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Imbedding of Lie Algebras in Nonassociative Structures.

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Imbedding of Lie Algebras in Nonassociative Structures (*).

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1. – As is known, there have been attempts to introduce new algebraic structures in physics other than *Lie algebras* (L.A.). One of the most interesting attempts is the Jordan investigation on the « r-number algebras », today called (*commutative*) *Jordan algebras* (C.J.A.) (1), which however have not been successful in their physical applications.

We personally think that a possible reason for this disappointment in elementary-particle physics may be the want of L.A. content in the C.J.A. In other words L.A. should not be abandoned, but might be expanded. For instance the validity of L.A. for free particles is well known. It may be interesting to investigate the possible validity of new algebraic structures for an interacting or decaying region but only in such a way that the standard procedures corresponding to the free states remain unchanged, that is preserving in any case a well-defined L.A. content.

In this connection in the present paper we introduce an imbedding of L.A. in more general nonassociative structures, we choose a suitable nonassociative algebra for our extension and we briefly discuss the possibilities of physical applications.

2. – In the *imbedding*

$$L \to A \tag{1}$$

of a given L.A. L into any algebra A, which we call the *extension* of L, a useful intermediate concept for preserving a Lie content is given by the concept of *Lie-admissible algebras* introduced by ALBERT (2). An algebra A with product ab is called

(*) Notes on a lecture given at the ICTP, Trieste, June 27, 1967.

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(1) The C.J.A. are nonassociative algebras defined by the relations: i) $ab = ba$, and ii) $(a^2b)a = a^2(ba)$. They are subdivided into: i) the *special* C.J.A., which are characterized by the product $ab = \frac{1}{2}(a\cdot b + b\cdot a) = \frac{1}{2}\{a, b\}$ (we call $a\cdot b$ the associative product), and ii) the *exceptional* C.J.A., *i.e.* the algebras which are not special. For an extensive bibliography on C.J.A. see H. BRAUN and M. KOCHER: *Jordan-Algebren* (Berlin, 1966).

(2) A. A. ALBERT: *Trans. A.M.S.*, **64**, 552 (1948).

Lie-admissible if the algebra A^-, which is the same vector space as A with the product $ab - ba = [a, b]$, is a Lie algebra. For example, if A is an associative algebra, then A^- is the Lie algebra in the standard form known by the physicist; if A is a L.A. with product $ab = a \cdot b - b \cdot a$, then A^- is still a L.A. with product $ab - ba = 2(a \cdot b - b \cdot a)$. Hence associative and Lie algebras are Lie-admissible. On the contrary C.J.A. are trivially Lie-admissible, since A^- is always a zero algebra (*i.e.* a nilpotent algebra of degree 2).

In the following we are interested in the general case where A is a nonassociative algebra. Then, by using the Lie-admissibility concept, *the imbedding* (1) *may be performed according to*

$$L \to A^- \to A \, , \tag{2}$$

that is by imbedding the considered L.A. L in a nonassociative extension A such that A^- is isomorphic to L. The insufficiency of the C.J.A. for this type of imbedding then appears clear because of the commutativity of the product. Hence we must search for large algebraic structures.

In order of to find the explicit condition for Lie-admissibility, we note that the product of A^- is anticommutative by construction. Hence A is Lie-admissible if and only if A^- satisfies the Jacoby identity, that is

$$[a, b, c] + [b, c, a] + [c, a, b] = [c, b, a] + [b, a, c] + [a, c, b] \, , \qquad a, b, c \in A \, , \tag{3}$$

where $[a, b, c] = (ab)c - a(bc)$ is the *associator*, a quantity which represents the amount by which the elements of a nonassociative algebra fail to obey the associative law of multiplication.

If we introduce *flexibility*, a weaker condition than associativity expressed by $(ab)a = a(ba)$ for every $a, b \in A$, then the Lie-admissibility condition is given by the reduced form

$$[a, b, c] + [b, c, a] + [c, a, b] = 0 \, , \tag{4}$$

which looks like a generalization of the Jacoby identity.

There is also the *Jordan-admissibility* concept ([3]) which will be useful for a more exhaustive characterization of the imbedding. An algebra A with product ab is said to be Jordan-admissible if the attached algebra A^+, which is the same vector space as A with the product $\frac{1}{2}(ab + ba) = \frac{1}{2}\{a, b\}$, is a C.J.A., that is the following relaton is verified:

$$(a^2 b) a + a(ba^2) + (ba^2) a + a(a^2 b) = a^2(ba) + (ab) a^2 + a^2(ab) + (ba) a^2 \, . \tag{5}$$

We note that associative and (commutative) Jordan algebras are Jordan-admissible, but L.A. are trivially Jordan-admissible since A^+ is a zero algebra. Moreover an algebra which is (nontrivially) Lie- and Jordan-admissible is the associative algebra.

3. – Clearly there is a great number of nonassociative Lie-admissible algebras. In order that our investigation may give rise to an explicit choice with interesting physical possibilities some suitable supplementary conditions on A must be intro-

duced. In the present paper we consider the case when A is power-associative, trace-admissible and normed; then the simple nonassociative extensions A are only the *noncommutative Jordan algebras* (N.C.J.A.) (3) and among them (4) the most interesting Lie-admissible algebras are the (split) *quasi-associative algebras* $A(\lambda)$ (2), that is algebras characterized by the free scalar λ and the product

$$ab = \lambda a\cdot b + (1-\lambda)b\cdot a = \lambda[a, b] + b\cdot a\,, \tag{6}$$

which constitute the basic algebras of the N.C.J.A.

Indeed (5) the only power-associative, simple and trace-admissible algebras are: i) the C.J.A.; ii) the quasi-associative algebras; iii) the flexible algebras of degree 2 (4). Furthermore (SCHAFER (1955) (3)) every N.C.J.A. is power-associative and trace-admissible, and every flexible Jordan-admissible algebra is a N.C.J.A., while (MCCRIMMON (1965) (3)) every normed algebra is a separable N.C.J.A. Finally we note that (SCHAFER (1965) (3)) the radical R of a N.C.J.A. coincides with the radical of the Jordan algebra A^+, $A \ominus R$ is semi-simple and may be expressed as a direct sum of simple algebras.

The Jordan-admissibility concept has the following property (5): when A is power-associative and trace-admissible, then A is simple if and only if A^+ is simple. Consequently the imbedding (2) may be used for simple L.A. L

$$L \to A^- \to A \leftarrow A^+\,; \tag{7}$$

the preservation of the simplicity for a power-associative trace-admissible extension A is guaranteed by the simplicity of A^+, A and A^+ also possessing the same radical. Our choice of power-associative trace-admissible algebras, that is the algebras of quasi-associative type, concerns algebras which are simultaneously (nontrivial) Lie- and Jordan-admissible as the associative algebras. Indeed $ab - ab = (2\lambda - 1)\cdot(a\cdot b - b\cdot a)$ and $\frac{1}{2}(ab + ba) = \frac{1}{2}(a\cdot b + b\cdot a)$. Furthermore $A(1)$ is isomorphic to an associative algebra; $A(0)$ is antiisomorphic to an associative algebra and $A(\frac{1}{2})$ is isomorphic to a special C.J.A. However in the $A(\lambda)$ algebra there is no finite value of λ to reduce the product (6) to the commutator (6), which lessens the physical interest. In this connection we now investigate a generalization of the $A(\lambda)$ algebras.

4. – Let A be any algebra with product ab over a field F and λ, μ be free scalars belonging to F. We define the algebra $A(\lambda, \mu)$ to be the (λ, μ)-*mutation* of the original

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(3) The N.C.J.A. are nonassociative algebras neither anticommutative nor commutative defined by the relations: i) $(ab)a = a(ba)$, and ii) $(a^2b)a = a^2(ba)$. They were first defined by R. D. SCHAFER: *Proc. A.M.S.*, **6**, 472 (1955). See also: BRAUN and KOECHER (1); R. D. SCHAFER: *An Introduction to Nonassociative Algebras* (New York, 1966); *Proc. A.M.S.*, **9**, 110 (1958); *Trans. A.M.S.*, **94**, 310 (1960); L. A. KOKORIS: *Proc. A.M.S.*, **9**, 164 (1958); *Canad. Journ. Math.*, **12**, 448 (1960); L. J. PAIGE: *Port. Math.*, **16**, 15 (1957); R. H. OEHMKE: *Trans. A.M.S.*, **87**, 226 (1958); *Proc. A.M.S.*, **12**, 151 (1961); K. MCCRIMMON: *Pacific Journ. Math.*, **15**, 925 (1965); *Proc. A.M.S.*, **17**, 1455 (1966); *Trans. A.M.S.*, **121**, 187 (1966).

(4) The simple N.C.J.A. of characteristic zero (we consider only algebras and fields of characteristic zero) have been classified by SCHAFER (1955) (3) according to: i) the simple C.J.A.; ii) the simple quasi-associative algebras; iii) the simple flexible algebras of degree 2.

(5) A. A. ALBERT: *Proc. N.A.S.*, **35**, 317 (1949).

(6) However, for $\lambda \to \infty$, $ab \to [a, b]$. The author is indebted to Prof. A. SALAM for this remark.

algebra, that is the same vector space as A but with the product [7]

$$(a, b) = \lambda ab + \mu ba = \varrho[a, b] + \sigma\{a, b\}, \qquad \begin{cases} \lambda = \sigma + \varrho, \\ \mu = \sigma - \varrho. \end{cases} \tag{8}$$

We see clearly that: i) $A(1, 0)$ is isomorphic to A; ii) $A(0, 1)$ is antiisomorphic to A; iii) $A(1, -1)$ is isomorphic to A^-; iv) $A(\frac{1}{2}, \frac{1}{2})$ is isomorphic to A^+; v) $A(\lambda, 1-\lambda)$ is isomorphic to the λ-mutations of A.

Theorem 1. $A(\lambda, \mu)$ is power-associative for every $\lambda \neq -\mu$ if and only if A is power-associative and for $\lambda = -\mu$ it is trivially power-associative. *Proof*: the identities $[a, a, a] = 0$ and $[a, a, a^2] = 0$ are sufficient to guarantee the power-associativity of an algebra (for fields of characteristic zero as in our case). The power-associativity of $A(\lambda, \mu)$ is then easily reduced to the validity of the above relations for A.

Let us also note that the algebras $A(\lambda, \mu)$ satisfy the relation

$$(a, a) = \gamma aa \qquad (\gamma = 2\sigma = \lambda + \mu), \tag{9}$$

namely powers in $A(\lambda, \mu)$ and A do not coincide for $\gamma \neq 1$. This is the first essential difference between the (λ, μ)- and λ-mutations of an algebra. If $\gamma = 1$ then the (λ, μ)- and λ-mutations are equivalent. Indeed

$$(a, b) = \varrho[a, b] + \tfrac{1}{2}\{a, b\} = (\varrho + \tfrac{1}{2})\, ab + (\tfrac{1}{2} - \varrho)\, ba = \lambda ab + (1 - \lambda)\, ba$$

for $\lambda = \varrho + \frac{1}{2}$.

Theorem 2. $A(\lambda, \mu)$ is flexible for every $\lambda, \mu \in F$ if and only if A is flexible *Proof*: we have

$$((a, b), a) = \lambda^2 (ab)\, a + \lambda\mu (ba)\, a + \lambda\mu a(ab) + \mu^2 a(ba)$$

and

$$(a, (b, a)) = \lambda^2 a(ba) + \lambda\mu a(ab) + \lambda\mu (ba)\, a + \lambda\mu (ba)\, a + \mu^2 (ab)\, a\,.$$

Hence, if $(ab)\, a = a(ba)$, then $((a, b), a) = (a, (a, a))$.

Theorem 3. $A(\lambda, \mu)$ is Lie-admissible for every $\lambda \neq \mu$ if and only if A is Lie-admissible. *Proof*: A^- and $[A(\lambda, \mu)]^-$ are defined by the respective products $ab - ba$ and $(a, b) - (b, a) = (\lambda - \mu)(ab - ba)$. Hence for $\lambda \neq \mu$ $[A(\lambda, \mu)]^-$ is isomorphic to the isotopic algebra A^{*-} [8] with product $a * b - b * a = (\lambda - \mu)\, ab - (\lambda - \mu)\, ba$.

For $\lambda = \mu$, $A(\lambda, \mu)$ is trivially Lie-admissible.

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(7) For the case with A=associative algebra see also: R. M. SANTILLI and G. SOLIANI: *A statistics and parastatistics formal unification*, to appear.

(8) Given an algebra A with product ab and an invertible element c we can form an algebra A^*, called the *isotope* of A, with the product $a * b = acb$. As a particular case we may have $c = \alpha 1$, where α is a free (nonzero) scalar. Then the new multiplication in A^* is simply α times the old multiplication in A: $a * b = \alpha ab$.

Theorem 4. $A(\lambda, \mu)$ is Jordan-admissible for every $\lambda \neq -\mu$ if and only if A is Jordan-admissible. *Proof*: A^+ and $[A(\lambda, \mu)]^+$ are characterized by the respective products $\frac{1}{2}(ab + ba)$ and $\frac{1}{2}[(a, b) + (b, a)] = \frac{1}{2}(\lambda + \mu)(ab + ba)$. Hence $[A(\lambda, \mu)]^+$ is isomorphic to the isotopic algebra A^{*+} with product $\frac{1}{2}(a*b + b*a) = \frac{1}{2}(\lambda + \mu)(ab + ba)$.

For $\lambda = -\mu$, $A(\lambda, \mu)$ is trivially Jordan-admissible.

Theorem 5. If $U = A(\lambda, \mu)$, then, for $\lambda \neq \pm\mu$, $A = U(\alpha, \beta)$, where $\alpha = \lambda/(\lambda^2 - \mu^2)$ and $\beta = \mu/(\mu^2 - \lambda^2)$. *Proof*: since

$$(a, b) - (b, a) = (\lambda - \mu)(ab - ba) \qquad \text{and} \qquad (a, b) + (b, a) = (\lambda + \mu)(ab + ba)\,,$$

we have

$$ab = \frac{\lambda}{\lambda^2 - \mu^2}(a, b) + \frac{\mu}{\mu^2 - \lambda^2}(b, a)\,. \tag{10}$$

Theorem 5 has the following consequences: as for the $A(\lambda)$ algebra [2], if R is a two-sided ideal of A, that is ba and $ab \in R$ for every $b \in R$ and $a \in A$, then (a, b) and $(b, a) \in R$. If $A = B \oplus R$, then $A(\lambda, \mu) = B(\lambda, \mu) \oplus R(\lambda, \mu)$. $R(\lambda, \mu)$ is solvable (nilpotent) if R is solvable (nilpotent), and the maximal solvable ideal of $A(\lambda, \mu)$ coincides with that of A. Hence $A(\lambda, \mu)$ is simple if A is simple, and $A(\lambda, \mu)$ can be given as a direct sum of simple algebras when the radical is zero, if the same occurs for A.

The possible interest of the (λ, μ)-mutations for physical applications is essentially given by the mutations of associative algebras A. In this case it is easy to show that $A(\lambda, \mu)$ is a realization of the N.C.J.A., since it is flexible and Jordan-admissible. Furthermore $A(\lambda, \mu)$ is an algebra of quasi-associative type (for $\lambda \neq -\mu$), indeed the associators in $A(\lambda, \mu)$ and $[A(\lambda, \mu)]^+$ are connected by the relation $[a, b, c] = (1 - \delta)[a, b, c]^+$, where $\delta = ((\lambda - \mu)/(\lambda + \mu))^2$ is the *discriminant* of the algebra (McCrimmon (1966) [3]) [9].

The (λ, μ)-mutations of an associative algebra satisfy the following essential relations:

$$\left\{\begin{array}{ll} \text{i)} & (ab)a = a(ba)\,, \\ \text{ii)} & (a^2 b)a = a^2(ba)\,, \\ \text{iii)} & a^2 = \gamma a \cdot a \qquad (\gamma \in F)\,, \\ \text{iv)} & [a, b, c] + [b, c, a] + [c, a, b] = 0\,, \end{array}\right. \tag{11}$$

where i) and ii) are the fundamental relations of the N.C.J.A., iii) connects powers in the associative algebra and powers in the corresponding mutation, and iv) represents the Lie-admissibility condition for flexible algebras.

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(9) The author is indebted to Prof. K. McCrimmon for a very kind letter (of June 12, 1967), where the connections between the $A(\lambda, \mu)$ and $A(\lambda)$ algebras are explicitly investigated. McCrimmon notes that for $\varrho, \sigma \neq 0$, by putting $\tau = 2\sigma$, $\lambda' = \lambda/\tau$ and $\mu' = \mu/\tau$, we have $\lambda' + \mu' = 1$. Then $(a, b) = \lambda'\tau a \cdot b + \mu'\tau b \cdot a = \lambda' a^* \cdot b + (1 - \lambda') b^* \cdot a$. Hence $A(\lambda, \mu)$ is just the $(\lambda/2\sigma)$-mutation of the isotopic algebra A^*, *i.e.* $A(\lambda, \mu)$ is isomorphic to $A^*(\lambda/2\sigma)$. In addition we note that, since $\sigma = \lambda + \mu$, the isotopic algebra A^* characterized by the product $a^* \cdot b = (\lambda + \mu) a \cdot b$ is the zero algebra for $\lambda = -\mu$. Furthermore for $\lambda = -\mu$ $A(\lambda, \mu)$ corresponds to the ∞-mutation of the A^* (zero) algebra (see also footnote (6)). Hence the $A(\lambda, \mu)$ and $A(\lambda)$ algebras are equivalent for every $\lambda \neq -\mu$, while the case $\lambda = -\mu$ corresponds to the explicit Lie content of the $A(\lambda, \mu)$ algebras which occurs when the discriminant has the degenerate value ∞.

Theorem 6. If $A(\lambda, \mu)$ is an algebra of quasi-associative type, then $A(\lambda, \mu)$ is solvable if it is a nilalgebra and it is strongly nilpotent if it is solvable; the radical $R(\lambda, \mu)$ is the maximal solvable ideal such that $A(\lambda, \mu) \ominus R(\lambda, \mu)$ is semi-simple and has no nonzero nilideals; every semi-simple $A(\lambda, \mu)$ algebra can be given as a direct sum of simple algebras. *Proof*: The above statements hold for λ-mutations of an associative algebra [2]. Hence they also hold for $[\lambda/(\lambda+\mu)]$-mutations of the isotopic algebras A^* with the product $a^* \cdot b = (\lambda+\mu) a \cdot b$.

For the case $\lambda = -\mu$ we prove the following

Theorem 7. If $\gamma = 0$, then relations (11) define a Lie algebra. *Proof*: If $\gamma = 0$, the condition $a^2 = 0$, which is the first relation for L.A., implies that the product is anticommutative, *i.e.* $ab = -ba$. Then the Lie-admissibility condition becomes the Jacoby identity, since $[a, b, c] + [b, c, a] + [c, a, b] = 2[(ab)c + (bc)a + (ca)b] = 0$. Furthermore flexibility becomes inessential since all Lie algebras are flexible, and condition ii) is trivially satisfied since $a^2 = 0$.

We conclude by noting that starting from a given Lie algebra which is the $(1, -1)$-mutation of an associative algebra, it is possible to perform an imbedding according to (7) by taking as extension the (λ, μ)-mutation of the original associative algebra, while the two-sided ideal, the derivations, the automorphisms and many other characteristics remain unchanged. In addition the given mathematical tool presents two free quantities belonging to the field which may have some physical interest. Particularly the (λ, μ)-product may be used: i) in the general form $(a, b) = \varrho[a, b] + \sigma\{a, b\}$, *i.e.* with two free scalars; ii) in the reduced form $(a, b) = \cos\alpha[a, b] + \sin\alpha\{a, b\}$, *i.e.* with the supplementary condition $\varrho^2 + \sigma^2 = 1$; iii) in the contracted form $(a, b) = [a, b] + \sigma\{a, b\}$, *i.e.* $\varrho = 1$ and only the scalar σ is free for « perturbation » of the Lie content.

Further investigations on the $A(\lambda, \mu)$ algebras particularly for what concerns the explicit construction of the basis, the classification of the matrix representations and the Pierce decomposition are in progress.

5. – Let us now discuss the possibilities of physical applications. At *a classical level* for nonconservative systems [10] there is already a physical application of imbedding (2) given by pseudo-Hamiltonian mechanics, introduced by DUFFIN, where the Poisson bracket may be imbedded in the more general Lie-admissible form [11] $(a, b)_G = \sum_{i=1}^{n} \big(\lambda(\partial a/\partial q_i)(\partial b/\partial p_i) + \mu(\partial a/\partial p_i)(\partial b/\partial q_i)\big)$.

At *a quantum-mechanical level* for elementary-particle interacting or decaying regions [12] there are many problems to be investigated in order to evaluate the possibilities of application of the given procedure. Among these problems one of the most crucial is the possibility that imbedding implies a change of the statistical Bose or Fermi character of the particles in interaction or decay [11,13].

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(10) Clearly for conservative systems the Hamiltonian mechanics and Lie algebra are completely satisfactory.

(11) R. M. SANTILLI: *Some remarks on pseudo-Hamiltonian mechanics*, to appear.

(12) Of course for free states Lie algebras are completely satisfactory. Their validity however becomes problematic for the same particles is interactions of for problems like the relativistic extension of the internal symmetries where infinite parameter Lie algebras occurs. See for instance: J. FORMÁNEK: *Czech. Journ. Phys.*, B 16, 1, 281 (1966).

(13) For instance, in the decay $\pi \to \mu + \nu$ there in the transition from bosons to fermions, which leaves open the problem of characterization of the decaying region from a statistical viewpoint [7].

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In this connection the following approaches are being attempted:

i) $\varrho \to 1$, $\sigma \to 0$ (or the « angle » $\alpha \to 0$). In this case it is possible to investigate the connection between the given procedure and the approximation methods or the « neighbouring algebras » by SEGAL [14].

ii) $\varrho, \sigma =$ *fixed quantities* (or $\alpha =$ fixed « angle »). In this case, by recalling that the fundamental representations of the SU_n Lie algebras are closed under both commutators and anticommutators (hence they are closed also for the $A(\lambda, \mu)$ algebra), it may be interesting to investigate the imbedding of the SU_3 (or SU_6) model in order to see if the given procedure may give a contribution to the problem of the (non) observability of the quarks and a more exact mathematical characterization of the physical numbers. Clearly it may be interesting also to investigate the imbedding of the equal-time commutation relations, current algebra and sum rules.

iii) $\varrho, \sigma =$ *variable quantities* (or $\alpha =$ variable « angle »). In this case it is possible, for instance, to consider a physical region with σ everywhere zero (*i.e.* we have L.A. everywhere) and only a well-defined (and limited) region of validity of the imbedding with $\sigma \neq 0$ (or $\alpha \neq 0$). In this last case the physical acceptance of the procedure might be allowed by the indetermination principle.

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(14) I. E. SEGAL: *Duke Math. Journ.*, 18, 221 (1951).

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