2. Gelfand Naimark theorem

This section is devoted to an exposition of a representation theorem of Gelfand-Naimark type for JB^*-triples.

The first subsection recalls the statements and ideas of the proofs of the earlier Gelfand-Naimark type theorems for the categories of C^*-algebras and JB^*-algebras. As in the proof of the Jordan algebra version and a later proof of the C^*-algebra version, a reduction is made to the case of the JB^*-triples which are dual spaces by passing to the second dual. Thus the structure theory of this special case is reviewed in the second subsection. The next two subsections outline proofs of the main theorem, due to Friedman–Russo and Dang–Friedman respectively. The section concludes with a candidate for the definition of real JB^*-triple, and a Gelfand representation theory in the commutative case. The infinite dimensional non-commutative real triples need to be further explored from the point of view of functional analysis.

2.1. Gelfand Naimark theorem for C^*-algebras and Jordan algebras

Let's begin by recalling the original Gelfand-Naimark theorem of 1943. Let A be a C^*-algebra, that is, a Banach *-algebra satisfying $\|x\|^2 = \|x^*x\|$. Each state $\varphi \in S(A)$ of A gives rise via the GNS (Gelfand–Naimark–Segal) construction to a representation (π_φ, H_φ). For each set S of states one forms a representation $\pi_S = \oplus_{\varphi \in S}\pi_\varphi$ which is faithful if $S \subset S(A)$ separates the points of A.

The following is the celebrated Gelfand–Naimark theorem for C^*-algebras. A complete proof can be found in most books on functional analysis, for example [97]

Theorem 2.1.1 (Gelfand–Naimark theorem for C^*-algebras). *Each C^*-algebra is isometrically isomorphic to a C^*-algebra of operators on a complex Hilbert space. Each commutative C^*-algebra is isometrically isomorphic to the C^*-algebra of all continuous complex valued functions vanishing at infinity on some locally compact Hausdorff space.*

As a by-product of the above proof we obtain the following, which was observed by Sherman and Takeda in 1954 [34]. Let $\pi_u = \oplus_{\varphi \in S(A)}\pi_\varphi$ be the *universal representation* of A. Then A^{**} is isometrically isomorphic to the von Neumann algebra which is the weak closure of $\pi_u(A)$.

Anticipating the rest of this subsection, we ask rhetorically at this point: Where do the axioms of a C^*-algebra come from?

In the middle of the 1960's, Topping [110] and Stormer [106], [107] began the study of real Jordan subalgebras of $\mathcal{L}(H)_{sa}$ (H a complex Hilbert space). These were called JC-algebras if norm closed and JW-algebras if weakly closed.

The abstract version of these Jordan operator algebras are the JB-algebras and were defined as early as 1948 by Segal [102]. Thus, the axioms of a JB-algebra came from physics; more recently, in view of Iochum's thesis 1982 [76], they can be said to come from geometry too.

The complexification of a JB-algebra is called a JB^*-algebra and previously went under the name of Jordan C^*-algebra (Kaplansky, Wright [120]).

In the 1930's two important steps on the algebraic side of Jordan algebra theory were the classification of finite dimensional formally real Jordan algebras over the reals (Jordan, von Neumann, and Wigner 1934 [79]) and the fact that exceptional Jordan algebras exist (Albert).

Since the finite dimensional JB-algebras coincide with the formally real ones, a Gelfand–Naimark theorem for Jordan Banach algebras must exclude the exceptional algebras. The following is due to Alfsen–Shultz–Størmer in 1978 [5], [67].

Theorem 2.1.2 (Gelfand–Naimark theorem for Jordan algebras). *If A is a JB-algebra, then there is unique closed ideal J such that A/J is isometrically isomorphic to a JC-algebra, and J is purely exceptional, that is, every representation of J into some $\mathcal{L}(H)_{sa}$ is zero.*

The original proof of this theorem follows a well known path, but is long, requiring new techniques to deal with the non-associativity. These techniques include the following:

- ordered Banach spaces
- topologies on the enveloping monotone completion $\tilde{A} \subset A^{**}$
- spectral theory (singly generated subalgebras are continuous function spaces)
- comparison and equivalence in the lattice of projections
- analysis of spin factors
- coordinatization.

2.2. Some structure theory for JBW^*-triples

We begin with the definition and some properties of JB^*-triples. The reader who finds this subsection uncomfortable might prefer the more leisurely discussion of the same topics in subsection 3.1. This subsection is essentially a summary of [51].

Definition 2.2.1. A Banach space U over $\mathbb{C}$ is said to be a *JB^*-triple* if it is equipped with a continuous triple product $(a,b,c) \mapsto \{abc\}$ mapping $U \times U \times U$ to U such that

(i) $\{abc\}$ is linear in a and c and conjugate linear in b;

(ii) $\{abc\}$ is symmetric in the outer variables, *i.e.*, $\{abc\} = \{cba\}$;

(iii) for any $x \in U$, the operator $\delta(x)$ from U to U defined by $\delta(x)y = \{xxy\}$, is hermitian (*i.e.*, $\exp it\delta$ is an isometry for all real t) with non-negative spectrum;

(iv) the triple product satisfies the following identity, called the "main identity":

$$\delta(x)\{abc\} = \{\delta(x)a, b, c\} - \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\}; \tag{2.1}$$

(v) the following norm condition holds:

$$\|\{xxx\}\| = \|x\|^3. \tag{2.2}$$

Define the quadratic operator $Q(x)$ by $Q(x)y = \{xyx\}$ and then set $Q(x,y)z = \{xzy\}$. A tripotent is an element e satisfying $e = Q(e)e = \{eee\}$. With each tripotent there are associated the Peirce projections:

- $P_2(e) = Q(e)^2$
- $P_1(e) = 2(\delta(e) - Q(e)^2)$
- $P_0(e) = I - 2\delta(e) + Q(e)^2$

so that we have

$$I = P_2(e) + P_1(e) + P_0(e) \text{ and } \delta(e) = P_2(e) + \frac{1}{2}P_1(e),$$

and the Peirce decomposition: $U = U_2(e) \oplus U_1(e) \oplus U_0(e)$.

The following are fundamental *algebraic* properties of a tripotent e in a Jordan triple system U.

- $\{U_i(e), U_j(e), U_k(e)\} \subset U_{i-j+k}(e)$
- $\{U_0(e), U_2(e), U\} = 0 = \{U_2(e), U_0(e), U\}$
- $U_2(e)$ is a complex unital Jordan algebra with involution:

$$x \circ y := \{xey\} \qquad x^\sharp := \{exe\}.$$

The following are fundamental *topological* properties of a tripotent e in a JB^*-triple U.

- $U_2(e)$ is a JB^*-algebra, and $U_2(e)_{sa}$ is a JB-algebra
- The Peirce projections are contractive, as is $P_2(e) + P_0(e)$
- The family $S_\lambda(e) := \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e)$ for $\lambda \in T$ is a one parameter group of *isometries*.

The following basic proposition has the interpretation that the Peirce spaces $U_2(e)$ and $U_0(e)$ have the *unique Hahn-Banach extension property*. It goes under the name "neutrality".

Proposition 2.2.2 (Proposition 1 of [51]). *Let U be a JB^*-triple, $e \in U$ a tripotent, and $f \in U^*$. Then $f \circ P_2(e) = f$ if and only if $\|f \circ P_2(e)\| = \|f\|$.*

A partial order for tripotents is defined by: $e \le e'$ if $e' - e$ is a tripotent orthogonal to e, where a orthogonal to b means $\{ab\cdot\} = 0$.

Corollary 2.2.3. *$e \le e'$ if and only if $P_2(e)e' = e$.*

A *JBW*-triple* is a JB^*-triple which is the dual of some Banach space. It is known that a JBW^*-triple has a unique predual and that the triple product in a JBW^*-triple is separately weak*-continuous. This will be discussed in subsection 2.3. The predual of U is denoted by U_*.

By the Gelfand theory of commutative JBW^*-triples, one has a polar and spectral decomposition of an arbitrary element of a JBW^*-triple. For functionals on the other hand, we have the following ([51, Proposition 2]):

Proposition 2.2.4 (Polar decomposition of a normal functional). *Let U be a JBW^*-triple and let $f \in U_*$. Then there is a unique tripotent e, called the support tripotent of f, such that $f \circ P_2(e) = f$ and $f|_{U_2(e)}$ is a faithful positive normal functional.*

Another key property of the predual of a JBW^*-triple is the fact ([51, Proposition 8]) that every norm exposed face of ball U_* is "projective" (cf. Theorem 1.3.1). In this context this simply means that for every norm exposed face F_x, that is, with $x \in U$ of norm 1,

$$F_x = \{\rho \in (U_*)_1 : \|\rho\| = 1 = \rho(x)\},$$

there is a tripotent w such that $F_x = F_w$.

The Peirce projections are fundamental operators on JB^*-triples. When do they commute? The most general condition has been given in [90]. For our purposes here, the following sufficient condition is adequate.

Proposition 2.2.5 (Lemma 1.10 of [51]). *If e and v are tripotents in a JB^*-triple U and if one of them belongs to one of the Peirce spaces of the other, then $[P_\alpha(e), P_\beta(v)] = 0$ for all $\alpha, \beta \in \{0, 1, 2\}$.*

Atomic decompositions

This subsection is devoted to the decompositions of a JBW^*-triple U and its predual into atomic and purely non-atomic parts. We therefore next explain some of the definitions and tools needed in the proofs.

A tripotent e in a JB^*-triple U is *minimal* if the Peirce 2-space $U_2(e)$ is one-dimensional. In any JBW^*-triple, there is a bijection between minimal tripotents and extreme points of the unit ball of the predual, and each such extreme point is norm exposed ([51, Proposition 4]).

The map

$$\sum_i \alpha_i f_i \mapsto \sum_i \overline{\alpha}_i e_i,$$

where e_i is the support tripotent of the extreme point f_i, is a conjugate-linear bijection of the finite span of the extreme points onto the finite span of the minimal tripotents ([51, Lemma 2.11]).

The following four properties, analogs of results for JB-algebras [4], are instrumental for the main decomposition theorems.

Symmetry of Transition Probabilities [51, Lemma 2.2]
If f_1, f_2 are extreme points with support tripotents e_1, e_2 respectively, then

$$f_1(e_2) = \overline{f_2(e_1)}.$$

Hilbert Ball Property [51, Proposition 5]
If u and v are minimal tripotents in any Jordan triple system, then the smallest Jordan triple system containing these two elements is of dimension at most 4, and is thus isomorphic to one of the following spaces:

$$\mathbb{C},\ M_{1,2}(\mathbb{C}),\ \mathbb{C}\oplus\mathbb{C},\ M_2(\mathbb{C}),\ S_2(\mathbb{C}).$$

Extreme Ray Property [51, Proposition 7]
If e is any tripotent and f is an extreme point, then $P_2(e)^*f$ is a scalar multiple of an extreme point.

Minimal Ray Property [51, Proposition 6]
If e is any tripotent and u is a minimal tripotent, then $P_2(e)u$ is a scalar multiple of a minimal tripotent.

Theorem 2.2.6 (Atomic decomposition of U_* [51, Theorem 1]). *If U is a JBW^*-triple, then*

$$U_* = \mathcal{A} \oplus^{\ell^1} \mathcal{N},$$

where $\mathcal{A}$ is the norm-closure of the span of the extreme points of the unit ball of U_*, and $\mathcal{N}$ is a closed subspace of U_* whose unit ball has no extreme points.

Proof. If $\varphi \in U_*$ has support tripotent e, then the polar decomposition says that φ restricts to a normal faithful state on the JBW^*-algebra $U_2(e)$. Thus φ decomposes locally. The minimal ray property is then used to show that this is a global decomposition. □

Note that the corresponding result for JBW^*-algebras is elementary since the projections form a complete lattice.

The original proof of the next theorem uses the map

$$\sum_i \alpha_i f_i \mapsto \sum_i \bar{\alpha}_i e_i,$$

and the extreme ray property. Later, proofs were given in [33], [91], and [29].

Theorem 2.2.7 (Atomic decomposition of U [51, Theorem 2]). *If U is a JBW^*-triple, then*

$$U = A \oplus^{\ell^\infty} N,$$

where A is an ideal which is the weak-closure of the span of the minimal tripotents of U, and N is a weak*-closed ideal with no minimal tripotents.*

Some further structure theory for JBW^*-triples will be discussed in subsections 2.3, 2.4 (type I) and 4.3 (continuous).

Facial structure in a JBW^*-triple and its predual

The facial structure of the closed unit balls in JBW-algebras and their preduals were described by Edwards–Rüttimann by means of elements of the complete lattice of idempotents. One of their main methods, which is also available in the complex case, is the use of the mappings $E \mapsto E'$ and $F \mapsto F_{\prime}$ between subsets of the unit balls ball V and ball V^* in a Banach space V and its dual V^* defined by

$$E' = \{a \in \text{ball}\, V^* : a(x) = 1, \forall x \in E\}$$

and

$$F_{\prime} = \{x \in \text{ball}\, V : a(x) = 1, \forall x \in F\}.$$

The following two theorems appear in [38]. Let $\mathcal{U}(A)$ denote the set of tripotents of the JBW^*-triple A with a largest member adjoined.

Theorem 2.2.8. *Let A be a JBW^*-triple with predual A_*. Then the mapping $u \mapsto u_{\prime}$ is an order isomorphism from the complete lattice $\mathcal{U}(A)$ onto the complete lattice of norm-closed faces of the unit ball of A_*. In particular, every norm closed face of ball A_* is norm-exposed.*

Theorem 2.2.9. *Let A be a JBW^*-triple. Then the mapping $u \mapsto u_{\prime}{}'$ is an anti-order isomorphism from the complete lattice $\mathcal{U}(A)$ onto the complete lattice of weak*-closed faces of the unit ball of A. Moreover, $u_{\prime}{}'$ coincides with $u +$ ball $A_0(u)$.*

2.3. Gelfand Naimark theorem for JB^*-triples

JB^*-triples are generalizations of JB^*-algebras and hence of C^*-algebras. The axioms can be said to come from geometry in view of Kaup's Riemann mapping theorem. JB^*-triples first arose in M. Koecher's proof ([86]) of the classification of bounded symmetric domains in $\mathbb{C}^n$. The original proof of this fact, done in the 1930's by Cartan, used Lie algebras and Lie groups, techniques which do not extend to infinite dimensions. On the other hand, to a large extent, the Jordan algebra techniques do so extend, as shown by Kaup and Upmeier.

The following is due to Friedman–Russo ([54]). The Cartan factors are defined in subsection 3.1.

Theorem 2.3.1 (Gelfand-Naimark for JB^*-triples). *Every JB^*-triple is isometrically isomorphic to a subtriple of an ℓ^∞-direct sum of Cartan factors.*

This theorem is not unexpected. However, the proof required new techniques because of the lack of an order structure on a JB^*-triple. Here is a chronology of the proof of the theorem. Some of the steps have been mentioned already.

step 1: February 1983 Friedman–Russo ([52])
Let $P : A \to A$ be a linear projection of norm 1 on a JC^*-triple A. Then $P(A)$ is a JB^*-triple under $\{xyz\}_{P(A)} := P(\{xyz\}_A)$ for $x, y, z \in P(A)$. (This was new for $A =$ a C^*-algebra.)

step 2: April 1983 Friedman–Russo ([50])
Same hypotheses. Then P is a conditional expectation, that is, for $a, b, c \in A$,

$$P\{PaPbPc\} = P\{PabPc\} \text{ and } P\{PaPbPc\} = P\{aPbPc\}.$$

step 3: May 1983 Kaup ([85])
Let $P : U \to U$ be a linear projection of norm 1 on a JB^*-triple U. Then $P(U)$ is a JB^*-triple under $\{xyz\}_{P(U)} := P(\{xyz\}_U)$ for $x, y, z \in P(U)$. Also $P\{PaPbPc\} = P\{PabPc\}$ for $a, b, c \in U$, which extends one of the formulas in the previous step.

step 4: February 1984 Friedman-Russo ([51])
Every JBW^*-triple splits into atomic and purely non-atomic ideals.

step 5: August 1984 Dineen ([32)]
The bidual of a JB^*-triple is a JB^*-triple.

step 6: October 1984 Barton–Timoney ([17])
The bidual of a JB^*-triple is a JBW^*-triple, that is, the triple product is separately weak*-continuous.

step 7: December 1984 Horn ([71],[72],73],74])
Every JBW^*-triple factor of type I is isomorphic to a Cartan factor. More generally, every JBW^*-triple of type I is isomorphic to an ℓ^∞-direct sum of L^∞ spaces with values in a Cartan factor.

step 8: March 1985 Friedman–Russo ([54])
Putting it all together:

$$\pi : U \to U^{**} = A \oplus N = (\oplus_\alpha C_\alpha) \oplus N = \sigma(U^{**}) \oplus N$$

implies that $\sigma \circ \pi : U \to A = \oplus_\alpha C_\alpha$ is an isometric isomorphism.

Here are some consequences of the Gelfand-Naimark theorem for JB^*-triples, found in [54].

- Every JB^*-triple is isomorphic to a subtriple of a JB^*-algebra.
- In every JB^*-triple, $\|\{xyz\}\| \leq \|x\|\|y\|\|z\|$.
- Every JB^*-triple U contains a unique norm-closed ideal J such that U/J is isomorphic to a JC^*-triple and J is purely exceptional, that is, every homomorphism of J into a C^*-algebra is zero.

The following two properties of JBW^*-triples, suggested by the Gelfand–Naimark theorem, were established by Barton–Dang–Horn [12].

- Every JBW^*-triple splits into a direct sum $U = J \oplus [U/J]$ where J is purely exceptional and U/J is isomorphic to a weakly closed JC^*-triple. (For JBW-algebras this is due to Shultz 1979 [103].)
- Every JBW^*-triple which is isomorphic to a JC^*-triple is isomorphic to a weakly closed JC^*-triple. (For W^*-algebras this is due to Sakai 1957 [101].)

2.4. Classification of atomic factors

There is a second proof of the Gelfand-Naimark theorem for JB^*-triples which is due to Dang–Friedman [29]. It relies on their new and transparent proof of the classification of Cartan factors of type I. This latter proof is based on the following three works:

- (idea) 1934 Jordan–von Neumann–Wigner [9]: classification of formally real Jordan algebras;
- (technique) 1978 Arazy–Friedman [9]: classification of the ranges of contractive projections on C_1 and C_∞;
- (relations between tripotents) 1985 Neher [91]: Jordan triple systems with enough tripotents.

The building blocks of the algebraic structure of a Jordan triple system are the tripotents and their corresponding Peirce projections, and there are important relations between pairs, triples, and quadruples of tripotents (orthogonal, colinear, governing, trangle, quadrangle,...). These terms will not be defined here. The relations are fundamental tools in the Dang–Friedman proof. Entirely similar ideas are instrumental in the proof of the main result of [59], which is the topic of section 3.

The Dang–Friedman classification of the Cartan factors of type I begins with an irreducible JBW^*-triple U with a minimal tripotent v.

Proposition 2.4.1 ([29]). *If u is any tripotent in the Peirce 1-space $U_1(v)$ of v, then one of the following holds:*

- *u is minimal in U* (this holds if and only if u and v are colinear);
- *u is minimal in $U_1(v)$ but not minimal in U* (this holds only if u governs v);
- *u is not minimal in $U_1(v)$* (this implies u is the sum of two minimal tripotents of U).

Corollary 2.4.2. *The rank of $U_1(v)$ is at most 2.*

Proposition 2.4.3 ([29]). *If $v, \tilde{v}, u, \tilde{u}$ are the minimal tripotents forming a quadrangle, and if $U_1(v + \tilde{v}) \neq \{0\}$, then $\dim U_2(v + \tilde{v}) \in \{4, 6, 8, 10\}$.*

The Dang–Friedman classification scheme ([29]) is now the following: let $J(v)$ denote the weak*-closed ideal generated by v. Then

case 0: Rank $U_1(v) = 0$; then $J(v) \simeq \mathbb{C}$;
case 1: Rank $U_1(v) = 1, u$ a tripotent of $U_1(v)$ minimal in U; then $J(v)$ is a Hilbert space (Cartan factor of type 1);
case 2: Rank $U_1(v) = 1, u$ a tripotent of $U_1(v)$ minimal in $U_1(v)$; then $J(v)$ is a Cartan factor of type 3 (symmetric operators).

In cases 3–7, Rank $U_1(v) = 2$, u is a non-minimal tripotent in $U_1(v)$, and $\tilde{v} := \{uvu\}$.
case 3: $U_1(v + \tilde{v}) = \{0\}$; then $J(v)$ is a Cartan factor of type 4 (spin factor).

In cases 4–7, $U_1(v + \tilde{v}) \neq \{0\}$ and so by Proposition 2.4.3, these are all the cases possible.

case 4: $\dim U_2(v + \tilde{v}) = 4$; then $J(v)$ is a Cartan factor of type 1 (all operators).
case 5: $\dim U_2(v + \tilde{v}) = 6$; then $J(v)$ is a Cartan factor of type 2 (anti-symmetric operators).
case 6: $\dim U_2(v + \tilde{v}) = 8$; then $J(v)$ is a Cartan factor of type 5 (1 by 2 matrices over the Octonions).
case 7: $\dim U_2(v + \tilde{v}) = 10$; then $J(v)$ is a Cartan factor of type 6 (3 by 3 Hermitians over the Octonions).

In each case, $J(v)$ is a summand in U.

The ideas just discussed have application to the study of isometries of real and complex triples and algebras.

- All the JB^*-triples for which every real linear surjective isometry preserves the triple product can be determined, and as a corollary it follows that all real linear isometries of any (complex) C^*-algebra preserve the triple product [28]. This will be discussed below in subsection 4.1.
- Surjective isometries of *real* C^*-algebras preserve the triple product [21]. This will be discussed below in subsection 4.1.
- Do the isometries of a real JB^*-triple preserve the triple product? Since we do not yet have a workable definition of real JB^*-triple, I won't say much here. However, see subsection 2.5.

This subsection also suggests the following problem, which is of great interest for C^*-algebras. If A is any C^*-algebra, then A and its bidual $M := A^{**}$ can be considered as JB^*-triples, M being a JBW^*-triple with predual A^*. As in any JBW^*-triple, there is a bijection between minimal tripotents of M and extreme points of the unit ball of A^*. Also M has an atomic part spanned in the weak*-topology by the minimal tripotents and equal to an ℓ^∞-sum of Cartan factors of types 1–6. By the work of Horn, Neher, and Dang–Friedman, each Cartan factor is spanned by a "grid", and thus elements of M may be considered as functions on the set S of extreme points of the unit ball of A^*.

Problem 1. *Put a "topological-like" structure on S so that $A \subset M$ is identified as the set of all "continuous-like" functions on S.*

A partial solution to this problem will be discussed in subsection 3.6. The problem is significant because even for C^*-algebras, it is necessary to take advantage of the triple product structure in order to guarantee that S will consist of extreme points. In other words, *colinearity* doesn't exist for *projections*, which are the building blocks for binary structures.

2.5. Real JB^*-triples

In contrast to the situation for JB^*-algebras (and to some extent for C^*-algebras), Jordan triple systems over the reals have played no role in the analytic theory of JB^*-triples. This is due to the history of the area: JB^*-triples were born of an investigation into certain aspects of several complex variables ([86]). However, a theory of real Jordan triples and real bounded symmetric domains in finite dimensions was developed by Loos ([88]). This, together with the observation that many of the more recent techniques in Jordan theory ([51], [84], [17]) rely on functional analysis and algebra rather than holomorphy, suggests that it may be possible to develop a real theory and to explore its relationship with the complex theory.

In this subsection we employ a Banach algebraic approach to real Banach Jordan triples. Because of our recent observation on commutative JB^*-triples (see a subsection below), we can now propose a new definition of real JB^*-triple, which we call *J^*B-triple.* Our J^*B-triples include real C^*-algebras and complex JB^*-triples. The main result of [31], which will be described in this subsection, is a structure theorem of Gelfand-Naimark type for commutative J^*B-triples.

Real Banach Jordan triples

Definition 2.5.1. A *Banach Jordan triple* is a real or complex Banach space U equipped with a continuous bilinear (sesquilinear in the complex case) map

$$U \times U \ni (x, y) \mapsto x \square y \in \mathcal{L}(U)$$

such that, with $\{xyz\} := x \square y(z)$ we have

$$\{xyz\} = \{zyx\}; \tag{2.3}$$

$$\{x, y, \{uvz\}\} + \{u, \{yxv\}, z\} = \{\{xyu\}, v, z\} + \{u, v, \{xyz\}\}. \tag{2.4}$$

Recall that a Banach Jordan triple U over $\mathbb{C}$ is said to be a *JB^*-triple* if

(a) for any $x \in U$, the operator $x \square x$ from U to U (that is, $x \square x(y) = \{xxy\}$, $y \in U$) is hermitian (i.e., $\exp it(x \square x)$ is an isometry for all real t) with non-negative spectrum;

(b) the following norm condition holds:

$$\|x \square x\| = \|x\|^2.$$

The proof of the following theorem was suggested by Jonathan Arazy. Since it is so short and elegant, we include it here.

Theorem 2.5.2. *Let U be a complex Banach Jordan triple. Suppose that*

1. $\|\{xxx\}\| = \|x\|^3$;
2. $\|\{xyz\}\| \leq \|x\| \, \|y\| \, \|z\|$;
3. *U is positive, i.e., $\sigma_{\mathcal{L}(U)}(x \square x) \subset [0, \infty)$ for each $x \in U$.*

Then U is a JB^*-triple.

Proof. We only need to show that $x \square x$ is hermitian, for each $x \in U$.

Since $\delta := ix \square x$ is a continuous derivation, $\alpha := e^{t\delta}$ is a continuous automorphism for each real t. Thus, for each $x \in U$,

$$\|\alpha(x)\|^3 = \|\{\alpha(x), \alpha(x), \alpha(x)\}\| = \|\alpha(\{xxx\})\| \leq \|\alpha\| \|x\|^3$$

and therefore, by iteration,

$$\|\alpha(x)\| \leq \|\alpha\|^{1/3^n} \|x\|,$$

that is, $\|\alpha\| \leq 1$. □

The terminology in the next definition was motivated by [7], and the spectral conditions were inspired by [115].

Definition 2.5.3. A *J^*B-triple* is a real Banach space A equipped with a structure of real Jordan triple system which satisfies

1. $\|\{xxx\}\| = \|x\|^3$;
2. $\|\{xyz\}\| \leq \|x\| \, \|y\| \, \|z\|$;
3. $\sigma^c_{\mathcal{L}(A)}(x \square x) \subset [0, \infty)$ for $x \in A$;
4. $\sigma^c_{\mathcal{L}(A)}(x \square y - y \square x) \subset i\mathbb{R}$ for x, y $\in A$.

Over the complex field, JB^*-triples are the same as J^*B-triples.

A closed subtriple B of a J^*B-triple A is a J^*B-triple. In particular, a closed *real* subtriple of a JB^*-triple is a J^*B-triple.

A real C^*-algebra is a closed subalgebra of its complexification, which is a complex C^*-algebra in some norm. Thus, a real C^*-algebra, with the triple product

$$\{xyz\} = \frac{1}{2}(xy^*z + zy^*x).$$

is a closed real subtriple of a JB^*-triple, and hence a real C^*-algebra is a J^*B-triple.

Two important and natural problems left open in the paper [31] are the following.

Problem 2. Is the complexification of a J^*B-triple is a JB^*-triple in some norm extending the original norm. (This is solved for commutative J^*B-triples in Theorem 2.5.8.)

Problem 3. Is the bidual of a J^*B-triple a J^*B-triple with a separately weak*-continuous triple product.

Commutative complex triples

We are going to use Theorem 2.5.2 to modify the treatment in [84, Section 1] by not requiring that $x \square x$ be hermitian. Theorem 2.5.5 below is needed to prove the main result of this subsection, namely Theorem 2.5.8, which leads to a Gelfand-Naimark Theorem for commutative real J^*B-triples.

Definition 2.5.4. A Banach Jordan triple is *commutative* if

$$\{\{xyz\}uv\} = \{xy\{zuv\}\} = \{x\{yzu\}v\}.$$

For example, any commutative C^*-algebra $C_0(\Omega)$ is a commutative Banach Jordan triple with $f \square g(h) = f\bar{g}h$.

Throughout this subsection U will denote a commutative complex Banach Jordan triple.

Let $B = B(U) :=$ the closed span of $U \square U$ in $\mathcal{L}(U)$. Then B is a commutative Banach subalgebra of $\mathcal{L}(U)$. Denote the Gelfand Transform of B by

$$\Gamma_B : B(U) \to C_0(X),$$

where $X = X_B$ is the maximal ideal space of B. Let $\Lambda = \Lambda(U) :=$ the set of all non-zero triple homomorphisms $\lambda : U \to \mathbb{C}$. Precisely,

$$\Lambda = \{\lambda : U \to \mathbb{C} : 0 \neq \lambda \text{ linear}, \lambda(\{abc\}) = \lambda(a)\overline{\lambda(b)}\lambda(c)\}.$$

According to [84, Lemma 1.6], Λ is a bounded subset of $\mathcal{L}(U, \mathbb{C})$. Thus, Λ is a weak*-locally compact space and a "principle $\mathbb{T}$-bundle" ($\mathbb{T}$=unit circle) under the action

$$T \times \Lambda \ni (t, \lambda) \mapsto t.\lambda \in \Lambda,$$

where $(t.\lambda)(x) = t\lambda(x)$.

Define a norm closed subtriple of $C_0(\Lambda)$:

$$C_{\text{hom}}(\Lambda) := \{f \in C_0(\Lambda) : f(t.\lambda) = tf(\lambda), \forall (t, \lambda) \in T \times \Lambda\}$$

and a Gelfand Transform $U \ni x \mapsto \hat{x} = \Gamma_U(x) \in C_{\text{hom}}(\Lambda)$ by $\Gamma_U(x)(\lambda) = \lambda(x)$. Thus

$$\Gamma_U : U \to C_{\text{hom}}(\Lambda)$$

is a continuous triple homomorphism. The proof of the following theorem is immediate from [84, §1], since the assumptions imply that U is a JB^*-triple.

Theorem 2.5.5. *Let U be a commutative complex Banach Jordan triple. Suppose that*
1. $\|\{xxx\}\| = \|x\|^3$;
2. $\|\{xyz\}\| \leq \|x\|\,\|y\|\,\|z\|$;
3. *U is positive, i.e., $\sigma_{\mathcal{L}(U)}(x \square x) \subset [0, \infty)$ for each $x \in U$.*

Then the Gelfand representation $U \to C_{hom}(\Lambda)$ is an isometric surjective triple isomorphism.

For a generalization of this theorem, see [48].

Commutative Real triples

Now let A be a commutative real Banach Jordan triple, that is, a real Banach space A, together with a tri-linear map

$$A \times A \times A \ni (x, y, z) \mapsto \{xyz\} \in A$$

which satisfies

$$\{xyz\} = \{zyx\};$$

$$\{\{xyz\}uv\} = \{xy\{zuv\}\} = \{x\{yzu\}v\}.$$

We shall define a natural Gelfand transform and state a representation theorem of Gelfand-Naimark type.

By analogy with the complex case, let $B(A)$ be the Banach subalgebra of $\mathcal{L}(A)$ generated by $A \square A$. Then $B(A)$ is a commutative real Banach algebra (not necessarily unital, cf. [62, p. 63]). Let $X^c_{B(A)}$ denote the space of complexified characters (cf. [62, p. 82]), that is

$$X^c_{B(A)} = \{\tau : B(A) \to \mathbb{C}, 0 \neq \tau \text{ real-linear}, \tau(ST) = \tau(S)\tau(T)\}.$$

By analogy we define Λ^c_A to be the collection of all non-zero real-linear triple homomorphisms of A into $\mathbb{C}$; precisely,

$$\Lambda^c_A = \{\lambda : A \to \mathbb{C} : \lambda \text{ real-linear}, \lambda \neq 0, \lambda(\{abc\}) = \lambda(a)\overline{\lambda(b)}\lambda(c)\}.$$

By the proof of [84, Lemma 1.6], each such λ is automatically continuous and Λ^c_A is contained in a bounded subset of $\mathcal{L}_{\mathbb{R}}(A, \mathbb{C})$. Note that $e^{i\theta}\Lambda^c_A = \Lambda^c_A$, that Λ^c_A is closed under complex conjugation, and that Λ^c_A is locally compact in the topology of pointwise convergence on A.

In order to obtain the analogue of Theorem 2.5.5 we need to consider the complexification of A.

Let $U := A^{\mathbb{C}} = \phi(A) + i\phi(A)$ be the complexification of A and let $\phi : A \to U$ be the natural embedding. The space U becomes a complex commutative Jordan triple system in the natural way and ϕ is a real-linear triple isomorphism into.

The given norm on A can be used to define a norm on U as described in [62]. With this norm, U is a commutative complex Banach Jordan triple.

As above, let $B(U)$ be the closed complex subalgebra of $\mathcal{L}(U)$ generated by $U \square U$ and define $B(\phi(A))$ to be the closed real subalgebra of $\mathcal{L}(U)$ generated by $\phi(A) \square \phi(A)$. Then $B(A^{\mathbb{C}}) = (B(A))^{\mathbb{C}}$.

Proposition 2.5.6. *Suppose that A is a commutative J^*B-triple. Then $B(A)$, with the norm of $\mathcal{L}(A)$, is a commutative real C^*-algebra with involution determined by $(x \square y)^* = y \square x$. Consequently, $B(U)$ is a C^*-algebra in some norm extending the norm on $B(A)$* (by [62, 12.4]).

Let $\Lambda(U)$ be defined as above.

Lemma 2.5.7. *With the above notation,*
(i) *for each $\lambda \in \Lambda(U)$ there is $\lambda' \in \Lambda_A^c$ such that $\lambda(\phi(x) + i\phi(y)) = \lambda'(x) + i\lambda'(y)$ for $x, y \in A$. This correspondence establishes a bijection $\Lambda(U) \leftrightarrow \Lambda_A^c$;*
(ii) *for each $\tau \in X_{B(U)}$ there is $\tau' \in X_{B(A)}^c$ such that $\tau(T + iS) = \tau'(T) + i\tau'(S)$ for $T, S \in B(A)$. This correspondence establishes a bijection $X_{B(U)} \leftrightarrow X_{B(A)}^c$.*

We can now state the main result of [31].

Theorem 2.5.8. *Let A be a commutative J^*B-triple. There is a norm on the complexification U of A extending the norm on A and for which U is a JB^*-triple.*

We conclude by describing the Gelfand transform and stating and proving a Gelfand-Naimark type theorem for commutative J^*B-triples.

As noted earlier, the space Λ_A^c is a locally compact Hausdorff space in the topology of pointwise convergence on A. The bijection in Lemma 2.5.7(i) is a homeomorphism. Now let

$$C^*_{\text{hom}}(\Lambda_A^c) = \{f \in C_0(\Lambda_A^c) : f(e^{i\theta}\lambda') = e^{i\theta} f(\lambda') \text{ and } f(\overline{\lambda'}) = \overline{f(\lambda')}\}$$

and define a Gelfand transform $\Gamma_A^{\mathbb{R}} : A \to C^*_{\text{hom}}(\Lambda_A^c)$ by $\Gamma_A^{\mathbb{R}}(x)(\lambda') = \lambda'(x)$. Let $\rho : \Lambda_U \to \Lambda_A^c$ be the restriction map implicit in Lemma 2.5.7 and let $\rho^* : C^*_{\text{hom}}(\Lambda_A^c) \to C_{\text{hom}}(\Lambda_U)$ be its transpose.

Note that $\rho^{-1}(\overline{\lambda})(\phi(x) + i\phi(y)) = \overline{\lambda(x)} + i\overline{\lambda(y)}$ and therefore $\Gamma_A^{\mathbb{R}}$ maps A into $C^*_{\text{hom}}(\Lambda_A^c)$. Since

$$\Gamma_U^{\mathbb{C}} \circ \phi = \rho^* \circ \Gamma_A^{\mathbb{R}},$$

$\Gamma_A^{\mathbb{R}}$ is an isometry.

Finally, if $f \in C^*_{\text{hom}}(\Lambda_A^c)$ and $x, y \in A$ are such that $\rho^* f = \Gamma_U(\phi(x) + i\phi(y))$, the fact that $f(\overline{\lambda'}) = \overline{f(\lambda')}$ implies that $y = 0$, hence $\Gamma_A^{\mathbb{R}}(A) = C^*_{\text{hom}}(\Lambda_A^c)$. This proves

Theorem 2.5.9. *Let A be a commutative J^*B-triple. Then the Gelfand transform is an isometric triple isomorphism of A onto $C^*_{\text{hom}}(\Lambda^c_A)$.*