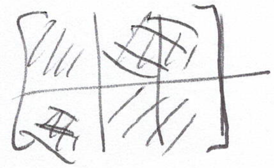
$$e = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad M_{2,3} =$$ 

$$M_{2,3}(\mathbb{C}) = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{matrix} 0 & a & b \\ c & 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 0 \\ 0 & d & e \end{matrix}$$

$$a = a_2 + a_1 + a_0$$

$$C = C_2(e) \quad C_1(e) \quad C_0(e)$$

$$a = \underbrace{\underbrace{\overset{2\times2\ \ 2\times3\ \ 3\times3}{ee^* a e^* e}}_{C_2(e)a}}_{a_2} + \overbrace{\overbrace{ee^* a(1-e^*e) + (1-ee^*)ae^*e}^{C_1(e)a}}^{a_1}$$

$$+ \underbrace{\underbrace{(1-ee^*)a(1-e^*e)}_{C_0(e)}}_{a_0}$$

$$e^* = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \qquad ee^* = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} = \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}$$

$$e^*e = \begin{matrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$a_2 = \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} = \begin{matrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$a_0 = \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} = \begin{matrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \end{matrix}$$

$$a_1 = \begin{matrix}1&0\\0&0\end{matrix} \quad \begin{matrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{matrix} \quad \begin{matrix}0&0&0\\0&1&0\\0&0&1\end{matrix} + \begin{matrix}0&0\\0&1\end{matrix} \quad \begin{matrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{matrix} \quad \begin{matrix}1&0&0\\0&0&0\\0&0&0\end{matrix}$$

$$= \begin{bmatrix}a_{11}&a_{12}&a_{13}\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} + \begin{bmatrix}0&0&0\\a_{21}&a_{22}&a_{23}\end{bmatrix}\begin{matrix}1&0&0\\0&0&0\\0&0&0\end{matrix}$$

$$\begin{bmatrix}0&a_{12}&a_{13}\\0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0\\a_{21}&0&0\end{bmatrix}$$

$$= \begin{bmatrix}0&a_{12}&a_{13}\\a_{21}&0&0\end{bmatrix}$$

$ee^*a\,e^*e) = a_{11}e$ as required for <u>minimal</u>.

$$ee^*e = \underbrace{\begin{matrix}1&0\\0&0\end{matrix}}_{ee^*} \ \underbrace{\begin{matrix}1&0&0\\0&0&0\end{matrix}}_{e} = \begin{matrix}1&0&0\\0&0&0\end{matrix} = e$$

Let $u \in M_{m,n}(\mathbb{C})$ $\qquad uu^*u = u \qquad$ $\overset{\text{minimal}}{ua^*u \in \mathbb{C}u}$

$$\Rightarrow \quad M_{m,n}(\mathbb{C}) = C_2(u) + C_1(u) + C_0(u)$$

u	$C_1(u)$
$C_1(u)$	$C_0(u)$

$$u = \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \qquad u^* = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

$$uu^* \cancel{u} = \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

u not minimal

$$uu^*u = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} = \begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix} = u$$

$$M_{3,2}(\mathbb{C}) = \begin{matrix} C_2(u) & + & C_1(u) & + & C_0(u) \\ \parallel & & \parallel & & \parallel \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 0 & 0 \end{bmatrix} & & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a_{31} & a_{32} \end{bmatrix} & & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{matrix}$$

C Cartan factor $u \in C$ $uu^*u = u$

$$\Rightarrow \quad C = \underbrace{C_2(u) + C_1(u) + C_0(u)}_{\text{subtriples}}$$

$$uu^* C u^*u + (1-uu^*) C u^*u + uu^* C(1-u^*u) + (1-uu^*) C(1-u^*u)$$

$$(uu^*au^*u)(uu^*bu^*u)^*(uu^*cu^*u)$$

$$= uu^*au^*u\, u^*ub^*uu^*\, uu^*cu^*u$$

$$= uu^*\underbrace{au^*u\, b^*uu^*c}_{\in C}u^*u$$

$ub^*u \in C$

$au^*(ub^*u) \in C$

$au^*(ub^*u)(uc^*u)^*u$

$au^*\underbrace{(ub^*u)}_{\in C}u^*c$

$\in C$

$\in C$

$\in C$

$\therefore$ $C_2(u)$ is a subtriple

$C_0(u)$ is a subtriple

$$(1-uu^*)x(1-u^*u)\big((1-uu^*)y(1-u^*u)\big)^*\big((1-uu^*)z(1-u^*u)\big)$$
$$\in$$
$$(1-uu^*)C(1-u^*u)$$

$C_1(u)$ is a subtriple

$$\begin{pmatrix}0&0\\0&0\\a&b\end{pmatrix}\begin{pmatrix}0&0&\bar a\\0&0&\bar b\end{pmatrix}\begin{pmatrix}0&0\\0&0\\a&b\end{pmatrix} = \begin{bmatrix}0&0\\0&0\\a&b\end{bmatrix}\begin{bmatrix}|a|^2&\bar a b\\ \bar b a&|b|^2\end{bmatrix} = \begin{bmatrix}0&0\\0&0\\a|a|^2+|b|^2a & |a|^2b+b|b|^2\end{bmatrix}$$