$ are related to the **signed number of sequences** of length t from x_i to x_j, i.e., the difference between the number of positive sequences from x_i to x_j of length t and the number of negative sequences from x_i to x_j of length t. Specifically, it is easy to prove the following theorem:

THEOREM 1. *In a simple pulse process starting at vertex x_i of a signed digraph D, the quantity $p_j(t)$ is given by the signed number of sequences from x_i to x_j of length equal to t and the quantity $v_j(t)$ is given by $v_j(\text{start}) + p_j^0(0) +$ the signed number of sequences from x_i to x_j of length less than or equal to t.*

The **adjacency matrix** of the weighted digraph D is the matrix $A = (a_{ij})$ with $a_{ij} = w(x_i, x_j)$. The following theorem is easy to prove for signed digraphs using Theorem 1. The proof generalizes readily to the case of weighted digraphs.

THEOREM 2. *Suppose D is a weighted digraph with adjacency matrix A. In a simple pulse process on D starting at vertex x_i, $p_j(t)$ is given by the i,j entry of A^t, while $v_j(t)$ is given by $v_j(\text{start})$ plus the i,j entry of $I + A + A^2 + \cdots + A^t$.*

It is fruitful to restate Theorem 2 in vector notation, in which case we see easily how to generalize it to autonomous pulse processes, obtaining

THEOREM 3. *In an autonomous pulse process on a weighted digraph, $P(t) = P(0)A^t$.*

5. Stability. The main qualitative property of a complex system which we shall study is stability. There are various notions of stability and we shall study only two of them here. We say that a vertex x_j of a weighted digraph D is **pulse stable** under a pulse process if the sequence

$$\{|p_j(t)|: t = 0, 1, 2, \cdots\}$$

is bounded, and **value stable** if the sequence

$$\{|v_j(t)|: t = 0, 1, 2, \cdots\}$$

is bounded. The weighted digraph D is **pulse (value) stable** under the pulse process if each vertex is. To give an example, in exponential growth such as $v_j(t) = 2^t$, variable x_j is both pulse and value unstable, as $p_j(t) = 2^{t-1}$. However, in linear growth such as $v_j(t) = 2t + 5$, variable x is still value unstable but it is now pulse stable.

Under any pulse process, value stability (at x_j) implies pulse stability (at x_j), since

$$|p_j(t)| = |v_j(t) - v_j(t-1)| \leqq |v_j(t)| + |v_j(t-1)|.$$

On the other hand, pulse stability does not imply value stability: consider, for example, the signed digraph of Fig. 4. Unsurprisingly, one can relate stability to the eigenvalues of D, i.e., those of D's adjacency matrix A. We shall do so in this section. Unfortunately, theorems relating stability to the eigenvalues of D are not always useful because they do not relate stability to the structure of the digraph.

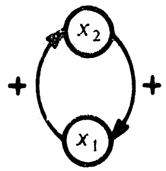

FIG. 4. Under the simple pulse process starting at vertex x_1, vertex x_2 is pulse stable but not value stable.

In the next section, we shall give a sample of a result which does relate stability to structure, and which can be exploited to choose stabilizing strategies. Finally, in Sec. 7, we shall use these results to study the examples of Figures 1 and 2. To begin with, we state a necessary condition for pulse stability.

THEOREM 4. *If a weighted digraph D has an eigenvalue greater than unity in magnitude, then D is pulse unstable under some simple pulse process.*

Proof. Let λ be an eigenvalue with $|\lambda| > 1$ and let U be an eigenvector corresponding to λ such that $\|U\| = 1$.* Write $U = \sum_{i=1}^{n} \alpha_i E_i$, where E_i is the vector with 1 in the ith component and 0 elsewhere. Then, for any integer $t > 0$,

$$|\lambda|^t = \|UA^t\| = \left\| \sum_{i=1}^{n} \alpha_i E_i A^t \right\| \leqq \sum_{i=1}^{n} |\alpha_i| \, \|E_i A^t\|.$$

Since $\|U\| = 1$, each $|\alpha_i| \leqq 1$. Thus

$$|\lambda|^t \leqq \sum_{i=1}^{n} \|E_i A^t\|.$$

It follows that for every $t > 0$, there is some i such that $\|E_i A^t\| \geqq 1/n|\lambda|^t$. Since there are only a finite number of E_i, we conclude that for at least one of them, $\|E_i A^t\| \geqq 1/n|\lambda|^t$ for arbitrarily large t. Pick $P(0) = E_i$. Then for arbitrarily large t, $\|P(t)\| = \|P(0)A^t\|$ is at least $1/n|\lambda|^t$. Since $|\lambda| > 1$ and n is fixed, we conclude that with $P(0) = E_i$, $\|P(t)\|$ gets arbitrarily large as $t \to \infty$, and hence the weighted digraph is pulse unstable under the simple pulse process starting at vertex x_i. Q.E.D.

* Throughout, if $M = (m_{ij})$, we use $\|M\| = \sqrt{\sum_{i,j} m_{ij}^2}$.

COROLLARY. *If an integer-weighted digraph D is pulse stable under all simple pulse processes, then each nonzero eigenvalue has magnitude equal to unity.*

Proof. By Theorem 4 each nonzero eigenvalue has magnitude at most unity. Let $\sum_{i=0}^{n} a_i\lambda^i$ be the characteristic polynomial of A. If i is the least integer such that $a_i \neq 0$, then the product of all the nonzero eigenvalues of D is $(\pm)$ times a_i. Since all entries of A are integers, a_i is an integer. Thus, each nonzero eigenvalue must have magnitude unity. Q.E.D.

Theorem 4 gives a necessary condition for pulse stability. This condition is also sufficient if D has no multiple eigenvalues (except possibly 0). This will follow from Theorem 5 below. To handle the case of multiple eigenvalues, let us consider the Jordan Canonical Form J corresponding to the matrix A. J can be written in the form

$$J = \begin{bmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \ddots & \\ 0 & & & \boxed{B_r} \end{bmatrix}, \tag{3}$$

where each B_j is an $(e_j + 1) \times (e_j + 1)$ diagonal block and B_j has the form

$$\begin{bmatrix} \lambda & \delta_j & & & & 0 \\ & \lambda & \delta_j & & & \\ & & \lambda & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \delta_j \\ & 0 & & & \lambda & \delta_j \\ & & & & & \lambda \end{bmatrix},$$

where λ is an eigenvalue of A and δ_j is 0 or 1, 0 if $e_j + 1 = 1$ and 1 otherwise. (Note that λ may appear in several of the B_j's.)

If J has the form (3), then J^t has the form

$$\begin{bmatrix} \boxed{B_1^t} & & & 0 \\ & \boxed{B_2^t} & & \\ & & \ddots & \\ 0 & & & \boxed{B_r^t} \end{bmatrix}.$$

Now if $\delta_j = 0$, then $B_j = (\lambda)$ and

$$B_j^t = (\lambda^t). \tag{4}$$

If $\delta_j = 1$, then it is easy to prove by induction on t that $b_{k,l}^{j,t}$, the k, l entry of B_j^t, is given by

$$b_{k,l}^{j,t} = \begin{cases} 0 & \text{if } k > l \\ \binom{t}{l-k}\lambda^{t-l+k} & \text{if } k \leqq l. \end{cases} \tag{5}$$

To state our necessary and sufficient condition for pulse stability, let us say that an eigenvalue λ of D is **linked in** J if there is an off-diagonal entry of 1 in some row of J in which λ appears as the diagonal element. Equivalently, λ is linked in J if it appears on the diagonal of some B_j in which $\delta_j = 1$.

THEOREM 5. *Suppose D is a weighted digraph and J is its Jordan Canonical Form. Then the following are equivalent:*

(a) *D is pulse stable under all autonomous pulse processes.*

(b) *D is pulse stable under all simple pulse processes.*

(c) *Every eigenvalue of D has magnitude less than or equal to unity and every eigenvalue of D which is linked in J has magnitude less than unity.*

The proof of Theorem 5 begins with two lemmas.

LEMMA 1. *If A is an $n \times n$ matrix and J is its Jordan Canonical Form, then*

$$\{\| A^t \| : t = 0, 1, 2, \cdots\} \tag{6}$$

is bounded if and only if

$$\{\| J^t \| : t = 0, 1, 2, \cdots\} \tag{7}$$

is bounded.

Proof. The proof uses a topological argument. The matrices A and J are related by a similarity transformation. Now such a transformation is bicontinuous under the topology induced by the norm $\| \cdot \|$, so it follows that $\{A^t\}$ is contained in a sphere if and only if $\{J^t\}$ is contained in a sphere.* Q.E.D.

LEMMA 2. *If a weighted digraph D is pulse stable under all simple pulse processes, then*

$$\{\| J^t \| : t = 0, 1, 2, \cdots\} \tag{7}$$

is bounded. If the sequence (7) *is bounded, then D is pulse stable under all autonomous pulse processes.*

* Our thanks to Garrett Birkhoff for suggesting this argument.

Proof. By Theorem 3,

$$P(t) = P(0)A^t.$$

Thus, pulse stability under all simple pulse processes implies that the sequence

$$\{\| A^t \| : t = 0,1,2,\cdots\} \tag{6}$$

is bounded and hence, by Lemma 1, we conclude that the sequence (7) is bounded as well. Conversely, if the sequence (7) is bounded, then by Lemma 1 the sequence (6) must be bounded. Hence by Theorem 3, D must be pulse stable under all autonomous pulse processes. Q.E.D.

To prove Theorem 5, let us observe that clearly (a) implies (b). We shall prove (b) implies (c) and (c) implies (a). We have already shown that if D is pulse stable under all simple pulse processes, then every eigenvalue of D has magnitude less than or equal to unity. To complete the proof of (b) implies (c), let us show that every linked eigenvalue in J has magnitude less than unity. By Lemma 2, we know from pulse stability that $\{\| J^t \|\}$ is bounded. But now suppose λ is an eigenvalue of D which is linked in J and has magnitude greater than or equal to unity. If λ appears on the diagonal of block B_j, then $b_{1,2}^{j,t} = t\lambda^{t-1}$, by Eq. (5). Since $|\lambda| \geqq 1$, $|b_{1,2}^{j,t}|$ gets arbitrarily large as t approaches ∞, and so $\| J^t \|$ becomes arbitrarily large as well. This completes the proof that (b) implies (c).

To prove that (c) implies (a), it is sufficient by Lemma 2 to show that under assumption (c), $\{\| J^t \|\}$ is bounded. To prove that $\{\| J^t \|\}$ is bounded, it is sufficient to prove that for each j, $\{\| B_j^t \|\}$ is bounded. Let λ be the diagonal entry of B_j^t. By hypothesis, $|\lambda| \leqq 1$. If $\delta_j = 0$, then B_j^t has the form (4) and so $\| B_j^t \|$ is bounded since $|\lambda|^t$ is bounded. If $\delta_j = 1$, then by hypothesis, $|\lambda| < 1$. We show that for $k \leqq l$, $\{|b_{k,l}^{j,t}| : t = 0,1,2,\cdots\}$ is bounded, where $b_{k,l}^{j,t}$ is given by Eq. (5). Let $e = e_j$, i.e., e is one less than the dimension of B_j. Observe that $l-k \leqq e$. Thus, for $t > 2e$, we have

$$\binom{t}{l-k} \leqq \binom{t}{e} = \frac{t(t-1)\cdots(t-e+1)(t-e)!}{e!(t-e)!} \leqq \frac{t^e}{e!}$$

and

$$|\lambda^{t-l+k}| = |\lambda|^{t-l+k} \leqq |\lambda|^{t-e},$$

since $|\lambda| < 1$. It follows that

$$|b_{k,l}^{j,t}| \leqq \frac{t^e}{e!}|\lambda|^{t-e}.$$

Since $t^e|\lambda|^{t-e}$ approaches 0 as t approaches ∞, we conclude that $\{|b_{k,l}^{j,t}|\}$ is bounded. Q.E.D.

COROLLARY. *Suppose D is an integer-weighted digraph and J is its Jordan Canonical Form. Then the following are equivalent:*

(a) *D is pulse stable under all autonomous pulse processes.*

(b) *D is pulse stable under all simple pulse processes.*

(c) *Every eigenvalue of D has magnitude less than or equal to unity and no nonzero eigenvalue of D is linked in J.*

(d) *Every nonzero eigenvalue of D has magnitude equal to unity and no nonzero eigenvalue of D is linked in J.*

Proof. Obviously, (d) implies (c). By Theorem (5), (c) implies (a). Again trivially, (a) implies (b). To prove that (b) implies (d), note that (b) implies part (c) of Theorem 5 and (b) implies, by the Corollary to Theorem 4, that each nonzero eigenvalue of D has magnitude equal to unity. Thus, (b) implies (d). Q.E.D.

The next theorem characterizes value stability.

THEOREM 6. *Suppose D is a weighted digraph. Then the following are equivalent:*

(a) *D is value stable under all autonomous pulse processes.*

(b) *D is value stable under all simple pulse processes.*

(c) *D is pulse stable under all simple pulse processes and unity is not an eigenvalue of D.*

Proof. The proof uses Lemmas analogous to Lemmas 1 and 2 for

$$\left\{\Big\| \sum_{t=0}^{N} A^t \Big\| : N = 0,1,2,\cdots\right\} \tag{8}$$

and

$$\left\{\Big\| \sum_{t=0}^{N} J^t \Big\| : N = 0,1,2,\cdots\right\}. \tag{9}$$

Clearly, (a) implies (b). We shall prove (b) implies (c) and (c) implies (a).

To prove the latter, it is sufficient to show that under assumption (c), $\{\| \sum_{t=0}^{N} J^t \|\}$ is bounded. To show that $\{\| \sum_{t=0}^{N} J^t \|\}$ is bounded, it is sufficient to show that for each j, $\{\| \sum_{t=0}^{N} B_j^t \|\}$ is bounded. Let λ be the eigenvalue appearing on the diagonal of B_j. If $\delta_j = 0$, then $\sum_{t=0}^{N} B_j^t$ is $\sum_{t=0}^{N} \lambda^t$. Since $|\lambda| \leqq 1$ by pulse stability and $\lambda \neq 1$ by hypothesis, $\sum_{t=0}^{N} \lambda^t$ converges. We conclude that $\{| \sum_{t=0}^{N} \lambda^t |\} = \{\| \sum_{t=0}^{N} B_j^t \|\}$ is bounded. Suppose next that $\delta_j = 1$. Then by pulse stability, $|\lambda| < 1$. If $k > l$, then $\sum_{t=0}^{N} b_{k,l}^{j,t} = 0$. If $k \leqq l$, then by the proof of Theorem 5, if $t > 2e$, we have

$$|b_{k,l}^{j,t}| \leqq \frac{t^e}{e!} |\lambda|^{t-e},$$

where e is one less than the dimension of B_j. Thus, for $t > 2e$,

$$\left| \sum_{t=0}^{N} b_{k,l}^{j,t} \right| \leqq \frac{1}{e!} \sum_{t=0}^{N} t^e |\lambda|^{t-e}. \tag{10}$$

Applying the ratio test to the sum on the right hand side of (10) and using the fact that $|\lambda| < 1$, we find that the series converges. Thus, $\{|\ \sum_{t=0}^{N} b_{k,l}^{j,t}|\}$ is bounded, and hence so is $\{\|\ \sum_{t=0}^{N} B_j^t\ \|\}$.

To complete the proof, we assume (b) and prove (c). Now value stability implies pulse stability. Finally, we shall assume that unity is an eigenvalue and reach a contradiction. It is sufficient to prove that $\{\|\ \sum_{t=0}^{N} J^t\|\}$ is unbounded. In particular, suppose $\lambda = 1$ and B_j is a diagonal block in which λ is the diagonal entry. By pulse stability λ is unlinked, and so $\sum_{t=0}^{N} B_j^t$ is $\sum_{t=0}^{N} \lambda^t = N + 1$. Thus, $\{\|\ \sum_{t=0}^{N} B_j^t\|\}$ is unbounded, and hence so is $\{\|\ \sum_{t=0}^{N} J^t\|\}$. This contradicts value stability. Q.E.D.

6. Rosettes. Although the results of Section 5 can be used to determine stability properties of a signed or weighted digraph, they do not relate stability to any structural properties of the digraph. If we could understand what it is about the structure which causes instabilities, we could exploit this knowledge to find stabilizing strategies, or to evaluate proposed strategies. Unfortunately, not much is known about the relation of structure to stability. In this section, we shall give a few sample theorems, which hold for a special class of digraphs called advanced rosettes. Although this class may appear very special, a surprisingly large number of systems which have been encountered by the authors belong to this category.

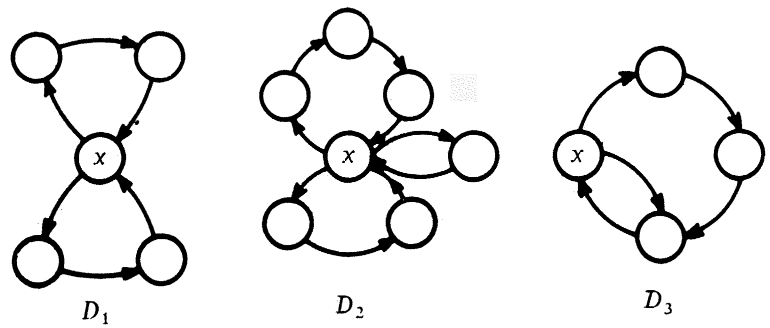

FIG. 5. Advanced rosettes with central vertex x. Digraphs D_1 and D_2 are rosettes, but D_3 is not.

A digraph D is a **rosette** if it consists of a central vertex x and nonintersecting cycles leading out of x. Digraphs D_1 and D_2 of Fig. 5 are rosettes. More generally, a digraph D is an **advanced rosette** if for every pair of vertices x and y of D, there is a sequence in D from x to y (we say D is **strongly connected**) and there is a **central vertex** x which is on all cycles of D. All the digraphs of Fig. 5 are advanced rosettes.

If D is a signed advanced rosette, let a_i denote the sum of the signs* of the cycles of length i and let s be the largest integer such that $a_s \neq 0$. If $a_i = 0$ for all i, we take $s = 0$. If $s = 0$, it is easy to show that D is pulse and value stable under all

* A sign is thought of as a number $+1$ or -1.

simple pulse processes. If $s > 0$ then the stability properties of D are mirrored in the properties of the **rosette sequence** $(a_1, a_2, \cdots, a_s)$. The next two theorems were first discovered by Joel Spencer.

THEOREM 7. *Suppose D is a signed advanced rosette with $s > 0$ and rosette sequence $(a_1, a_2, \cdots, a_s)$. If D is pulse stable under all simple pulse processes, then*
(a) $a_s = \pm 1$, *and*
(b) $a_i = (-a_s)a_{s-i}$, $1 \leqq i \leqq s-1$.

Proof. Let

$$R(\lambda) = \lambda^s - \sum_{i=1}^{s} a_i \lambda^{s-i}.$$

It is not hard to prove that the characteristic polynomial $C(\lambda)$ of D is given by

$$C(\lambda) = \lambda^{n-s} R(\lambda).$$

Note that $R(0) = a_s \neq 0$. Thus, 0 is not a root of $R(\lambda)$ and so $R(\lambda)$ has as roots exactly the nonzero eigenvalues of D. By the corollary to Theorem 4, if D is pulse stable under all simple pulse processes, then every root of $R(\lambda)$ has magnitude 1. The product of the roots is $\pm a_s$. Since a_s is an integer, it follows that a_s is ± 1, which proves (a).

An old lemma of Kronecker's states that if $f(x)$ is a monic polynomial with integer coefficients each of whose roots has magnitude unity, then each root of $f(x)$ is a root of unity. Thus, each root of $R(\lambda)$ is a root of unity. We shall prove that $R(\alpha) = 0$ if and only if $R(\alpha^{-1}) = 0$. This result will give us (b). For, applying it, one proves that $R(\lambda) = R(0)\lambda^s R(\lambda^{-1})$. By comparing the coefficients of the terms λ^{s-i}, one derives (b).

It is left to show that $R(\alpha) = 0$ if and only if $R(\alpha^{-1}) = 0$. Factor $R(\lambda)$ into irreducible factors $P_j(\lambda)$. Let β be any root of P_j. Then P_j is the unique irreducible polynomial with β as a root. If β is a primitive n_j-th root of unity, then P_j is the n_j-th cyclotomic polynomial, the unique monic polynomial whose roots are the primitive n_j-th roots of unity. Thus, if α is a root of R, then α is a root of some P_j, whence α^{-1} is a root of P_j, whence α^{-1} is a root of R. And conversely. Q.E.D.

THEOREM 8. *Suppose D is a signed advanced rosette with $s > 0$ and rosette sequence $(a_1, a_2, \cdots, a_s)$, and suppose D is pulse stable under all simple pulse processes. Then D is value stable under all simple pulse processes if and only if $\sum_{i=1}^{s} a_i \neq 1$.*

Proof: By Theorem 6, D is value stable under all simple pulse processes if and only if 1 is not an eigenvalue, i.e., if and only if $R(1) \neq 0$, i.e., if and only if $1 - \sum_{i=1}^{s} a_i \neq 0$, i.e., if and only if $\sum_{i=1}^{s} a_i \neq 1$. Q.E.D.

7. Applications. In this section, we shall apply the theorems about stability to the digraphs of Fig.'s 1 and 2. Let us consider first the signed digraph D of Fig. 1, which represents energy demand in the electrical power area. A calculation shows that the characteristic polynomial $C(\lambda)$ is $\lambda^2(\lambda^5-\lambda^3-\lambda^2-1)$. Now $f(\lambda)=\lambda^5-\lambda^3-\lambda^2-1$ has a real root strictly between 1 and 2, since $f(1)=-2$ and $f(2)=19$. Thus, Theorem 4 implies that the signed digraph is pulse unstable under some simple pulse process. If one believes the model, one can interpret the instability result as follows: the system is sensitive enough to external influences that certain external influences lead to arbitrarily large values at some of the variables, and indeed to increasingly large changes in value. These results are not too surprising after our analysis in Section 2 of the cycles of D, or in view of observation of exponential increases in such variables as energy use. However, the results are based on our oversimplified pulse process model and should be tested using other methods. The same will be true of the other conclusions in this section. In practice, one usually assumes that no variable in a real-life system can reach arbitrarily large levels. In particular, no variable can change by larger and larger amounts in successive time periods. Long before changes (pulses) or values reach very large levels, the structure of the system itself will become modified. It would be wise to try to minimize the impact of that modification by foreseeing it or consciously choosing one of a possible set of modifications.

In any case, it has been suggested that one promising strategy is to invert the rate structure, that is, charge large users of electricity more rather than less. This corresponds to changing the sign of arc(U,R) from $-$ to $+$. Dealing with the new signed digraph obtained from that of Fig. 1 by making this change of sign, one calculates that $C(\lambda)$ is given by $\lambda^2(\lambda^5+\lambda^3-\lambda^2-1)=\lambda^2(\lambda-1)(\lambda^2+1)(\lambda^2+\lambda+1)$. Hence, the eigenvalues are $0,0,1,\pm i,-1/2\pm(\sqrt{3}/2)i$. Each nonzero eigenvalue has magnitude 1. Since all the nonzero eigenvalues are distinct, none of them can be linked. We conclude, by Theorem 5, that the new signed digraph is pulse stable under all autonomous pulse processes. However, 1 is an eigenvalue, so Theorem 6 implies that we still have value instability. The strategy of inverting the rate structure does not prevent the system from being sufficiently sensitive to external changes that some of these changes can lead to arbitrarily large values. What inverting the rate structure has accomplished is to make sure that changes at any given time cannot be too large.

The strategy of inverting the rate structure is value-stabilizing as well as pulse-stabilizing if we also make a second change, changing the sign of arc (C,F) from $+$ to $-$. This corresponds to forcing factories to move out of an area every time a new power plant is constructed! (Whether or not this strategy corresponds to a feasible real-world strategy is highly doubtful.) To show that it is value-stabilizing, let us consider the signed digraph of Fig. 1 with both (U,R) changed from $-$ to $+$ and (C,F) changed from $+$ to $-$. Now the characteristic polynomial is $C(\lambda)=\lambda^2(\lambda^5+\lambda^3+\lambda^2+1)=\lambda^2(\lambda+1)(\lambda^2+1)(\lambda^2-\lambda+1)$, and we have as roots $0,0,-1,\pm i, 1/2\pm(\sqrt{3}/2)i$. By Theorems 5 and 6, the new signed digraph is pulse and value stable under all autonomous pulse processes. Once again, the reader is reminded that this conclusion depends on our oversimplified model, and should be subjected to test using other techniques.

The theorems of Sec. 6 apply to the signed digraph D of Fig. 1, since it is an advanced rosette with central vertex U. A simple calculation shows that the rosette sequence is $(0,1,1,0,1)$. Now $a_2 \neq -a_5a_3$, so Theorem 7 implies that D is pulse unstable under some simple pulse process. If we change (U,R) from $-$ to $+$, the rosette sequence turns out to be $(0,-1,1,0,1)$. The necessary conditions of Theorem 7 are satisfied, though this does not verify pulse stability, which must be checked by calculating eigenvalues. Value instability follows from Theorem 8, since $a_1 + a_2 + a_3 + a_4 + a_5 = 1$. Finally, if (U,R) is changed from $-$ to $+$ and (C,F) from $+$ to $-$, then the rosette sequence is $(0,-1,-1,0,-1)$. The conditions of Theorem 7 are again satisfied, and here $a_1 + a_2 + a_3 + a_4 + a_5 = -3 \neq 1$. Thus, one discovers that changing (C,F) from $+$ to $-$ could be value-stabilizing by modifying the rosette sequence so that the necessary conditions of Theorem 7 and the condition of Theorem 8 are satisfied. It is left to check this conclusion by calculating eigenvalues. Other value-stabilizing strategies can be discovered by using the conditions of Theorems 7 and 8 as necessary conditions. Any change of sign in the signed digraph of Fig. 1 will leave $a_1 = 0$ and $a_4 = 0$. Moreover, $a_2 = \pm 1$, $a_3 = \pm 1$ or ± 3, and $a_5 = \pm 1$. The necessary conditions $a_2 = -a_5a_3$ and $a_1 + a_2 + a_3 + a_4 + a_5 \neq 1$ then imply that $a_2 = a_3 = a_5 = -1$. Potential value-stabilizing strategies are those corresponding to change of signs which set $a_2 = a_3 = a_5 = -1$. Using this observation, one discovers that there is no single sign change which will value-stabilize the signed digraph. The only other value-stabilizing strategies which involve change of two signs are changing signs of arcs (R,U) and (J,P) or changing signs of arcs (R,U) and (F,J).

Turning next to the weighted digraph of Fig. 2, we find that the characteristic polynomial is $C(\lambda) = \lambda^8(\lambda^4 + .19)$. The roots are 0 with multiplicity 8, $-.47 \pm (.47)i$, $.47 \pm (.47)i$. By Theorems 5 and 6, the digraph is pulse and value stable under all autonomous pulse processes. The reader will recall that Fig. 2 describes energy use and air pollution resulting in the short-term from San Diego's transportation system. As we remarked earlier, in recent years San Diego has been exhibiting rising rates of fuel consumption by automobiles and rising levels of air pollution. The pulse process model suggests that any spiralling of fuel consumption or air pollution, at least in the short-term, would have to come from repeated external impulses, rather than from the operation of feedback within the system. Indeed, it is fairly easy to see from the weighted digraph why this is the case. There is only one cycle, that from Number of People-trips T to Vehicle Miles V to Accidents A to Average Delay D to People-trips T. This is an inhibiting (negative feedback) cycle. Such cycles, as a general rule, produce oscillations. Here, the product of the weights of the arcs on this cycle is small enough so that the oscillations eventually die out. In any case, the results suggest that to stabilize the rising levels of variables in this system, we shall have to search for external influences and perhaps counter these influences by introducing countervailing pulses into the system.

Although these conclusions about transportation once again depend on our specific model, they suggest something interesting about the system of energy use in transportation. Namely, they suggest that the source of rising levels of energy use in transportation is different from the source of rising levels of usage of electrical power, where the operation of feedback within the system tends to lead to spiralling levels. Perhaps this conclusion is true only of San Diego, but even then it is interesting.

Signed and weighted digraphs like those of Fig.'s 1 and 2 have been constructed for a wide variety of problems. Other signed digraphs for energy use in transportation are studied in Roberts [7]. Kruzic [6] builds a weighted digraph for the energy and environmental impact of deep water ports; Coady, *et al.* [1] build a weighted digraph for assessing the use of the coastal zone for urban recreation; Kane, *et al.* [4] build a weighted digraph for analyzing the allocation of scarce resources to health care delivery; and the Organization for Economic Cooperation and Development is building signed digraphs for analyzing the impact of various governmental funding decisions on the scientific community. (In references [1], [4] and, [6], the analysis was done using the KSIM model of Julius Kane, which differs in a number of respects from the specific change of value rule we have adopted. However, the general spirit of the approach is the same.) These examples and others are summarized in Chapter 4 of the forthcoming book Roberts [8].

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TOPICS IN ORTHOGONAL FUNCTIONS

J. J. PRICE

1. Introduction. The study of general orthogonal functions has been a field of active research since about 1920. It arose from the classical theory of Fourier series, Bessel functions, orthogonal polynomials, etc., that branch of analysis concerned with detailed investigation of orthonormal sets derived from physics. By 1900, the literature on these sets was already extensive and well on the way to becoming vast.

As the new ideas of functional analysis circulated in the early part of this century, mathematicians began looking at orthonormal sets more abstractly, as bases of certain spaces. Working from this point of view, they were led to investigate general properties of orthonormal systems as well as special properties of particular systems.

In this article, we discuss two types of questions in the general theory. The first deals with completeness of orthonormal sets, the second with rearrangements of orthogonal series. To convey the flavor of these topics as quickly as possible, let us state several typical theorems at once, leaving the necessary definitions for later.

THEOREM (Talalyan). *Suppose you delete one function from a complete orthonormal set of functions in* $L^2[0,1]$. *The remaining functions, although no longer complete, have the following property: For each* $\varepsilon > 0$, *there is a set* $S_\varepsilon \subset [0,1]$ *of measure exceeding* $1-\varepsilon$ *on which they are complete.*

The same conclusion holds if any finite number of functions are deleted. Roughly speaking, the remaining orthonormal set is nearly complete, or is complete in some weaker sense. We intend to discuss several types of near completeness and relations among them. In particular, we shall consider possible extensions of Talalyan's theorem: can you delete *infinitely* many functions from a complete orthonormal set and still retain some kind of near completeness? If so, then "how many"?

To illustrate our interest in rearrangements, we recall the celebrated theorem of Carleson: