Structure of JB^*-Triples

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Abstract. This paper is a summary of the known structure of JB^*-triples. Its central focus consists of two structure theorems, due jointly to the author and Yaakov Friedman. Sections 2 and 3 discuss these results in detail. Sections 1 and 4 play somewhat different roles. The former discusses some general aspects of the subject, and gives some background. The latter discusses some topics which are especially interesting to the author.

The style of this survey is informal. Proofs of some major theorems are given in outline, together with the background material. In several places, a preview of forthcoming work is described and some problems are proposed.

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Introduction and overview

Why study JB^*-triples?

Here are two reasons from the functional analyst's point of view. In the first place, by the fundamental result of Kaup ([84]), JB^*-triples are in one to one correspondence with bounded symmetric domains in complex Banach spaces. In the finite dimensional case this is due to Koecher [86], cf. Loos [88]. Operators on Hilbert spaces of analytic functions is a mainstream topic in operator theory and has blossomed in the last 25 years. Although most attention has been focused on functions defined on the unit disk, recently much attention has been devoted to functions defined on the unit ball or on the unit polydisk in $\mathbb{C}^n$. More generally, Toeplitz, Hankel, and composition operators have been considered on spaces of holomorphic functions defined on Cartan domains (the finite dimensional bounded symmetric domains). A fundamental result here is the complete structure theory of the Toeplitz C^*-algebra of a Cartan domain, due to Upmeier (cf. Lecture 8 of [116]). More recently, the notion of quantization is playing a key role here (Upmeier [117] and Coburn).

A second justification for the study of JB^*-triples arises from the fact that the category includes C^*-algebras, JB^*-algebras, Hilbert spaces and spin factors. There is reason to believe that the well known Jordan algebra approach to quantum mechanics can be broadened in such a way that Jordan structures other than the binary ones will be significant. This is currently being explored by Friedman and Naimark [46].

Jordan Algebras (Oberwolfach 1992), 209-280, de Gruyter, Berlin, 1994

In a lecture at the second U.S.-Japan conference on operator algebras in 1977 ([39]), E. Effros introduced these three categories to study injectivity and nuclearity:

$\mathcal{N}$: real normed spaces, contractions;
$\mathcal{F}$: function systems, positive unital maps;
$\mathcal{O}$: operator systems, completely positive unital maps.

It was known for a long time that the appropriate algebraic models for the categories $\mathcal{F}$ and $\mathcal{O}$ are Jordan C^*-algebras and C^*-algebras, respectively. Investigation into the extent that the algebraic structure within these categories is influenced by geometric properties alone was carried out by Friedman–Russo ([49]). This investigation suggested that in the categories $\mathcal{F}$ and $\mathcal{O}$, geometric properties do not depend on the order structure, and at the same time introduced the appropriate algebraic model for the category $\mathcal{N}$, namely the JB^*-triple.

In order to see why a triple product rather than a binary one determines the geometry, we focus our attention on mappings of the algebraic models. If A is a unital C^*-algebra and w is any fixed unitary element in A, then T defined by $Tx = wx, x \in A$ is a linear isometry. Moreover, $\{Tx, Ty, Tz\} = \{xyz\}$, where

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x),$$

i.e., T preserves the Jordan triple structure. However, T does not preserve the order structure on A. In general, it is known that a completely positive unital isometry of a C^*-algebra preserves the (associative) C*-structure (Choi, Størmer, unpublished), a positive unital isometry of a C^*-algebra (more generally of a "JB-algebra") preserves the Jordan structure (Kadison [80], Wright-Youngson [121]), and an arbitrary isometry of a JC^*-triple (resp. JB^*-triple) preserves the Jordan triple structure (Harris [68], resp. Kaup [84]). Thus in each algebraic model the (surjective) norm preserving linear maps respect the algebraic structure.

Consider next the norm decreasing idempotent mappings in each model. Before 1980, it was known that the image of a completely positive unital projection on a C^*-algebra is a C^*-algebra ([20]), and that the image of a positive unital projection on a "JC-algebra" is a "JC-algebra" ([41]). Friedman–Russo have shown that the image of an arbitrary contractive projection on a JC^*-triple (a "concrete" JB^*-triple) is a JC^*-triple ([52]). Since in particular, the image of a contractive projection on a C^*-algebra has, in general, only a Jordan triple structure, this is further evidence that a JB^*-triple is the appropriate algebraic model in the category $\mathcal{N}$.

These three categories are the framework, and the following three tables, in which [ ] refers to literature references, constitute a guide to this survey of "classical associative and non-associative operator systems".

Table 1. Representation theorems

C^*-algebra	JB^*-algebra	JB^*-triple
	Gelfand Naimark theorems	
[60] Gelfand & Naimark 1943	[5] Alfsen, Shultz & Stormer 1978	[54] Friedman & Russo 1986
	State space characterization	
[6] Alfsen–Shultz & H.–Olsen 1980	[4] Alfsen & Shultz 1978	[59] Friedman & Russo 1992
	Equivalent geometric category	
[25] Connes	[76] Iochum 1984	[84][118][115] [82] Kaup 1983 Upmeier, Vigué 1976

Table 2. Structure theory

C^*-algebra	JB^*-algebra	JB^*-triple
	Factor classification	
[35][24][26][65] von Neumann 1929 Connes 1976 Haagerup 1983	[110][106][107] Topping 1965 Stormer 1968	[71][73][74][29] Horn 1984 Dang & Friedman 1987
	Duality	
[40] Effros 1963 [101] Sakai 1960	[103] Shultz 1979 [51] Friedman–Russo 1985	[17] Barton–Timoney 1986 [72] Horn 1987

Table 3. Linear mappings

C^*-algebra	JB^*-algebra	JB^*-triple
	Isometries	
[80] Kadison 1951	[80] Kadison 1951 [121] Wright & Youngson 1978	[84] Kaup 1983 [68] Harris 1973
	Contractive projections	
[20] Choi & Effros 1977	[41] Effros & Stormer 1979	[52][85] Kaup 1983 Friedman–Russo 1985
	Derivations	
[99][100][81] Kadison, Sakai 1966	[114] Upmeier 1980	[14][70] Ho 1992 Barton–Friedman 1990
	Amenability	
[27][63][19] Connes 1978 Haagerup 1983 Bunce–Paschke 1980		[70] Ho 1992
	Bilinear forms	
[64][93] Pisier 1978 Haagerup 1985	[23] Chu–Iochum & Loupias 1989	Barton–Friedman 1987 [13][15] B.F.–Russo 1992

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CONTENTS