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Right and left solvable extensions of an associative Leibniz algebra

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ABSTRACT
A sequence of nilpotent Leibniz algebras denoted by $N_{n,18}$ is introduced. Here n denotes the dimension of the algebra defined for $n \geq 4$; the first term in the sequence is $\mathscr{R}_{18}$ in the list of four-dimensional nilpotent Leibniz algebras introduced by Albeverio et al. [4]. Then all possible right and left solvable indecomposable extensions over the field $\mathbb{R}$ are constructed so that $N_{n,18}$ serves as the nilradical of the corresponding solvable algebras. The construction continues Winternitz' and colleagues' program established to classify solvable Lie algebras using special properties rather than trying to extend one dimension at a time.

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1. Introduction

Leibniz algebras were discovered by Bloch in 1965 [21] who called them $D-$ algebras. Later on they were considered by Loday and Cuvier [43, 57–59] as a non-antisymmetric analogue of Lie algebras. It makes every Lie algebra be a Leibniz algebra, but the converse is not true. Exactly Loday named them Leibniz algebras after Gottfried Wilhelm Leibniz.

Since then many researchers are working on them and as one of the results they found analogs of important theorems in Lie theory such as the analogue of Levi's theorem which was proved by Barnes [15]. He showed that any finite-dimensional complex Leibniz algebra is decomposed into a semidirect sum of the solvable radical and a semisimple Lie algebra. Therefore the biggest challenge in the classification problem of finite-dimensional complex Leibniz algebras is to study the solvable part. And to classify solvable Leibniz algebras, we need nilpotent Leibniz algebras used as the nilradicals same as in the case of Lie algebras [61].

There are also other important analogous theorems such as Engel's theorem has been generalized by several researchers like Ayupov and Omirov [8] in a stronger form by Patsourakos [68], and finally Barnes [14] gave a simple proof of this theorem. Many other authors work on proving analogous properties adjusted to Leibniz algebras and their work could be found in the citations [2, 5, 12, 16, 18, 19, 22, 39, 44, 46, 51, 60, 64, 65].

Every Leibniz algebra satisfies a generalized version of the Jacobi identity called the Leibniz identity. There are two Leibniz identities: the left and right Leibniz identity. We call Leibniz algebras right Leibniz algebras if they satisfy the right Leibniz identity, and left if they satisfy the left Leibniz identity. A left Leibniz algebra is not necessarily a right Leibniz algebra [44]. In this paper we construct both right and left solvable Leibniz algebras.[1]

Leibniz algebras inherit an important property of Lie algebras which is that the right (left) multiplication operator of a right (left) Leibniz algebra is a derivation [40]. Besides the algebra of right (left) multiplication operators is endowed with a structure of a Lie algebra by means of the commutator [40].

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[1] When we say a Leibniz algebra, we mean a right Leibniz algebra, unless the choice is understood from the context.

Another important property is that the quotient algebra by the two-sided ideal generated by the square elements of a Leibniz algebra is a Lie algebra [67], where such ideal is the minimal, abelian and in the case of non-Lie Leibniz algebras it is always nontrivial.

The classification of Leibniz algebras started in 1993 by Loday [58]. In that paper he gave in particular the examples of Leibniz algebras in dimensions one and two over the field that does not have any zero divisors, such that the Leibniz algebra in dimension one is abelian and coincides with a Lie algebra. In dimension two there are two non Lie Leibniz algebras: one is nilpotent (null-filiform) and one is solvable.

Leibniz algebras in dimension three over the complex numbers were classified by Omirov and Ayupov in 1998 [8, 9] and was reviewed in 2012 by Casas et al. [38], where they noticed that one isomorphism class does not have a Leibniz algebra structure. Therefore there are four nilpotent (one is null-filiform, three are filiform) non Lie Leibniz algebras and seven solvable, such that one nilpotent and two solvable algebras contain a parameter.

Two-dimensional and three-dimensional left Leibniz algebras were classified by Demir et al. [44] over the field of characteristic 0. In dimension two they found two non Lie Leibniz algebras: one is nilpotent, same as the right Leibniz algebra [58], and one is a solvable left Leibniz algebra. In dimension three there are five nilpotent Leibniz algebras (one of them depends on a parameter) isomorphic to the right Leibniz algebras [38]. Also there are seven solvable non Lie left Leibniz algebras, where two of them are one parametric families.

Four-dimensional nilpotent complex Leibniz algebras, one of them is $\mathscr{R}_{18}$, which is the first term in the sequence of the nilpotent Leibniz algebras $N_{n,18}$, $(n \geq 4)$, were classified by Albeverio et al. [4]. Altogether they found 21 nilpotent complex non Lie Leibniz algebras, where the first 10 of them are either null-filiform or filiform [10] and remaining 11 are associative algebras. In one of them the parameter just takes two values 0 or 1, so we think that such algebra does not contain a parameter, therefore there are three nilpotent one parametric families among the associative algebras in [4].

Four-dimensional solvable Leibniz algebras in dimension four over the field of complex numbers were classified by Cañete and Khudoyberdiyev [37]. They found 38 non Lie Leibniz algebras, where 16 of them contain up to two parameters.

Besides in dimension four Omirov et al. [67] proved that there only exists one non solvable 4-dimensional Leibniz algebra $\mathfrak{sl}_2 \oplus \mathbb{C} = \{e, f, h, x\}$ defined as follows

$$\mathfrak{sl}_2 \oplus \mathbb{C} : [e, h] = -[h, e] = 2e,\ [h, f] = -[f, h] = 2f,\ [e, f] = -[f, e] = h,$$

which is a decomposable Lie algebra at the same time.

In dimension five Khudoyberdiyev et al. [56] studied solvable Leibniz algebras with 3-dimensional nilradicals. They showed that a five-dimensional solvable Leibniz algebra with a three-dimensional Heisenberg nilradical is a Lie algebra. Also they found 22 non isomorphic classes of solvable Leibniz algebras including one with the Heisenberg nilradical, where 12 of them are parametric families depending on up to four parameters.

Few words about the classification of semisimple and simple Leibniz algebras. The definition of simple Leibniz algebras was suggested by Dzhumadil'daev and Abdykassymova [45]. According to them L is called a simple Leibniz algebra if all two-sided ideals of L are 0, L^{ann} and L. Semisimple Leibniz algebras were defined by Demir et al. [44] as well as an analogue of the Killing form for Leibniz algebras. They also showed if the Leibniz algebra is semisimple then the Killing form is nondegenerate, but the converse is not true. Further work on semisimple and simple complex Leibniz algebras with the classification of some particular cases was performed by Camacho et al. [35], Gómez-Vidal et al. [51], Omirov et al. [67], Rakhimov et al. [74].

In other dimensions (starting from dimension 5) the classification of nilpotent Leibniz algebras has attempts to move to any finite dimension over the field of characteristic zero, where there are the following directions of the research based on the classes of nilpotent algebras: Ayupov and Omirov [10] proved that in every finite dimension n there is the only null-filiform non Lie Leibniz algebra defined as

$$[x_i, x_1] = x_{i+1},\ (1 \leq i \leq n-1).$$

Two kinds of filiform Leibniz algebras were classified in any finite dimension such as naturally graded filiform Leibniz algebras studied by Vergne [90] and Ayupov and Omirov [10] and non characteristically nilpotent filiform Leibniz algebras, which were classified by Ayupov and Omirov [10] and Khudoyberdiyev and Ladra [54]. The filiform Leibniz algebras whose natural gradation is a filiform Lie algebra were considered by Omirov and Rakhimov [66]. This class is denoted by $TLeib_{n+1}$. They presented an effective algorithm to control the behavior of the structure constants under adapted transformations of basis in the arbitrary fixed dimension. Also in one particular case, they gave the formulas in dimensions less than 10. Rakhimov and Hassan [71] considered a subtype of $TLeib_{n+1}$ denoted by $Ced(\mu_n)$ and provided the isomorphism criteria for specific 5 and 6-dimensional cases. Their other work on the classification of 5 and 6-dimensional filiform algebras from the family $TLeib_{n+1}$ could be found in [72]. The filiform Leibniz algebras which are one-dimensional central extensions of a filiform Lie algebras up to dimension nine were classified by Rakhimov and Hassan [73]. If the natural gradation is a non-Lie filiform Leibniz algebra, then such case was studied by Ayupov [10] and Omirov and Gómez [50], where Omirov and Gómez [50] noticed that because of the type of their natural gradation such filiform Leibniz algebras belong to two classes either $FLeib_{n+1}$ or $SLeib_{n+1}$. Rakhimov and Bekbaev [70] introduced a method to classify such Leibniz algebras in any finite fixed dimension based on algebraic invariants and gave the application of the developed method from dimension 5 up to dimension 9. Rakhimov and Said Husain [75, 76] classified up to isomorphism the subclass $FLeib_{n+1}$ for $n = 4, 5, 6$ and dealt with the classification in low dimensions of the remaining subclass $SLeib_{n+1}$. Altogether filiform Leibniz algebras were only classified up to dimension less than 10 with some other work not mentioned above given in the citations [1, 3, 25, 49, 63, 69, 77–79]. Naturally graded quasi-filiform Leibniz algebras were studied by Camacho et al. [31]. The study of naturally graded Leibniz algebras of the certain nilindex and different characteristic sequences could be found in the following references [24, 29]. There is an ongoing study presented by Camacho et al. [26, 28, 32–34] on p-filiform and naturally graded p-filiform Leibniz algebras [25].

The subject of this paper is to add to the program of classifying solvable Leibniz algebras over the field of real numbers in any finite dimension applying Winternitz and his colleagues Snobl, Rubin, Karasek, Tremblay, Ndogmo [62, 80, 85–89] approach, which was established to classify solvable Lie algebras, whereby they start with a particular nilpotent Lie algebra to find all its possible solvable extensions. Some other work on solvable Lie algebras following the same approach could be found in the following references [82, 84, 91]. Moreover, that approach is based on what was shown by Mubarakzyanov in [61]: the dimension of the complimentary vector space to the nilradical does not exceed the number of nil-independent derivations of the nilradical. That result was extended to Leibniz algebras by Casas, Ladra, Omirov and Karimjanov in [41] with the help of the paper [8], where they classified solvable Leibniz algebras with null-filiform nilradical. Similarly to Lie algebras, the inequality $\dim \mathfrak{nil}(L) \geq \frac{1}{2} \dim L$ [61] is also true for solvable Leibniz algebra L.

There is already a lot of work performed on such approach to classify solvable Leibniz algebras over the field of a characteristic zero: Omirov and his colleagues Casas, Khudoyberdiyev, Ladra, Karimjanov, Camacho and Masutova classified solvable Leibniz algebras whose nilradicals are a direct sum of null-filiform algebras [55], naturally graded filiform [40], triangular [53] and finally filiform [36]. Bosko-Dunbar, Dunbar, Hird and Stagg attempted to classify left solvable Leibniz algebras with Heisenberg nilradical [23].

Some other work and kinds of Leibniz algebras not discussed are shown in the citations [6, 7, 11, 13, 17, 20, 27, 30, 42, 47, 48, 52, 92].

The starting point of the present article is the four-dimensional nilpotent non Lie right Leibniz algebras [4] introduced by Albeverio et al. [4] of which there are 21 such algebras over the field of complex numbers. As we have said, the first 10 of them are either null-filiform or filiform [10] and remaining 11 are associative algebras, and there are three nilpotent one parametric families among the associative algebras. Our choice is to work with one associative Leibniz algebra from the remaining 11 $\mathscr{R}_{18}$. An interesting observation is that it is left and right Leibniz algebra at the same time, which is true for all those associative Leibniz algebras. Also it does not depend on the parameter and over the field of real numbers it remains the same. Other nilpotent Leibniz algebras in dimensions two, three and four are either null-filiform or filiform and solvable extensions of such algebras were studied by Omirov and his colleagues [36, 40, 41].

We create a sequence of nilpotent Leibniz algebras $N_{n,18}$, $(n \geq 4)$, where the first term in the sequence is the four-dimensional algebra $N_{4,18}$, which is exactly $\mathscr{R}_{18}$ using the notation of the paper [4].

For a sequence $N_{n,18}$ we find all possible solvable indecomposable extensions as left and right Leibniz algebras in every finite dimension. Such extensions of dimensions one and, in one particular case two, are possible. Right solvable extensions with a codimension one nilradical $N_{n,18}$ are found following the steps in Theorems 5.1, 5.2 and 5.3 with the main result summarized in Theorem 5.3, where it is shown there are four such solvable algebras. The major result for the right solvable extensions with a codimension two nilradical $N_{n,18}$ is stated in Theorem 5.4, where it is proved there is only one such Leibniz algebra. We follow the steps in Theorems 6.1, 6.2 and 6.3 to find one-dimensional left solvable extensions. We notice in Theorem 6.3 that there is the same number of them, but one algebra is left and right at the same time. We find the only two-dimensional solvable extension as well stated in Theorem 6.4.

As regards notation, we use $\langle e_1, e_2, \ldots, e_r \rangle$ to denote the r-dimensional subspace generated by $e_1, e_2, \ldots, e_r$, where $r \in \mathbb{N}$, LS is the lower central series and DS is the derived series and refer to them collectively as the characteristic series. Besides $\mathfrak{g}$ is used to denote solvable right Leibniz algebras and l is solvable left Leibniz algebras.

Throughout the paper all the algebras are finite dimensional over the field of real numbers and if the bracket is not given, then it is assumed to be zero, except the brackets for the nilradical, which are not given at all (see Remark 5.1) to save space.

We use Maple software to compute the Leibniz identity, the "absorption" (see [81, 83] and Section 3.2), the change of basis for solvable Leibniz algebras in some particular dimensions, which are generalized and proved in an arbitrary finite dimension.

2. Preliminaries

In this section we give some basic definitions encountered working with Leibniz algebras.

Definition 2.1.

(1) A vector space L over a field F with a bilinear operation $[-,-] : L \to L$ is called a Leibniz algebra if for any $x, y, z \in L$ the so called Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]]$$

holds. This Leibniz identity is known as the right Leibniz identity and we call L in this case a right Leibniz algebra.

(2) There exists the version corresponding to the left Leibniz identity

$$[z,[x,y]] = [[z,x],y] + [x,[z,y]],$$

and a Leibniz algebra L is called a left Leibniz algebra.

Remark 2.1. In addition, if L satisfies $[x,x] = 0$ for every $x \in L$, then it is a Lie algebra. Therefore every Lie algebra is a Leibniz algebra, but the converse is not true.

Definition 2.2. The two-sided ideal $C(L) = \{x \in L : [x,y] = [y,x] = 0\}$ is said to be the center of L.

Definition 2.3. A linear map $d : L \to L$ of a Leibniz algebra L is a derivation if for all $x, y \in L$

$$d([x,y]) = [d(x),y] + [x,d(y)].$$

In terms of this definition for $x \in L$, where L is a right Leibniz algebra, the right multiplication operator $\mathcal{R}_x : L \to L$ defined as $\mathcal{R}_x(y) = [y, x]$, $y \in L$ is a derivation (for a left Leibniz algebra L, the left multiplication operator $\mathcal{L}_x : L \to L$, $\mathcal{L}_x(y) = [x, y]$, $y \in L$ is a derivation).

Any right Leibniz algebra L is associated with the algebra of right multiplications $\mathcal{R}(L) = \{\mathcal{R}_x \mid x \in L\}$ endowed with the structure of a Lie algebra by means of the commutator $[\mathcal{R}_x, \mathcal{R}_y] = \mathcal{R}_x\mathcal{R}_y - \mathcal{R}_y\mathcal{R}_x = \mathcal{R}_{[y,x]}$, which defines an antihomomorphism between L and $\mathcal{R}(L)$.

For a left Leibniz algebra L, the corresponding algebra of left multiplications $\mathcal{L}(L) = \{\mathcal{L}_x \mid x \in L\}$ is endowed with the structure of a Lie algebra by means of the commutator as well $[\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_x\mathcal{L}_y - \mathcal{L}_y\mathcal{L}_x = \mathcal{L}_{[x,y]}$. In this case we have a homomorphism between L and $\mathcal{L}(L)$.

Definition 2.4. For a given Leibniz algebra L, we define the sequence of two-sided ideals as follows:

$$L^0 = L,\ L^{k+1} = [L^k, L],\ (k \geq 0) \qquad\qquad L^{(0)} = L,\ L^{(k+1)} = [L^{(k)}, L^{(k)}],\ (k \geq 0),$$

are said to be the lower central series and the derived series of L, respectively.

A Leibniz algebra L is said to be nilpotent (solvable) if there exists $n \in \mathbb{N}$ such that $L^n = 0$ ($L^{(n)} = 0$). The minimal such number n is said to be the index of nilpotency (solvability).

It follows that if L is a Leibniz algebra with $L^2 = 0$, then it is associative.

3. Constructing solvable Leibniz algebras with a given nilradical

Every solvable Leibniz algebra L contains a unique maximal nilpotent ideal called the nilradical and denoted $\mathfrak{nil}(L)$ such that $\dim \mathfrak{nil}(L) \geq \frac{1}{2}\dim(L)$ [61]. Let us consider the problem of constructing solvable Leibniz algebras L with a given nilradical $N = \mathfrak{nil}(L)$. Suppose $\{e_1, e_2, e_3, \ldots, e_n\}$ is a basis for the nilradical and $\{e_{n+1}, \ldots, e_p\}$ is a basis for a subspace complementary to the nilradical.

If L is a solvable Leibniz algebra [8], then

$$[L, L] \subseteq N \tag{3.1}$$

and we have the following structure equations

$$[e_i, e_j] = C_{ij}^k e_k, [e_a, e_i] = A_{ai}^k e_k, [e_i, e_a] = A_{ia}^k e_k, [e_a, e_b] = B_{ab}^k e_k, \tag{3.2}$$

where $1 \leq i, j, k, m \leq n$ and $n + 1 \leq a, b \leq p$.

3.1. Solvable right Leibniz algebras

Calculation shows that the right Leibniz identity is equivalent to the following conditions:

$$A_{ai}^k C_{kj}^m = A_{aj}^k C_{ki}^m + C_{ij}^k A_{ak}^m,\ A_{ia}^k C_{kj}^m = C_{ij}^k A_{ka}^m + A_{aj}^k C_{ik}^m,\ C_{ij}^k A_{ka}^m = A_{ia}^k C_{kj}^m + A_{ja}^k C_{ik}^m, \tag{3.3}$$

$$B_{ab}^k C_{ki}^m = A_{ai}^k A_{kb}^m + A_{bi}^k A_{ak}^m,\ A_{ai}^k A_{kb}^m = B_{ab}^k C_{ki}^m + A_{ib}^k A_{ak}^m,\ B_{ab}^k C_{ik}^m = A_{kb}^m A_{ia}^k - A_{ka}^m A_{ib}^k. \tag{3.4}$$

Then the entries of the matrices A_a, which are $(A_i^k)_a$, must satisfy the equations (3.3) which come from all the possible Leibniz identities between the triples $\{e_a, e_i, e_j\}$. In (3.4) we have relations on structure constants obtained from all Leibniz identities between the triples $\{e_a, e_b, e_i\}$.

Since N is the nilradical of L, no nontrivial linear combination of the matrices A_a, $(n + 1 \leq a \leq p)$ is nilpotent which means that the matrices A_a must be "nil-independent", that is, no non trivial combination of them is nilpotent [41, 61].

Let us now consider the right multiplication operator $\mathcal{R}_{e_a}$ and restrict it to N, $(n+1 \leq a \leq p)$. Notice we shall get outer derivations of the nilradical $N = \mathfrak{nil}(L)$ [41]. Then finding the matrices A_a is the same as finding outer derivations $\mathcal{R}_{e_a}$ of N. Further the commutators $[\mathcal{R}_{e_b}, \mathcal{R}_{e_a}] = \mathcal{R}_{[e_a,e_b]}$, $(n+1 \leq a, b \leq p)$ due to (3.1) consist of inner derivations of N. So those commutators give the structure constants B_{ab}^k as shown in the last equation of (3.4) but only up to the elements in the center of the nilradical N, because if e_i, $(1 \leq i \leq n)$ is in the center of N then $\left(\mathcal{R}_{e_i}\right)_{|N} = 0$, where $\left(\mathcal{R}_{e_i}\right)_{|N}$ is an inner derivation of the nilradical N. Notice that outer derivations can be nilpotent whereas inner derivations (of N) must be nilpotent.

3.2. Solvable left Leibniz algebras

The left Leibniz identity is equivalent to the following conditions:

$$A^k_{ai}C^m_{jk} = A^k_{ja}C^m_{ki} + C^k_{ji}A^m_{ak},\ A^k_{ia}C^m_{jk} = C^k_{ji}A^m_{ka} + A^k_{ja}C^m_{ik},\ C^k_{ij}A^m_{ak} = A^k_{ai}C^m_{kj} + A^k_{aj}C^m_{ik}, \tag{3.5}$$

$$B^k_{ab}C^m_{ik} = A^k_{ia}A^m_{kb} + A^k_{ib}A^m_{ak},\ A^k_{ib}A^m_{ak} = B^k_{ab}C^m_{ik} + A^k_{ai}A^m_{kb},\ B^k_{ab}C^m_{ki} = A^m_{ak}A^k_{bi} - A^m_{bk}A^k_{ai}. \tag{3.6}$$

Then the entries of the matrices A_a, which are $(A_a)^k_i$, must satisfy the equations (3.5) which come from all the possible Leibniz identities between the triples $\{e_a, e_i, e_j\}$. In this case $A_a = \left(\mathcal{L}_{e_a}\right)_{|N}$, $(n+1 \leq a \leq p)$ are outer derivations and the commutators $[\mathcal{L}_{e_a}, \mathcal{L}_{e_b}] = \mathcal{L}_{[e_a,e_b]}$ give the structure constants B^k_{ab} as shown in the last equation of (3.6) but only up to the elements in the center of the nilradical N.

Once the left or right Leibniz identities are satisfied in the most general possible way and the outer derivations are found:

(i) We can carry out the technique of "absorption" [81, 83], which means we can simplify a solvable Leibniz algebra without affecting the nilradical in (3.2) applying the transformation

$$e'_i = e_i, \qquad (1 \leq i \leq n), \qquad\qquad e'_a = e_a + \sum_{k=1}^{n} d^k e_k, \qquad (n+1 \leq a \leq p).$$

(ii) We can change basis such that the brackets for the nilradical in (3.2) are unchanged to remove all the possible parameters.

4. The nilpotent sequence $N_{n,18}$

4.1. $N_{n,18}$

In $N_{n,18}$, $(n \geq 4)$ the positive integer n denotes the dimension of the algebra. The center of this algebra is $C(N_{n,18}) = \langle e_{n-1}, e_n\rangle$. $N_{n,18}$ can be described explicitly as follows: in the basis $\{e_1, e_2, e_3, e_4, e_5, \dots, e_n\}$ it has only the following non-zero brackets

$$[e_1, e_i] = e_{n-1}, \qquad [e_i, e_1] = -e_{n-1}, \qquad [e_i, e_i] = e_n, \qquad (2 \leq i \leq n-2,\ n \geq 4). \tag{4.1}$$

The dimensions of the ideals in the characteristic series are

$$LS = DS = [n,\ 2,\ 0], \qquad (n \geq 4).$$

It gives that $N_{n,18}$ is an associative Leibniz algebra and satisfies the left and right Leibniz identities, so it is the left and right Leibniz algebra at the same time.

The notation for $N_{n,18}$ could be explained as follows: a subscript n, $(n \geq 4)$ denotes the dimension of the algebra and 18 is the numbering based on the four-dimensional nilpotent Leibniz algebra $\mathscr{R}_{18}$ using the notation of the paper [4], which is exactly this algebra for $n = 4$. Besides N emphasizes the fact that this nilpotent Leibniz algebra is at the same time the nilradical of the solvable indecomposable left and right Leibniz algebras we construct in this paper.

It is shown below that solvable indecomposable right (left) Leibniz algebras with the nilradical $N_{n,18}$ only exist for $\dim \mathfrak{g} = n+1$ and $\dim \mathfrak{g} = n+2$ ($\dim l = n+1$ and $\dim l = n+2$).

For the solvable indecomposable right Leibniz algebras with a codimension one nilradical, we use the notation $\mathfrak{g}_{n+1,i}$ where $n+1$ indicates the dimension of the algebra $\mathfrak{g}$ and i its numbering within the list of algebras. There are four types of such algebras up to isomorphism so $1 \leq i \leq 4$. There is the only solvable indecomposable right Leibniz algebra up to isomorphism with a codimension two nilradical $\mathfrak{g}_{n+2,1}$.

We also have four solvable indecomposable left Leibniz algebras up to isomorphism with a codimension one nilradical $N_{n,18}$ denoted $l_{n+1,1}, \mathfrak{g}_{n+1,2}, l_{n+1,3}$ and $l_{n+1,4}$, where $\mathfrak{g}_{n+1,2}$ is left and right Leibniz algebra at the same time. There is the only solvable indecomposable left Leibniz with a codimension two nilradical as well denoted $l_{n+2,1}$.

There are only two right (left) algebras that contain parameters, such as $\mathfrak{g}_{n+1,2}, \mathfrak{g}_{n+1,3}$ $(\mathfrak{g}_{n+1,2}, l_{n+1,3})$. Altogether right (left) algebras depend on at most n, $(n \geq 4)$ parameters, where $\mathfrak{g}_{n+1,2}$ depends on $n-1$ parameters and $\mathfrak{g}_{n+1,3}, l_{n+1,3}$ depend on only one. In this case we do not consider $\epsilon = 0, 1$ to be a parameter as it is a discrete value. Therefore those algebras define a continuous family of algebras. We have also restricted the values of the parameters, where needed, as much as possible so as to make each algebra unique within its class.

5. Classification of solvable indecomposable right Leibniz algebras with a nilradical $N_{n,18}$

Our goal in this section and Section 6 is to find all the possible right and left solvable indecomposable extensions of the nilpotent Leibniz algebra $N_{n,18}$, which serves as the nilradical of the extended algebra.

Remark 5.1. It is assumed throughout this section and next section that the solvable right Leibniz algebras $\mathfrak{g}$ and the solvable left Leibniz algebras l have the nilradical $N_{n,18}$; however, for the sake of simplicity the brackets of the nilradical will be mostly omitted.

5.1. Solvable indecomposable right Leibniz algebras with a codimension one nilradical $N_{n,18}$

The nilpotent Leibniz algebra $N_{n,18}$, $(n \geq 4)$ is defined in (4.1). Suppose $\{e_{n+1}\}$ is in the complementary subspace to the nilradical $N_{n,18}$ and $\mathfrak{g}$ is the corresponding right solvable Leibniz algebra. Since $[\mathfrak{g}, \mathfrak{g}] \subseteq N_{n,18}$, therefore we have

$$\begin{cases} [e_1, e_i] = e_{n-1}, [e_i, e_1] = -e_{n-1}, [e_i, e_i] = e_n, & (2 \leq i \leq n-2), \\ [e_r, e_{n+1}] = \displaystyle\sum_{j=1}^{n} a_{j,r} e_j, [e_{n+1}, e_k] = \sum_{j=1}^{n} b_{j,k} e_j, & (1 \leq k \leq n, 1 \leq r \leq n+1). \end{cases} \tag{5.1}$$

Theorem 5.1. *Set $a_{2,2} := a$ in (5.1). To satisfy the right Leibniz identity, there are the following cases based on the conditions involving parameters, each gives a continuous family of solvable Leibniz algebras.*
(1) *If $a \neq 0$, $a_{1,1} + a \neq 0$, $2a - a_{1,1} \neq 0$, then*

$$\begin{cases} [e_1, e_{n+1}] = a_{1,1}e_1 + a_{n-1,1}e_{n-1} + a_{n,1}e_n, [e_i, e_{n+1}] = a_{1,2}e_1 + ae_i + \displaystyle\sum_{k=n-1}^{n} a_{k,i}e_k, \\ [e_{n-1}, e_{n+1}] = (a_{1,1} + a)e_{n-1}, [e_n, e_{n+1}] = 2ae_n, [e_{n+1}, e_{n+1}] = a_{n,n+1}e_n, \\ [e_{n+1}, e_1] = -a_{1,1}e_1 - a_{n-1,1}e_{n-1} + \dfrac{a_{1,1}a_{n,1}}{2a - a_{1,1}}e_n, [e_{n+1}, e_i] = -a_{1,2}e_1 - ae_i \\ \quad - a_{n-1,i}e_{n-1} + \left(a_{n,i} + \dfrac{2a_{1,2}a_{n,1}}{2a - a_{1,1}}\right)e_n, \quad (2 \leq i \leq n-2), \quad [e_{n+1}, e_{n-1}] = (-a_{1,1} - a)e_{n-1}. \end{cases}$$

(2) *If* $a_{1,1} := -a,\ a \neq 0$, *then the brackets for the algebra are*

$$\begin{cases} [e_1,e_{n+1}] = -ae_1 + a_{n-1,1}e_{n-1} - 3b_{n,1}e_n, [e_i,e_{n+1}] = a_{1,2}e_1 + ae_i + \sum_{k=n-1}^{n} a_{k,i}e_k, \\ [e_n,e_{n+1}] = 2ae_n, [e_{n+1},e_{n+1}] = a_{n-1,n+1}e_{n-1} + a_{n,n+1}e_n, [e_{n+1},e_1] = ae_1 - a_{n-1,1}e_{n-1} \\ \quad + b_{n,1}e_n, [e_{n+1},e_i] = -a_{1,2}e_1 - ae_i - a_{n-1,i}e_{n-1} \\ + \left(a_{n,i} - \frac{2a_{1,2}b_{n,1}}{a}\right)e_n, \quad (2 \leq i \leq n-2). \end{cases}$$

(3) *If* $a = 0,\ a_{1,1} \neq 0$, *then we have*

$$\begin{cases} [e_1,e_{n+1}] = a_{1,1}e_1 + a_{n-1,1}e_{n-1} + a_{n,1}e_n, [e_i,e_{n+1}] = a_{1,2}e_1 + \sum_{k=n-1}^{n} a_{k,i}e_k, \\ [e_{n-1},e_{n+1}] = a_{1,1}e_{n-1}, [e_{n+1},e_{n+1}] = a_{n,n+1}e_n, [e_{n+1},e_1] \\ \quad = -a_{1,1}e_1 - a_{n-1,1}e_{n-1} - a_{n,1}e_n, \\ [e_{n+1},e_i] = -a_{1,2}e_1 - a_{n-1,i}e_{n-1} + b_{n,i}e_n, \qquad (2 \leq i \leq n-2), \\ \quad [e_{n+1},e_{n-1}] = -a_{1,1}e_{n-1}. \end{cases}$$

(4) *If* $a_{1,1} := 2a,\ a \neq 0$, *then*

$$\begin{cases} [e_1,e_{n+1}] = 2ae_1 + a_{n-1,1}e_{n-1}, [e_i,e_{n+1}] = a_{1,2}e_1 + ae_i + \sum_{k=n-1}^{n} a_{k,i}e_k, \\ [e_{n-1},e_{n+1}] = 3ae_{n-1}, [e_n,e_{n+1}] = 2ae_n, [e_{n+1},e_{n+1}] = a_{n,n+1}e_n, [e_{n+1},e_1] = -2ae_1 \\ \quad -a_{n-1,1}e_{n-1} + b_{n,1}e_n, [e_{n+1},e_i] = -a_{1,2}e_1 - ae_i - a_{n-1,i}e_{n-1} + \left(a_{n,i} + \frac{a_{1,2}b_{n,1}}{a}\right)e_n, \\ \quad (2 \leq i \leq n-2), \qquad [e_{n+1},e_{n-1}] = -3ae_{n-1}. \end{cases}$$

Proof. Let us denote $\mathcal{R}_{e_{n+1}}$ by $\mathcal{R}$. To satisfy the right Leibniz identity in (5.1), we have to consider in $\mathbb{R}^2$ as the $(a, a_{1,1})-$ plane the following three linear equations: $a_{1,1} = -a,\ a = 0,\ a_{1,1} = 2a$. They have one point of the intersection $(0,0)$, which is excluded in order to avoid nilpotent Leibniz algebras. As a result, we consider below four regions of $\mathbb{R}^2$; collectively their union is $\mathbb{R}^2 \backslash (0,0)$ corresponding to all possible ordered pairs $(a, a_{1,1})$ excluding the origin. The first region given in (1) is the open subset of $\mathbb{R}^2$ for which all of the three equations are not satisfied and is a generic case. The equation $a_{1,1} = -a$ corresponds to region (2), the equation $a = 0$ corresponds to region (3) and finally the equation $a_{1,1} = 2a$ corresponds to region (4)

(1) Suppose $a \neq 0,\ a_{1,1} + a \neq 0,\ 2a - a_{1,1} \neq 0$. The proof is off-loaded to Table 1.
(2) Suppose $a_{1,1} := -a,\ a \neq 0$. Following similar computations as in steps (1)–(17) of the generic case and after that considering the right Leibniz identity $\mathcal{R}([e_{n+1},e_i]) = [\mathcal{R}(e_{n+1}),e_i] + [e_{n+1},\mathcal{R}(e_i)],\ (2 \leq i \leq n-2)$, we derive the result.
(3) Suppose $a = 0,\ a_{1,1} \neq 0$. We apply steps (1)–(18) of case (1) Then for steps (19) and (20), we consider the identities $\mathcal{R}_{e_1}([e_{n+1},e_{n+1}]) = [\mathcal{R}_{e_1}(e_{n+1}),e_{n+1}] + [e_{n+1},\mathcal{R}_{e_1}(e_{n+1})]$, $\mathcal{R}([e_{n+1},e_i]) = [\mathcal{R}(e_{n+1}),e_i] + [e_{n+1},\mathcal{R}(e_i)],\ (2 \leq i \leq n-2)$ and prove the result.
(4) Suppose $a_{1,1} := 2a,\ a \neq 0$. We repeat steps (1)–(18) of the generic case. Then considering the identity $\mathcal{R}([e_{n+1},e_i]) = [\mathcal{R}(e_{n+1}),e_i] + [e_{n+1},\mathcal{R}(e_i)],\ (2 \leq i \leq n-2)$, we deduce the result. □

Table 1. Right Leibniz identities in the generic case in Theorem 5.1.

Steps	Ordered triple	Result
1.	$\mathcal{R}[e_1, e_2]$	$[e_{n-1}, e_{n+1}] = \left(a_{1,1} + a + \sum_{k=3}^{n-2} a_{k,2}\right) e_{n-1} + a_{2,1}e_n \implies$ $a_{k,n-1} = 0,\ (1 \le k \le n-2), a_{n,n-1} := a_{2,1},$ $a_{n-1,n-1} := a_{1,1} + a + \sum_{k=3}^{n-2} a_{k,2}.$
2.	$\mathcal{R}[e_2, e_2]$	$[e_n, e_{n+1}] = 2ae_n \implies a_{k,n} = 0,\ (1 \le k \le n-1), a_{n,n} := 2a.$
3.	$\mathcal{R}[e_2, e_{n+1}]$	$a_{1,n+1} = a_{2,n+1} = 0 \implies [e_{n+1}, e_{n+1}] = \sum_{k=3}^{n} a_{k,n+1}e_k.$
4.	$\mathcal{R}[e_i, e_{n+1}]$	$a_{i,n+1} = 0,\ (3 \le i \le n-2)$. Combining with 3., $[e_{n+1}, e_{n+1}] = a_{n-1,n+1}e_{n-1} + a_{n,n+1}e_n.$
5.	$\mathcal{R}[e_2, e_1]$	$a_{2,1} = 0 \implies [e_{n-1}, e_{n+1}] = \left(a_{1,1} + a + \sum_{k=3}^{n-2} a_{k,2}\right) e_{n-1},$ $[e_1, e_{n+1}] = a_{1,1}e_1 + \sum_{k=3}^{n} a_{k,1}e_k.$
6.	$\mathcal{R}[e_1, e_i]$	$a_{i,1} = 0,\ (3 \le i \le n-2) \implies [e_1, e_{n+1}] = a_{1,1}e_1 + a_{n-1,1}e_{n-1} + a_{n,1}e_n.$
7.	$\mathcal{R}[e_i, e_i]$	$a_{i,i} := a,\ (3 \le i \le n-2) \implies$ $[e_j, e_{n+1}] = \sum_{k=1}^{j-1} a_{k,j}e_k + ae_j + \sum_{k=j+1}^{n} a_{k,j}e_k,\ (2 \le j \le n-2).$
8.	$\mathcal{R}[e_{i-k}, e_i]$	$a_{1,i} := a_{1,i-k},\ a_{i-k,i} := -a_{i,i-k},\ (2+k \le i \le n-2,\ 1 \le k \le n-4)$, where k is fixed $\implies [e_j, e_{n+1}] = a_{1,2}e_1 - \sum_{m=2}^{j-1} a_{j,m}e_m + ae_j +$ $\sum_{m=j+1}^{n} a_{m,j}e_m,\ (2 \le j \le n-2).$
9.	$\mathcal{R}_{e_2}\left([e_{n+1}, e_1]\right)$	$b_{k,n-1} = 0,\ (1 \le k \le n-2), b_{n,n-1} := b_{2,1}, b_{n-1,n-1} := b_{1,1} + \sum_{k=2}^{n-2} b_{k,2}$ $\implies [e_{n+1}, e_{n-1}] = \left(b_{1,1} + \sum_{k=2}^{n-2} b_{k,2}\right) e_{n-1} + b_{2,1}e_n.$
10.	$\mathcal{R}_{e_2}\left([e_{n+1}, e_2]\right)$	$[e_{n+1}, e_n] = 0 \implies b_{k,n} = 0,\ (1 \le k \le n).$
11.	$\mathcal{R}_{e_1}\left([e_2, e_{n+1}]\right)$	$b_{2,1} = 0,\ b_{1,1} := -a_{1,1} \implies b_{n-1,n-1} := -a_{1,1} + \sum_{k=2}^{n-2} b_{k,2}$ and $[e_{n+1}, e_{n-1}] = \left(-a_{1,1} + \sum_{k=2}^{n-2} b_{k,2}\right) e_{n-1},$ $[e_{n+1}, e_1] = -a_{1,1}e_1 + \sum_{k=3}^{n} b_{k,1}e_k.$
12.	$\mathcal{R}_{e_1}\left([e_i, e_{n+1}]\right)$	$b_{i,1} = 0,\ (3 \le i \le n-2) \implies$ $[e_{n+1}, e_1] = -a_{1,1}e_1 + b_{n-1,1}e_{n-1} + b_{n,1}e_n.$
13.	$\mathcal{R}_{e_i}\left([e_i, e_{n+1}]\right)$	$b_{1,i} := -a_{1,2},\ b_{i,i} := -a,\ (2 \le i \le n-2).$
14.	$\mathcal{R}_{e_k}\left([e_{n+1}, e_i]\right)$	$b_{i,k} := b_{k,i},\ (2 \le i \le n-3,\ 1+i \le k \le n-2)$, where i is fixed $\implies$ $[e_{n+1}, e_j] = -a_{1,2}e_1 + \sum_{m=2}^{j-1} b_{j,m}e_m - ae_j + \sum_{m=j+1}^{n} b_{m,j}e_m,\ (2 \le$ $j \le n-2).$
15.	$\mathcal{R}_{e_k}\left([e_i, e_{n+1}]\right)$	$b_{k,i} := a_{k,i},\ (2 \le i \le n-3,\ 1+i \le k \le n-2)$, where i is fixed $\implies$ $b_{n-1,n-1} := -a - a_{1,1} + \sum_{m=3}^{n-2} a_{m,2},$ $[e_{n+1}, e_j] = -a_{1,2}e_1 + \sum_{m=2}^{j-1} a_{j,m}e_m - ae_j + \sum_{m=j+1}^{n-2} a_{m,j}e_m +$ $\sum_{m=n-1}^{n} b_{m,j}e_m,\ (2 \le j \le n-2)$, and $[e_{n+1}, e_{n-1}] = \left(-a - a_{1,1} + \sum_{m=3}^{n-2} a_{m,2}\right) e_{n-1}.$
16.	$\mathcal{R}_{e_i}\left([e_k, e_{n+1}]\right)$	$a_{k,i} = 0,\ (2 \le i \le n-3,\ 1+i \le k \le n-2)$, where i is fixed $\implies$ $[e_j, e_{n+1}] = a_{1,2}e_1 + ae_j + \sum_{m=n-1}^{n} a_{m,j}e_m,\ [e_{n+1}, e_j] =$ $-a_{1,2}e_1 - ae_j + \sum_{m=n-1}^{n} b_{m,j}e_m,\ (2 \le j \le n-2),$ $[e_{n-1}, e_{n+1}] = (a_{1,1} + a)\, e_{n-1}, [e_{n+1}, e_{n-1}] = (-a - a_{1,1})\, e_{n-1}.$
17.	$\mathcal{R}[e_{n+1}, e_1]$	$b_{n-1,1} := -a_{n-1,1}, b_{n,1} := \frac{a_{1,1}a_{n,1}}{2a - a_{1,1}} \implies$ $[e_{n+1}, e_1] = -a_{1,1}e_1 - a_{n-1,1}e_{n-1} + \frac{a_{1,1}a_{n,1}}{2a - a_{1,1}}e_n.$
18.	$\mathcal{R}[e_{n+1}, e_{n+1}]$	$a_{n-1,n+1} = 0 \implies [e_{n+1}, e_{n+1}] = a_{n,n+1}e_n.$
19.	$\mathcal{R}_{e_i}\left([e_{n+1}, e_{n+1}]\right)$	$b_{n-1,i} := -a_{n-1,i},\ b_{n,i} := a_{n,i} + \frac{2a_{1,2}a_{n,1}}{2a - a_{1,1}} \implies [e_{n+1}, e_i] = -a_{1,2}e_1 -$ $ae_i - a_{n-1,i}e_{n-1} + \left(a_{n,i} + \frac{2a_{1,2}a_{n,1}}{2a - a_{1,1}}\right) e_n,\ (2 \le i \le n-2).$

Remark 5.2. In the tables throughout the paper an ordered triple is a shorthand notation for a derivation property of the multiplication operators, which are either $\mathscr{R}_z\left([x, y]\right) = [\mathscr{R}_z(x), y] + [x, \mathscr{R}_z(y)]$ or $\mathscr{L}_z\left([x, y]\right) = [\mathscr{L}_z(x), y] + [x, \mathscr{L}_z(y)]$. We assign $\mathscr{R}_{e_{n+1}} := \mathscr{R}$ and $\mathscr{L}_{e_{n+1}} := \mathscr{L}$.

Theorem 5.2. *Applying the technique of "absorption" (see Section 3.2), we can further simplify the algebras in each of the four cases in Theorem 5.1 as follows:*

(1) *Suppose* $a \neq 0,\ a_{1,1}+a \neq 0,\ 2a-a_{1,1} \neq 0$. *We have*

$$\begin{cases} [e_1,e_{n+1}]=a_{1,1}e_1+a_{n,1}e_n, [e_2,e_{n+1}]=a_{1,2}e_1+ae_2+a_{n-1,2}e_{n-1}+a_{n,2}e_n,\\ [e_i,e_{n+1}]=a_{1,2}e_1+ae_i+a_{n-1,i}e_{n-1}, [e_{n-2},e_{n+1}]=a_{1,2}e_1+ae_{n-2},\\ [e_{n-1},e_{n+1}]=(a+a_{1,1})e_{n-1}, [e_n,e_{n+1}]=2ae_n, [e_{n+1},e_1]=-a_{1,1}e_1+\dfrac{a_{1,1}a_{n,1}}{2a-a_{1,1}}e_n,\\ [e_{n+1},e_2]=-a_{1,2}e_1-ae_2-a_{n-1,2}e_{n-1}+\left(a_{n,2}+\dfrac{2a_{1,2}a_{n,1}}{2a-a_{1,1}}\right)e_n,\\ [e_{n+1},e_i]=-a_{1,2}e_1-ae_i-a_{n-1,i}e_{n-1}+\dfrac{2a_{1,2}a_{n,1}}{2a-a_{1,1}}e_n, (3\leq i\leq n-3),\\ [e_{n+1},e_{n-2}]=-a_{1,2}e_1-ae_{n-2}+\dfrac{2a_{1,2}a_{n,1}}{2a-a_{1,1}}e_n, [e_{n+1},e_{n-1}]=(-a-a_{1,1})e_{n-1}. \end{cases}$$

(2) *Suppose* $a_{1,1}:=-a,\ a\neq 0$. *Then the brackets for the algebra are*

$$\begin{cases} [e_1,e_{n+1}]=-ae_1-3b_{n,1}e_n, [e_2,e_{n+1}]=a_{1,2}e_1+ae_2+a_{n-1,2}e_{n-1}+a_{n,2}e_n,\\ [e_i,e_{n+1}]=a_{1,2}e_1+ae_i+a_{n-1,i}e_{n-1}, [e_{n-2},e_{n+1}]=a_{1,2}e_1+ae_{n-2}, [e_n,e_{n+1}]=2ae_n,\\ [e_{n+1},e_{n+1}]=a_{n-1,n+1}e_{n-1}, [e_{n+1},e_1]=ae_1+b_{n,1}e_n, [e_{n+1},e_2]=-a_{1,2}e_1-ae_2\\ \qquad -a_{n-1,2}e_{n-1}+\left(a_{n,2}-\dfrac{2a_{1,2}b_{n,1}}{a}\right)e_n,\\ [e_{n+1},e_i]=-a_{1,2}e_1-ae_i-a_{n-1,i}e_{n-1}-\dfrac{2a_{1,2}b_{n,1}e_n}{a},\\ (3\leq i\leq n-3),\quad [e_{n+1},e_{n-2}]=-a_{1,2}e_1-ae_{n-2}-\dfrac{2a_{1,2}b_{n,1}}{a}e_n. \end{cases}$$

(3) *If* $a=0,\ a_{1,1}\neq 0$, *then*

$$\begin{cases} [e_1,e_{n+1}]=a_{1,1}e_1+a_{n,1}e_n, [e_2,e_{n+1}]=a_{1,2}e_1+a_{n-1,2}e_{n-1}+a_{n,2}e_n, [e_i,e_{n+1}]=a_{1,2}e_1\\ \qquad +a_{n-1,i}e_{n-1},\quad (3\leq i\leq n-3),\quad [e_{n-2},e_{n+1}]=a_{1,2}e_1, [e_{n-1},e_{n+1}]=a_{1,1}e_{n-1},\\ [e_{n+1},e_{n+1}]=a_{n,n+1}e_n, [e_{n+1},e_1]=-a_{1,1}e_1-a_{n,1}e_n, [e_{n+1},e_j]=-a_{1,2}e_1-a_{n-1,j}e_{n-1}\\ \qquad +b_{n,j}e_n, (2\leq j\leq n-3), [e_{n+1},e_{n-2}]=-a_{1,2}e_1+b_{n,n-2}e_n,\quad [e_{n+1},e_{n-1}]=-a_{1,1}e_{n-1}. \end{cases}$$

(4) *If* $a_{1,1}:=2a,\ a\neq 0$, *then the brackets for the solvable Leibniz algebra are*

$$\begin{cases} [e_1,e_{n+1}]=2ae_1, [e_2,e_{n+1}]=a_{1,2}e_1+ae_2+a_{n-1,2}e_{n-1}+a_{n,2}e_n, [e_i,e_{n+1}]=a_{1,2}e_1\\ \qquad +ae_i+a_{n-1,i}e_{n-1}, [e_{n-2},e_{n+1}]=a_{1,2}e_1+ae_{n-2}, [e_{n-1},e_{n+1}]=3ae_{n-1}, [e_n,e_{n+1}]=2ae_n,\\ [e_{n+1},e_1]=-2ae_1+b_{n,1}e_n, [e_{n+1},e_2]=-a_{1,2}e_1-ae_2-a_{n-1,2}e_{n-1}+\left(a_{n,2}+\dfrac{a_{1,2}b_{n,1}}{a}\right)e_n,\\ [e_{n+1},e_i]=-a_{1,2}e_1-ae_i-a_{n-1,i}e_{n-1}+\dfrac{a_{1,2}b_{n,1}}{a}e_n,\qquad (3\leq i\leq n-3),\\ [e_{n+1},e_{n-2}]=-a_{1,2}e_1-ae_{n-2}+\dfrac{a_{1,2}b_{n,1}}{a}e_n, [e_{n+1},e_{n-1}]=-3ae_{n-1}. \end{cases}$$

Proof. We show the generic case corresponding to region (1) Other cases are proved applying applicable transformations and properly renaming the entries.

(1) Suppose $a \neq 0$, $a_{1,1}+a \neq 0$, $2a-a_{1,1} \neq 0$. The right (derivation) and left (not a derivation) multiplication operators restricted to the nilradical are

$$\mathcal{R}_{e_{n+1}} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,2} & a_{1,2} & a_{1,2} & \cdots & a_{1,2} & 0 & 0 \\ 0 & a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & a_{n-1,5} & \cdots & a_{n-1,n-2} & a_{1,1}+a & 0 \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & a_{n,5} & \cdots & a_{n,n-2} & 0 & 2a \end{bmatrix},$$

$$\mathcal{L}_{e_{n+1}} = \begin{bmatrix} -a_{1,1} & -a_{1,2} & -a_{1,2} & -a_{1,2} & -a_{1,2} & \cdots & -a_{1,2} & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & 0 & -a & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & 0 & 0 & -a & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & -a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -a & 0 & 0 \\ -a_{n-1,1} & -a_{n-1,2} & -a_{n-1,3} & -a_{n-1,4} & -a_{n-1,5} & \cdots & -a_{n-1,n-2} & -a_{1,1}-a & 0 \\ \frac{a_{1,1}a_{n,1}}{2a-a_{1,1}} & a_{n,2}+\frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & a_{n,3}+\frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & a_{n,4}+\frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & a_{n,5}+\frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & \cdots & a_{n,n-2}+\frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & 0 & 0 \end{bmatrix}.$$

- The transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = e_{n+1} + a_{n-1,n-2}e_1$ removes $a_{n-1,n-2}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{n-1,n-2}$ in $\mathcal{L}_{e_{n+1}}$, but it affects other entries as well, such as the entries in the $(n-1,2), (n-1,3), \dots, (n-1,n-3)$ positions in $\mathcal{L}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+1}}$, which we rename by $-a_{n-1,2}, -a_{n-1,3}, \dots, -a_{n-1,n-3}$ and $a_{n-1,2}, a_{n-1,3}, \dots, a_{n-1,n-3}$, respectively. Consecutively we rename the coefficient in front of e_n in the bracket $[e_{n+1}, e_{n+1}]$ back by $a_{n,n+1}$.
- We fix j and apply the transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = e_{n+1} - a_{n,j}e_j$, $(3 \leq j \leq n-2)$, which removes $a_{n,j}$ in $\mathcal{R}_{e_{n+1}}$ and $a_{n,j}$ from the $(n,j)^{th}$ entries in $\mathcal{L}_{e_{n+1}}$. Then we change the entry in the $(n-1,1)$ position in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ to $a_{n-1,1}$ and $-a_{n-1,1}$, respectively, and the coefficient in front of e_n in the bracket $[e_{n+1}, e_{n+1}]$ back to $a_{n,n+1}$.
- We apply the transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = e_{n+1} - a_{n-1,1}e_2$ to remove $a_{n-1,1}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{n-1,1}$ in $\mathcal{L}_{e_{n+1}}$. As a consequence, we rename the entry in the $(n,2)$ position in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ back by $a_{n,2}$ and $a_{n,2} + \frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}}$, respectively. We also change the coefficient in front of e_n in the bracket $[e_{n+1}, e_{n+1}]$ back to $a_{n,n+1}$.
- Finally applying the transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = e_{n+1} - \frac{a_{n,n+1}}{2a}e_n$, we remove $a_{n,n+1}$ in the bracket $[e_{n+1}, e_{n+1}]$ and prove the result. □

Theorem 5.3. *There are four solvable indecomposable right Leibniz algebras up to isomorphism with a codimension one nilradical $N_{n,18}$, $(n \geq 4)$, which are given below:*

(i)
$$\mathfrak{g}_{n+1,1}: [e_{n+1}, e_1] = e_1, [e_1, e_{n+1}] = -e_1, [e_{n+1}, e_i] = -e_i, [e_i, e_{n+1}] = e_i,$$
$$(2 \leq i \leq n-2), [e_n, e_{n+1}] = 2e_n, [e_{n+1}, e_{n+1}] = e_{n-1},$$
$$DS = [n+1,\ n,\ 2,\ 0], LS = [n+1,\ n,\ n, \ldots],$$

(ii)
$$\mathfrak{g}_{n+1,2}: [e_{n+1}, e_i] = -e_i, \quad [e_i, e_{n+1}] = e_i, (i = 1,\ n-1),$$
$$[e_2, e_{n+1}] = \epsilon e_n, (\epsilon = 0, 1),$$
$$[e_{n+1}, e_j] = a_{j-1} e_n, (2 \leq j \leq n-2),$$
$$[e_k, e_{n+1}] = b e_n, (3 \leq k \leq n-2), [e_{n+1}, e_{n+1}] = c e_n,$$
$$|a_{m-1}| \leq |a_m|, \qquad (3 \leq m \leq n-3),$$
$$DS = [n+1,\ 3,\ 0], \qquad LS = [n+1,\ 3,\ 2,\ 2, \ldots].$$

Remark 5.3. To guarantee the algebras are unique within its class, we should always have that $|a_{m-1}| \leq |a_m|, (3 \leq m \leq n-3)$, otherwise permuting $e_3, e_4, \ldots, e_{n-2}$, we obtain isomorphic algebras. In case when $\epsilon = 0$ and $b = 0$ by permuting e_2 and e_k, we may assume that $|a_1| \leq |a_{k-1}|, (3 \leq k \leq n-2)$. If $\epsilon = 1, b = 1, (n \geq 6)$, then we should have as well that $|a_1| \leq |a_{k-1}|, (3 \leq k \leq n-2)$, because permuting e_2 and e_k gives isomorphic algebras. Finally if $n = 5$ and $\epsilon = 1, b \neq 0$, then applying the transformation $e_1' = e_1, e_i' = b e_i, (2 \leq i \leq 4), e_5' = b^2 e_5, e_6' = e_6$, we may assume that either $|b| \leq 1$ or $|a_1| \leq |a_2|$.

(iii)
$$\mathfrak{g}_{n+1,3}: [e_{n+1}, e_1] = -a e_1, \quad [e_1, e_{n+1}] = a e_1,$$
$$[e_{n+1}, e_i] = -e_i, [e_i, e_{n+1}] = e_i, \quad (2 \leq i \leq n-2),$$
$$[e_{n+1}, e_{n-1}] = (-a-1) e_{n-1}, [e_{n-1}, e_{n+1}] = (a+1) e_{n-1},$$
$$[e_n, e_{n+1}] = 2e_n,$$
$$DS = [n+1,\ n,\ 2,\ 0], LS = [n+1,\ n,\ n, \ldots],$$

(iv)
$$\mathfrak{g}_{n+1,4}: [e_{n+1}, e_1] = -e_1, [e_1, e_{n+1}] = e_1,$$
$$[e_{n+1}, e_i] = -e_1 - e_i, [e_i, e_{n+1}] = e_1 + e_i, \qquad (2 \leq i \leq n-2),$$
$$[e_{n+1}, e_{n-1}] = -2e_{n-1}, \quad [e_j, e_{n+1}] = 2e_j, \qquad (n-1 \leq j \leq n),$$
$$DS = [n+1,\ n,\ 2,\ 0], LS = [n+1,\ n,\ n, \ldots].$$

Proof. One applies the change of basis transformations keeping the nilradical $N_{n,18}$ given in (4.1) unchanged. We show the case corresponding to region (1) and case (4). Other cases are proved applying applicable transformations of case (1) 1. which is a generic, and carefully observing how they affect other entries, so the details for those cases will be mostly omitted.

(1) Suppose $a \neq 0$, $a_{1,1} + a \neq 0$, $2a - a_{1,1} \neq 0$. The right (derivation) and left (not a derivation) multiplication operators are

$$\mathcal{R}_{e_{n+1}} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,2} & a_{1,2} & a_{1,2} & \cdots & a_{1,2} & a_{1,2} & 0 & 0 \\ 0 & a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & a_{n-1,5} & \cdots & a_{n-1,n-3} & 0 & a+a_{1,1} & 0 \\ a_{n,1} & a_{n,2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2a \end{bmatrix},$$

$$\mathcal{L}_{e_{n+1}} = \begin{bmatrix} -a_{1,1} & -a_{1,2} & -a_{1,2} & -a_{1,2} & -a_{1,2} & \cdots & -a_{1,2} & -a_{1,2} & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & -a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & -a & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & -a & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -a & 0 & 0 \\ 0 & -a_{n-1,2} & -a_{n-1,3} & -a_{n-1,4} & -a_{n-1,5} & \cdots & -a_{n-1,n-3} & 0 & -a-a_{1,1} & 0 \\ \frac{a_{1,1}a_{n,1}}{2a-a_{1,1}} & a_{n,2}+\frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & \frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & \frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & \frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & \cdots & \frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & \frac{2a_{1,2}a_{n,1}}{2a-a_{1,1}} & 0 & 0 \end{bmatrix}.$$

(1) Let $a_{1,1} \neq a$.

- The transformation $e'_1 = e_1$, $e'_i = e_i + \frac{a_{1,2}}{a-a_{1,1}} e_1$, $(2 \leq i \leq n-2)$, $e'_j = e_j$, $(n-1 \leq j \leq n+1)$ removes $a_{1,2}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{1,2}$ in $\mathcal{L}_{e_{n+1}}$ from the entries in the $(1,2)$, $(1,3), \ldots, (1, n-2)$ positions, respectively. This transformation affects other entries as well. As the result we change the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ to $a_{n,2} + \frac{a_{1,2}a_{n,1}}{a-a_{1,1}}$ and name the entries in the $(n,3)$, $(n,4), \ldots, (n, n-2)$ positions in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ by $\frac{a_{1,2}a_{n,1}}{a-a_{1,1}}$.
- Applying the transformation $e'_1 = e_1 - \frac{a_{n,1}}{2a-a_{1,1}} e_n$, $e'_i = e_i - \frac{a_{n-1,i}}{a_{1,1}} e_{n-1}$, $(2 \leq i \leq n-3)$, $e'_j = e_j$, $(n-2 \leq j \leq n+1)$, we remove $a_{n-1,2}, a_{n-1,3}, \ldots, a_{n-1,n-3}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{n-1,2}, -a_{n-1,3}, \ldots, -a_{n-1,n-3}$ in $\mathcal{L}_{e_{n+1}}$; $a_{n,1}$ and $\frac{a_{1,1}a_{n,1}}{2a-a_{1,1}}$ from the $(n,1)^{st}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$, respectively, keeping other entries unchanged.
- We apply the transformation $e'_1 = e_1$, $e'_2 = e_2 - \left(\frac{a_{n,2}}{a} + \frac{a_{1,2}a_{n,1}}{a(a-a_{1,1})}\right) e_n$, $e'_i = e_i - \frac{a_{1,2}a_{n,1}}{a(a-a_{1,1})} e_n$, $(3 \leq i \leq n-2)$, $e'_k = e_k$, $(n-1 \leq k \leq n+1)$ to remove $a_{n,2} + \frac{a_{1,2}a_{n,1}}{a-a_{1,1}}$ from the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$; $\frac{a_{1,2}a_{n,1}}{a-a_{1,1}}$ from the $(n,i)^{th}$ entries in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$.

- Finally applying the transformation $e'_i = e_i$, $(1 \leq i \leq n)$, $e'_{n+1} = \frac{e_{n+1}}{a}$, we scale a to unity. Then we rename $\frac{a_{1,1}}{a}$ by $a_{1,1}$ and obtain a Leibniz algebra

$$\begin{cases} [e_1, e_{n+1}] = a_{1,1}e_1, [e_i, e_{n+1}] = e_i, [e_{n-1}, e_{n+1}] = (a_{1,1}+1)e_{n-1}, [e_n, e_{n+1}] = 2e_n, \\ [e_{n+1}, e_1] = -a_{1,1}e_1, [e_{n+1}, e_i] = -e_i, [e_{n+1}, e_{n-1}] = (-a_{1,1}-1)e_{n-1}, \\ \qquad (2 \leq i \leq n-2, a_{1,1} \neq 1, a_{1,1} \neq -1, a_{1,1} \neq 2). \end{cases} \tag{5.2}$$

(2) Suppose $a_{1,1} = a$, $a \neq 0$. We start with applying the second transformation of the previous case, carefully watching how it affects the entries. Then we consider the transformation $e'_1 = e_1$, $e'_2 = e_2 - \frac{a \cdot a_{n,2} + a_{n,1}a_{1,2}}{a^2} e_n$, $e'_i = e_i - \frac{a_{1,2}a_{n,1}}{a^2} e_n$, $(3 \leq i \leq n-2)$, $e'_k = e_k$, $(n-1 \leq k \leq n+1)$ to remove $a_{n,2} + \frac{a_{1,2}a_{n,1}}{a}$ from the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$; $\frac{a_{1,2}a_{n,1}}{a}$ from the $(n,i)^{th}$ entries in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$. Further applying the transformation $e'_j = e_j$, $(1 \leq j \leq n)$, $e'_{n+1} = \frac{e_{n+1}}{a}$, we scale a to unity and renaming $\frac{a_{1,2}}{a}$ by $a_{1,2}$, we obtain a Leibniz algebra

$$\begin{cases} [e_1, e_{n+1}] = e_1, [e_i, e_{n+1}] = a_{1,2}e_1 + e_i, [e_j, e_{n+1}] = 2e_j, (n-1 \leq j \leq n), \\ [e_{n+1}, e_1] = -e_1, [e_{n+1}, e_i] = -a_{1,2}e_1 - e_i, (2 \leq i \leq n-2), [e_{n+1}, e_{n-1}] = -2e_{n-1}. \end{cases}$$

If $a_{1,2} = 0$ then we have a limiting case of (5.2) with $a_{1,1} = 1$. If $a_{1,2} \neq 0$ then applying the transformation $e'_1 = a_{1,2}e_1$, $e'_i = e_i$, $(2 \leq i \leq n-2)$, $e'_{n-1} = a_{1,2}e_{n-1}$, $e'_j = e_j$, $(n \leq j \leq n+1)$, we derive a Leibniz algebra $\mathfrak{g}_{n+1,4}$.

(2) Suppose $a_{1,1} := -a$, $a \neq 0$. In this case we have

$$\begin{cases} [e_1, e_{n+1}] = -e_1, [e_i, e_{n+1}] = e_i, [e_n, e_{n+1}] = 2e_n, [e_{n+1}, e_{n+1}] = a_{n-1,n+1}e_{n-1}, \\ [e_{n+1}, e_1] = e_1, [e_{n+1}, e_i] = -e_i, \qquad (2 \leq i \leq n-2). \end{cases}$$

If $a_{n-1,n+1} = 0$ then we obtain a limiting case of (5.2) with $a_{1,1} = -1$. If $a_{n-1,n+1} \neq 0$ then applying the transformation $e'_1 = a_{n-1,n+1}e_1$, $e'_i = e_i$, $(2 \leq i \leq n-2)$, $e'_{n-1} = a_{n-1,n+1}e_{n-1}$, $e'_j = e_j$, $(n \leq j \leq n+1)$, we have a Leibniz algebra $\mathfrak{g}_{n+1,1}$.

(3) Suppose $a = 0$ and $a_{1,1} \neq 0$. We have

$$\begin{cases} [e_1, e_{n+1}] = e_1, [e_2, e_{n+1}] = a_{n,2}e_n, [e_k, e_{n+1}] = be_n, (3 \leq k \leq n-2), \\ [e_{n-1}, e_{n+1}] = e_{n-1}, [e_{n+1}, e_{n+1}] = ce_n, [e_{n+1}, e_1] = -e_1, [e_{n+1}, e_j] = a_{j-1}e_n, \\ \qquad (2 \leq j \leq n-2), [e_{n+1}, e_{n-1}] = -e_{n-1}. \end{cases} \tag{5.3}$$

If $a_{n,2} \neq 0$ then applying the transformation $e'_1 = e_1$, $e'_i = a_{n,2}e_i$, $(2 \leq i \leq n-1)$, $e'_n = (a_{n,2})^2 e_n$, $e'_{n+1} = e_{n+1}$, we scale $a_{n,2}$ to unity. Changing the other entries to what they were originally, we obtain a Leibniz algebra

$$\begin{cases} [e_1, e_{n+1}] = e_1, [e_2, e_{n+1}] = e_n, [e_k, e_{n+1}] = be_n, \qquad (3 \leq k \leq n-2), \\ [e_{n-1}, e_{n+1}] = e_{n-1}, [e_{n+1}, e_{n+1}] = ce_n, [e_{n+1}, e_1] = -e_1, [e_{n+1}, e_j] = a_{j-1}e_n, \\ \qquad (2 \leq j \leq n-2), [e_{n+1}, e_{n-1}] = -e_{n-1}. \end{cases} \tag{5.4}$$

Altogether (5.3) with $a_{n,2} = 0$ and (5.4) give us a Leibniz algebra $\mathfrak{g}_{n+1,2}$.

(4) Suppose $a_{1,1} := 2a$, $a \neq 0$. We apply the following transformations.

- The transformation $e'_1 = e_1$, $e'_i = e_i - \frac{a_{1,2}}{a} e_1$, $(2 \leq i \leq n-2)$, $e'_j = e_j$, $(n-1 \leq j \leq n+1)$ removes $a_{1,2}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{1,2}$ in $\mathcal{L}_{e_{n+1}}$ from the entries in the $(1,2), (1,3), \ldots, (1,n-2)$ positions, respectively. In $\mathcal{L}_{e_{n+1}}$ this transformation also removes the part $\frac{a_{1,2}b_{n,1}}{a}$ from the $(n,2)^{nd}$ entry $a_{n,2} + \frac{a_{1,2}b_{n,1}}{a}$ and $\frac{a_{1,2}b_{n,1}}{a}$ from the entries in the $(n,3), (n,4), \ldots, (n,n-2)$ positions.
- Applying the transformation $e'_1 = e_1 - \frac{b_{n,1}}{2a} e_n$, $e'_i = e_i - \frac{a_{n-1,i}}{2a} e_{n-1}$, $(2 \leq i \leq n-3)$, $e'_j = e_j$, $(n-2 \leq j \leq n+1)$, we remove $a_{n-1,2}, a_{n-1,3}, \ldots, a_{n-1,n-3}$ in $\mathcal{R}_{e_{n+1}}$ and

$-a_{n-1,2}, -a_{n-1,3}, \dots, -a_{n-1,n-3}$ in $\mathcal{L}_{e_{n+1}}$; $b_{n,1}$ from the $(n,1)^{st}$ entry in $\mathcal{L}_{e_{n+1}}$ without affecting other entries.

- The transformation $e'_1 = e_1$, $e'_2 = e_2 - \frac{a_{n,2}}{a} e_n$, $e'_i = e_i$, $(3 \le i \le n+1)$ removes $a_{n,2}$ from the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$.
- Finally we apply the transformation $e'_i = e_i$, $(1 \le i \le n)$, $e'_{n+1} = \frac{e_{n+1}}{a}$ to scale a to unity and obtain a limiting case of (5.2) with $a_{1,1} = 2$.

Altogether (5.2) and all its limiting cases after replacing $a_{1,1}$ with a give us a Leibniz algebra $\mathfrak{g}_{n+1,3}$. □

5.2. Solvable indecomposable right Leibniz algebras with a codimension two nilradical $N_{n,18}$

The non-zero inner derivations of $N_{n,18}$, $(n \ge 4)$ are given by

$$\mathcal{R}_{e_1} = -\sum_{i=2}^{n-2} E_{n-1,i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{R}_{e_i} = E_{n-1,1} + E_{n,i}, \ (2 \le i \le n-2),$$

where $E_{n-1,i}$ is the matrix which has 1 in the $(n-1,i)^{th}$ entry and all other entries are zero, $E_{n-1,1}$ is the matrix with 1 in the $(n-1,1)^{st}$ position and zeros in all other places and $E_{n,i}$ is the matrix that has 1 in the $(n,i)^{th}$ position and zeros everywhere else.

The nilradical $N_{n,18}$ in the basis $\{e_1, e_2, e_3, \dots, e_n\}$ is defined in (4.1), where e_{n-1} and e_n are in the center of $N_{n,18}$. The vectors complementary to the nilradical are e_{n+1} and e_{n+2}.

One could notice that two outer derivations $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$ are "nil-independent" [41, 61] only in case (1) in Theorem 5.2.

Remark 5.4. If we have three or more outer derivations, then they are "nil-dependent" means a linear combination of them is nilpotent, which cannot happen as we construct solvable Leibniz algebras. Therefore the solvable algebras we are constructing are of codimension at most two.

By taking a linear combination of $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$ and keeping in mind that no nontrivial linear combination of the matrices $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$ can be a nilpotent matrix, one could set $\begin{pmatrix} a_{1,1} \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} b_{1,1} \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, because at the same time we have that $a \neq 0$, $a_{1,1} + a \neq 0$, $2a - a_{1,1} \neq 0$ (the same is true for b and $b_{1,1}$). Therefore the vector space of outer derivations as the $n \times n$ matrices are $\mathcal{R}_{e_{n+1}} =$

$$\begin{bmatrix} 1 & a_{1,2} & a_{1,2} & a_{1,2} & a_{1,2} & \cdots & a_{1,2} & a_{1,2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & a_{n-1,5} & \cdots & a_{n-1,n-3} & 0 & 2 & 0 \\ a_{n,1} & a_{n,2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$\mathcal{R}_{e_{n+2}} = \begin{bmatrix} 0 & b_{1,2} & b_{1,2} & b_{1,2} & b_{1,2} & \cdots & b_{1,2} & b_{1,2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & b_{n-1,2} & b_{n-1,3} & b_{n-1,4} & b_{n-1,5} & \cdots & b_{n-1,n-3} & 0 & 1 & 0 \\ b_{n,1} & b_{n,2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \end{bmatrix}.$$

General approach to find solvable right Leibniz algebras with a codimension two nilradical $N_{n,18}$.[2]

(i) We consider $\mathcal{R}_{[e_r,e_s]} = [\mathcal{R}_{e_s}, \mathcal{R}_{e_r}] = \mathcal{R}_{e_s}\mathcal{R}_{e_r} - \mathcal{R}_{e_r}\mathcal{R}_{e_s}$, $(n+1 \le r \le n+2,\ 1 \le s \le n+2)$ and compare with $\sum_{k=1}^{n-2} c_k \mathcal{R}_{e_k}$, because e_{n-1} and e_n are in the center of $N_{n,18}$, $(n \ge 4)$ defined in (4.1).

(ii) We write down $[e_r, e_s]$, $(n+1 \le r \le n+2,\ 1 \le s \le n+2)$ including a linear combination of e_{n-1} and e_n as well. We add the brackets from the nilradical $N_{n,18}$, $(n \ge 4)$ and outer derivations $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$.

(iii) One satisfies the right Leibniz identity: $[[e_r, e_s], e_t] = [[e_r, e_t], e_s] + [e_r, [e_s, e_t]]$ or, equivalently, $\mathcal{R}_{e_t}([e_r, e_s]) = [\mathcal{R}_{e_t}(e_r), e_s] + [e_r, \mathcal{R}_{e_t}(e_s)]$, $(1 \le r, s, t \le n+2)$ for all the brackets obtained in step *(ii)*

(iv) We apply the change of basis transformations without affecting the nilradical to remove as many parameters as possible.

(i) One could find that

$$\begin{cases} \mathcal{R}_{[e_{n+1},e_i]} = -\mathcal{R}_{e_i}, (1 \le i \le n-2), \mathcal{R}_{[e_{n+1},e_k]} = 0, \qquad (n-1 \le k \le n+1), \\ \mathcal{R}_{[e_{n+1},e_{n+2}]} = (n-4)a_{n,1}b_{1,2}\mathcal{R}_{e_2} - a_{n,1}b_{1,2}\left(\sum_{j=3}^{n-2} \mathcal{R}_{e_j}\right), \qquad \mathcal{R}_{[e_{n+2},e_1]} = 0, \\ \mathcal{R}_{[e_{n+2},e_m]} = -b_{1,2}\mathcal{R}_{e_1} - \mathcal{R}_{e_m}, \qquad (2 \le m \le n-2), \qquad \mathcal{R}_{[e_{n+2},e_{n-1}]} = \mathcal{R}_{[e_{n+2},e_n]} = 0, \\ \mathcal{R}_{[e_{n+2},e_{n+1}]} = (4-n)a_{n,1}b_{1,2}\mathcal{R}_{e_2} + a_{n,1}b_{1,2}\left(\sum_{j=3}^{n-2} \mathcal{R}_{e_j}\right), \qquad \mathcal{R}_{[e_{n+2},e_{n+2}]} = 0. \end{cases}$$

At the same time, we deduce that $a_{1,2} = 0$, $b_{n,1} := 2a_{n,1}$, $b_{n,2} := a_{n,2} - (n-3)a_{n,1}b_{1,2}$, $b_{n-1,i} = 0$, $(2 \le i \le n-3)$ and we make such changes in the outer derivations $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$.

(ii) We include a linear combination of e_{n-1} and e_n and obtain

$$\begin{cases} [e_{n+1}, e_i] = -e_i + c_{n-1,i}e_{n-1} + c_{n,i}e_n, \qquad (1 \le i \le n-2), [e_{n+1}, e_k] = c_{n-1,k}e_{n-1} + c_{n,k}e_n, \\ \quad (n-1 \le k \le n+1), \quad [e_{n+1}, e_{n+2}] = (n-4)a_{n,1}b_{1,2}e_2 - \sum_{j=3}^{n-2} a_{n,1}b_{1,2}e_j + c_{n-1,n+2}e_{n-1} \\ \quad + c_{n,n+2}e_n, [e_{n+2}, e_1] = d_{n-1,1}e_{n-1} + d_{n,1}e_n, \quad [e_{n+2}, e_m] = -b_{1,2}e_1 - e_m + d_{n-1,m}e_{n-1} \\ \quad + d_{n,m}e_n, \qquad (2 \le m \le n-2), \quad [e_{n+2}, e_{n-1}] = d_{n-1,n-1}e_{n-1} + d_{n,n-1}e_n, \\ [e_{n+2}, e_n] = d_{n-1,n}e_{n-1} + d_{n,n}e_n, \quad [e_{n+2}, e_{n+1}] = (4-n)a_{n,1}b_{1,2}e_2 + \sum_{j=3}^{n-2} a_{n,1}b_{1,2}e_j \\ \quad + d_{n-1,n+1}e_{n-1} + d_{n,n+1}e_n, [e_{n+2}, e_{n+2}] = d_{n-1,n+2}e_{n-1} + d_{n,n+2}e_n. \end{cases}$$

[2]When we work with the left Leibniz algebras, we interchange s and r in step *(i)* and *(ii)*, the right multiplication operator to the left, and the right Leibniz identity to the left Leibniz identity in step *(iii)*

Besides we have the brackets from outer derivations $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$ and $N_{n,18}$, $(n \geq 4)$.

(iii) We satisfy the right Leibniz identity by checking the derivation property or right Leibniz identities for all possible triples. As a consequence, we obtain that $\mathcal{R}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+2}}$ restricted to the nilradical as $n \times n$ matrices do not change and other brackets are modified to

$$\begin{cases} [e_{n+1}, e_1] = -e_1 + a_{n,1}e_n, \quad [e_{n+1}, e_2] = -e_2 - a_{n-1,2}e_{n-1} + a_{n,2}e_n, \quad [e_{n+1}, e_i] = -e_i \\ \quad - a_{n-1,i}e_{n-1}, \quad (3 \leq i \leq n-3), \quad [e_{n+1}, e_{n-2}] = -e_{n-2}, \quad [e_{n+1}, e_{n-1}] = -2e_{n-1}, \\ [e_{n+1}, e_{n+1}] = c_{n,n+1}e_n, \quad [e_{n+1}, e_{n+2}] = (n-4)a_{n,1}b_{1,2}e_2 - \sum_{j=3}^{n-2} a_{n,1}b_{1,2}e_j + c_{n-1,n+2}e_{n-1} \\ \qquad + \left(c_{n,n+1} - (n-4)a_{n,1}b_{1,2}a_{n,2}\right) e_n, \\ \quad [e_{n+2}, e_2] = -b_{1,2}e_1 - e_2 + \left(a_{n,2} - (n-5)a_{n,1}b_{1,2}\right) e_n, \\ [e_{n+2}, e_i] = -b_{1,2}e_1 - e_i + 2a_{n,1}b_{1,2}e_n, \quad (3 \leq i \leq n-2), \quad [e_{n+2}, e_{n-1}] = -e_{n-1}, \\ [e_{n+2}, e_{n+1}] = (4-n)a_{n,1}b_{1,2}e_2 + \sum_{j=3}^{n-2} a_{n,1}b_{1,2}e_j - c_{n-1,n+2}e_{n-1} + d_{n,n+1}e_n, \\ [e_{n+2}, e_{n+2}] = \left(d_{n,n+1} + (n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2 - (n-4)a_{n,1}b_{1,2}a_{n,2}\right) e_n. \end{cases}$$

Altogether the nilradical $N_{n,18}$, $(n \geq 4)$, the outer derivations and the remaining brackets give us a continuos family of solvable right Leibniz algebras depending on the parameters.

(iv) Finally we apply the change of basis transformations given below to remove all parameters keeping the nilradical $N_{n,18}$ given in (4.1) unchanged.

- The transformation $e_1' = e_1$, $e_i' = e_i + b_{1,2}e_1$, $(2 \leq i \leq n-2)$, $e_j' = e_j$, $(n-1 \leq j \leq n+2)$ removes $b_{1,2}$ in $\mathcal{R}_{e_{n+2}}$ and $-b_{1,2}$ in $\mathcal{L}_{e_{n+2}}$ from the entries in the $(1,2)$, $(1,3), \ldots, (1, n-2)$ positions, respectively. Then we change the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+2}}$ to $a_{n,2} - (n-5)a_{n,1}b_{1,2}$, in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ to $a_{n,2} + a_{n,1}b_{1,2}$. We name the entries in the $(n,3)$, $(n,4), \ldots, (n, n-2)$ positions in $\mathcal{R}_{e_{n+2}}$ by $2a_{n,1}b_{1,2}$, in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ by $a_{n,1}b_{1,2}$.
- Applying the transformation $e_1' = e_1$, $e_i' = e_i - a_{n-1,i}e_{n-1}$, $(2 \leq i \leq n-3)$, $e_j' = e_j$, $(n-2 \leq j \leq n+2)$, we remove $a_{n-1,2}, a_{n-1,3}, \ldots, a_{n-1,n-3}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{n-1,2}, -a_{n-1,3}, \ldots, -a_{n-1,n-3}$ in $\mathcal{L}_{e_{n+1}}$. This transformation affects the coefficients in front of e_{n-1} in $[e_{n+2}, e_{n+1}]$ and $[e_{n+1}, e_{n+2}]$. We change them back to $-c_{n-1,n+2}$ and $c_{n-1,n+2}$, respectively.
- The transformation $e_1' = e_1 - a_{n,1}e_n$, $e_2' = e_2 + \left((n-5)a_{n,1}b_{1,2} - a_{n,2}\right) e_n$, $e_i' = e_i - 2a_{n,1}b_{1,2}e_n$, $(3 \leq i \leq n-2)$, $e_k' = e_k$, $(n-1 \leq k \leq n+2)$ removes the element from the $(n,1)$ position in $\mathcal{R}_{e_{n+1}}$, $\mathcal{R}_{e_{n+2}}$ and $\mathcal{L}_{e_{n+1}}$, the element from the $(n,2)^{nd}$ position in $\mathcal{R}_{e_{n+2}}$ and $\mathcal{L}_{e_{n+2}}$, but it affects the $(n,2)^{nd}$ entries in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$, so we change them to $(n-4)a_{n,1}b_{1,2}$. It also removes the elements from the $(n,i)^{th}$ positions in $\mathcal{R}_{e_{n+2}}$ and $\mathcal{L}_{e_{n+2}}$, but changes them in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ to $-a_{n,1}b_{1,2}$. As the final affect of this transformation, it changes the coefficients in front of e_n in $[e_{n+1}, e_{n+2}]$ and $[e_{n+2}, e_{n+1}]$ to $c_{n,n+1} - (n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2$ and $d_{n,n+1} + (n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2 - (n-4)a_{n,1}b_{1,2}a_{n,2}$, respectively.

We finish the change of basis with the technique of "absorption".

- Applying the transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = e_{n+1} + (4-n)a_{n,1}b_{1,2}e_2 + \sum_{j=3}^{n-2} a_{n,1}b_{1,2}e_j$, $e_{n+2}' = e_{n+2}$, we remove the elements from the entries in the $(n,2)$, $(n,3)$, $(n,4), \ldots, (n, n-2)$ positions in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$. Then we change the coefficient in front of e_n in $[e_{n+1}, e_{n+1}]$ to $c_{n,n+1} - (n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2$. This transformation also removes the coefficients in front of $e_2, e_3, e_4, \ldots, e_{n-2}$ in $[e_{n+2}, e_{n+1}]$ and $[e_{n+1}, e_{n+2}]$.
- We apply transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = e_{n+1} - c_{n-1,n+2}e_{n-1}$, $e_{n+2}' = e_{n+2}$ to remove the coefficients $c_{n-1,n+2}$ and $-c_{n-1,n+2}$ in front of e_{n-1} in $[e_{n+1}, e_{n+2}]$ and $[e_{n+2}, e_{n+1}]$, respectively.

- The transformation $e_i'=e_i,\ (1\le i\le n),\ e_{n+1}'=e_{n+1}+\frac{\left((n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2-c_{n,n+1}\right)e_n}{2}$, $e_{n+2}'=e_{n+2}-\frac{1}{2}\left(d_{n,n+1}+(n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2-(n-4)a_{n,1}b_{1,2}a_{n,2}\right)e_n$ removes the remaining coefficient $c_{n,n+1}-(n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2$ in front of e_n in $[e_{n+1},e_{n+1}]$, $[e_{n+1},e_{n+2}]$, and $d_{n,n+1}+(n-3)(n-4)\left(a_{n,1}b_{1,2}\right)^2-(n-4)a_{n,1}b_{1,2}a_{n,2}$ in front of e_n in $[e_{n+2},e_{n+1}]$, $[e_{n+2},e_{n+2}]$. We obtain the algebra $\mathfrak{g}_{n+2,1}$. To summarize, we formulate the following theorem:

Theorem 5.4. *There is one solvable indecomposable right Leibniz algebra up to isomorphism with a codimension two nilradical $N_{n,18}$, $(n\ge 4)$, which is given below:*

$$\begin{aligned}
&\mathfrak{g}_{n+2,1}:[e_i,e_{n+1}]=e_i,\ [e_{n+1},e_i]=-e_i,\qquad (1\le i\le n-2),\\
&[e_j,e_{n+1}]=2e_j,\qquad (n-1\le j\le n),\\
&[e_{n+1},e_{n-1}]=-2e_{n-1},\quad [e_k,e_{n+2}]=e_k,\quad [e_{n+2},e_k]=-e_k,\quad (2\le k\le n-1),\\
&[e_n,e_{n+2}]=2e_n,\quad DS=[n+2,\ n,\ 2,\ 0],\quad LS=[n+2,\ n,\ n,\ldots].
\end{aligned}$$

6. Classification of solvable indecomposable left Leibniz algebras with a nilradical $N_{n,18}$

6.1. Solvable indecomposable left Leibniz algebras with a codimension one nilradical $N_{n,18}$

Classification of one-dimensional left solvable extensions of $N_{n,18}$, $(n\ge 4)$ follows the same steps as we used for the right solvable Leibniz algebras. We consider the same equation (5.1) and we have the following theorem:

Theorem 6.1. *Set $a_{2,2}:=a$ in (5.1). To satisfy the left Leibniz identity, there are the following cases based on the conditions involving parameters, each gives a continuous family of solvable Leibniz algebras.*

(1) *If $a\ne 0$, $a_{1,1}+a\ne 0$, $2a-a_{1,1}\ne 0$, then*

$$\begin{cases}
[e_1,e_{n+1}]=a_{1,1}e_1+a_{n-1,1}e_{n-1}+\dfrac{a_{1,1}b_{n,1}}{2a-a_{1,1}}e_n,\quad [e_i,e_{n+1}]=a_{1,2}e_1+ae_i+\displaystyle\sum_{k=n-1}^{n}a_{k,i}e_k,\\
[e_{n-1},e_{n+1}]=(a+a_{1,1})e_{n-1},\quad [e_{n+1},e_{n+1}]=a_{n,n+1}e_n,\quad [e_{n+1},e_1]=-a_{1,1}e_1-a_{n-1,1}e_{n-1}\\
\quad +b_{n,1}e_n,[e_{n+1},e_i]=-a_{1,2}e_1-ae_i-a_{n-1,i}e_{n-1}+\left(a_{n,i}-\dfrac{2a_{1,2}b_{n,1}}{2a-a_{1,1}}\right)e_n,\quad (2\le i\le n-2),\\
[e_{n+1},e_{n-1}]=(-a-a_{1,1})e_{n-1},[e_{n+1},e_n]=-2ae_n.
\end{cases}$$

(2) *If $a_{1,1}:=-a$, $a\ne 0$, then the brackets for the algebra are*

$$\begin{cases}
[e_1,e_{n+1}]=-ae_1+a_{n-1,1}e_{n-1}+a_{n,1}e_n,\qquad [e_i,e_{n+1}]=a_{1,2}e_1+ae_i+\displaystyle\sum_{k=n-1}^{n}a_{k,i}e_k,\\
[e_{n+1},e_{n+1}]=a_{n-1,n+1}e_{n-1}+a_{n,n+1}e_n,\qquad [e_{n+1},e_1]=ae_1-a_{n-1,1}e_{n-1}-3a_{n,1}e_n,\\
[e_{n+1},e_i]=-a_{1,2}e_1-ae_i-a_{n-1,i}e_{n-1}+\left(a_{n,i}+\dfrac{2a_{1,2}a_{n,1}}{a}\right)e_n,\qquad (2\le i\le n-2),\\
[e_{n+1},e_n]=-2ae_n.
\end{cases}$$

(3) *If $a=0$, $a_{1,1}\ne 0$, then*

$$\begin{cases}
[e_1,e_{n+1}]=a_{1,1}e_1+a_{n-1,1}e_{n-1}+a_{n,1}e_n,\quad [e_i,e_{n+1}]=a_{1,2}e_1+a_{n-1,i}e_{n-1}+a_{n,i}e_n,\\
[e_{n-1},e_{n+1}]=a_{1,1}e_{n-1},\quad [e_{n+1},e_{n+1}]=a_{n,n+1}e_n,\quad [e_{n+1},e_1]=-a_{1,1}e_1-a_{n-1,1}e_{n-1}-a_{n,1}e_n,\\
[e_{n+1},e_i]=-a_{1,2}e_1-a_{n-1,i}e_{n-1}+b_{n,i}e_n,\quad (2\le i\le n-2),\quad [e_{n+1},e_{n-1}]=-a_{1,1}e_{n-1}.
\end{cases}$$

Remark 6.1. This algebra is the same as the right Leibniz algebra in case (3) in Theorem 5.1.

(4) *If* $a_{1,1} := 2a,\ a \neq 0$, *then*

$$\begin{cases} [e_1, e_{n+1}] = 2ae_1 + a_{n-1,1}e_{n-1} + a_{n,1}e_n, \qquad [e_i, e_{n+1}] = a_{1,2}e_1 + ae_i + \sum_{k=n-1}^{n} a_{k,i}e_k, \\ [e_{n-1}, e_{n+1}] = 3ae_{n-1}, \qquad [e_{n+1}, e_{n+1}] = a_{n,n+1}e_n, \qquad [e_{n+1}, e_1] = -2ae_1 - a_{n-1,1}e_{n-1}, \\ [e_{n+1}, e_i] = -a_{1,2}e_1 - ae_i - a_{n-1,i}e_{n-1} + \left(a_{n,i} - \dfrac{a_{1,2}a_{n,1}}{a}\right)e_n, \qquad (2 \leq i \leq n-2), \\ [e_{n+1}, e_{n-1}] = -3ae_{n-1}, \qquad [e_{n+1}, e_n] = -2ae_n. \end{cases}$$

Proof.

(1) Suppose $a \neq 0,\ a_{1,1} + a \neq 0,\ 2a - a_{1,1} \neq 0$. The proof of this case could be found in Table 2.

(2) Suppose $a_{1,1} := -a,\ a \neq 0$. Considering the same left Leibniz identities as in the generic case, where the identity in step (19) does not give any extra conditions on the parameters, we prove the result.

(3) Suppose $a = 0,\ a_{1,1} \neq 0$. We apply steps (1)–(19) of case (1) and for the last two steps (20) and (21), we consider the identities: $\mathcal{L}_{e_1}([e_{n+1}, e_{n+1}]) = [\mathcal{L}_{e_1}(e_{n+1}), e_{n+1}] + [e_{n+1}, \mathcal{L}_{e_1}(e_{n+1})]$, $\mathcal{L}[e_i, e_{n+1}] = [\mathcal{L}(e_i), e_{n+1}] + [e_i, \mathcal{L}(e_{n+1})]$, $(2 \leq i \leq n-2)$.

(4) Suppose $a_{1,1} := 2a,\ a \neq 0$. We repeat steps (1)–(20) of the generic case and derive the result.

□

Theorem 6.2. *Applying the technique of "absorption" (see Section 3.2), we can further simplify the algebras given in Theorem 6.1 as follows:*

(1) *Suppose* $a \neq 0,\ a_{1,1} + a \neq 0,\ 2a - a_{1,1} \neq 0$. *We have*

$$\begin{cases} [e_1, e_{n+1}] = a_{1,1}e_1 + \dfrac{a_{1,1}b_{n,1}}{2a - a_{1,1}}e_n, [e_2, e_{n+1}] = a_{1,2}e_1 + ae_2 + a_{n-1,2}e_{n-1} + a_{n,2}e_n, \\ [e_i, e_{n+1}] = a_{1,2}e_1 + ae_i + a_{n-1,i}e_{n-1}, [e_{n-2}, e_{n+1}] = a_{1,2}e_1 + ae_{n-2}, [e_{n-1}, e_{n+1}] \\ \qquad = (a + a_{1,1})e_{n-1}, [e_{n+1}, e_1] = -a_{1,1}e_1 + b_{n,1}e_n, [e_{n+1}, e_2] = -a_{1,2}e_1 - ae_2 - a_{n-1,2}e_{n-1} \\ \qquad + \left(a_{n,2} - \dfrac{2a_{1,2}b_{n,1}}{2a - a_{1,1}}\right)e_n, [e_{n+1}, e_i] = -a_{1,2}e_1 - ae_i - a_{n-1,i}e_{n-1} - \dfrac{2a_{1,2}b_{n,1}e_n}{2a - a_{1,1}}, \\ \qquad (3 \leq i \leq n-3), [e_{n+1}, e_{n-2}] = -a_{1,2}e_1 - ae_{n-2} - \dfrac{2a_{1,2}b_{n,1}}{2a - a_{1,1}}e_n, \\ [e_{n+1}, e_{n-1}] = (-a - a_{1,1})e_{n-1}, [e_{n+1}, e_n] = -2ae_n. \end{cases}$$

(2) *Suppose* $a_{1,1} := -a,\ a \neq 0$. *Then the brackets for the algebra are*

$$\begin{cases} [e_1, e_{n+1}] = -ae_1 + a_{n,1}e_n, [e_2, e_{n+1}] = a_{1,2}e_1 + ae_2 + a_{n-1,2}e_{n-1} + a_{n,2}e_n, \\ [e_i, e_{n+1}] = a_{1,2}e_1 + ae_i + a_{n-1,i}e_{n-1}, [e_{n-2}, e_{n+1}] = a_{1,2}e_1 + ae_{n-2}, \\ [e_{n+1}, e_{n+1}] = a_{n-1,n+1}e_{n-1}, [e_{n+1}, e_1] = ae_1 - 3a_{n,1}e_n, [e_{n+1}, e_2] = -a_{1,2}e_1 - ae_2 \\ \quad -a_{n-1,2}e_{n-1} + \left(a_{n,2} + \dfrac{2a_{1,2}a_{n,1}}{a}\right)e_n, [e_{n+1}, e_i] = -a_{1,2}e_1 - ae_i - a_{n-1,i}e_{n-1} + \dfrac{2a_{1,2}a_{n,1}e_n}{a}, \\ \quad (3 \leq i \leq n-3), [e_{n+1}, e_{n-2}] = -a_{1,2}e_1 - ae_{n-2} + \dfrac{2a_{1,2}a_{n,1}}{a}e_n, [e_{n+1}, e_n] = -2ae_n. \end{cases}$$

Table 2. Left Leibniz identities in the generic case in Theorem 6.1.

Steps	Ordered triple	Result
1.	$\mathcal{L}_{e_1}([e_2,e_{n+1}])$	$a_{k,n-1}=0,\ (1\le k\le n-2), a_{n,n-1}:=-a_{2,1}$, $a_{n-1,n-1}:=a_{1,1}+a+\sum_{k=3}^{n-2}a_{k,2}\implies$ $[e_{n-1},e_{n+1}]=\left(a_{1,1}+a+\sum_{k=3}^{n-2}a_{k,2}\right)e_{n-1}-a_{2,1}e_n$.
2.	$\mathcal{L}_{e_2}([e_2,e_{n+1}])$	$[e_n,e_{n+1}]=0\implies a_{k,n}=0,\ (1\le k\le n)$.
3.	$\mathcal{L}[e_{n+1},e_2]$	$a_{1,n+1}=a_{2,n+1}=0\implies[e_{n+1},e_{n+1}]=\sum_{k=3}^{n}a_{k,n+1}e_k$.
4.	$\mathcal{L}[e_{n+1},e_i]$	$a_{i,n+1}=0,\ (3\le i\le n-2)\implies$ $[e_{n+1},e_{n+1}]=a_{n-1,n+1}e_{n-1}+a_{n,n+1}e_n$.
5.	$\mathcal{L}_{e_1}([e_i,e_{n+1}])$	$a_{i,1}:=a_{2,1},\ (3\le i\le n-2)\implies$ $[e_1,e_{n+1}]=a_{1,1}e_1+\sum_{k=2}^{n-2}a_{2,1}e_k+a_{n-1,1}e_{n-1}+a_{n,1}e_n$.
6.	$\mathcal{L}_{e_{i-k}}([e_i,e_{n+1}])$	$a_{1,i}:=a_{1,i-k},\ a_{i-k,i}:=a_{i,i-k},\ (2+k\le i\le n-2,\ 1\le k\le n-4)$, where k is fixed $\implies[e_2,e_{n+1}]=a_{1,2}e_1+ae_2+\sum_{m=3}^{n}a_{m,2}e_m$, $[e_j,e_{n+1}]=$ $a_{1,2}e_1+\sum_{m=2}^{j-1}a_{j,m}e_m+a_{j,j}e_j+\sum_{m=j+1}^{n}a_{m,j}e_m,\ (3\le j\le n-2)$.
7.	$\mathcal{L}[e_1,e_2]$	$b_{k,n-1}=0,\ (1\le k\le n-2), b_{n,n-1}:=b_{2,1}$, $b_{n-1,n-1}:=b_{1,1}+b_{2,2}+\sum_{k=3}^{n-2}b_{k,2}\implies$ $[e_{n+1},e_{n-1}]=\left(b_{1,1}+b_{2,2}+\sum_{k=3}^{n-2}b_{k,2}\right)e_{n-1}+b_{2,1}e_n$.
8.	$\mathcal{L}[e_2,e_2]$	$[e_{n+1},e_n]=2b_{2,2}e_n\implies b_{k,n}=0,\ (1\le k\le n-1),\ b_{n,n}:=2b_{2,2}$.
9.	$\mathcal{L}_{e_2}([e_{n+1},e_1])$	$b_{2,1}=0\implies[e_{n+1},e_{n-1}]=\left(b_{1,1}+b_{2,2}+\sum_{k=3}^{n-2}b_{k,2}\right)e_{n-1}$, $[e_{n+1},e_1]=b_{1,1}e_1+\sum_{k=3}^{n}b_{k,1}e_k$.
10.	$\mathcal{L}_{e_i}([e_{n+1},e_1])$	$b_{i,1}=0,\ (3\le i\le n-2)\implies[e_{n+1},e_1]=b_{1,1}e_1+b_{n-1,1}e_{n-1}+b_{n,1}e_n$.
11.	$\mathcal{L}_{e_i}([e_{n+1},e_i])$	$b_{1,i}:=-a_{1,2},\ (2\le i\le n-2),\ b_{2,2}:=-a,\ b_{j,j}:=a_{j,j}-2a,\ (3\le j\le n-2)$ $\implies b_{n-1,n-1}:=b_{1,1}-a+\sum_{k=3}^{n-2}b_{k,2}$ and $[e_{n+1},e_{n-1}]=(b_{1,1}-a+\sum_{k=3}^{n-2}b_{k,2})e_{n-1},\ [e_{n+1},e_n]=-2ae_n$.
12.	$\mathcal{L}_{e_1}([e_{n+1},e_1])$	$a_{2,1}=0\implies[e_1,e_{n+1}]=a_{1,1}e_1+a_{n-1,1}e_{n-1}+a_{n,1}e_n,\ [e_{n-1},e_{n+1}]=$ $\left(a_{1,1}+a+\sum_{k=3}^{n-2}a_{k,2}\right)e_{n-1}$.
13.	$\mathcal{L}[e_i,e_k]$	$b_{i,k}:=-b_{k,i},\ (2\le i\le n-3,\ 1+i\le k\le n-2)$, where i is fixed.
14.	$\mathcal{L}[e_i,e_i]$	$a_{i,i}:=a,\ (3\le i\le n-2)\implies$ $[e_{n+1},e_j]=-a_{1,2}e_1-\sum_{k=2}^{j-1}b_{j,k}e_k-ae_j+\sum_{k=j+1}^{n}b_{k,j}e_k,\ [e_j,e_{n+1}]=$ $a_{1,2}e_1+\sum_{k=2}^{j-1}a_{j,k}e_k+ae_j+\sum_{k=j+1}^{n}a_{k,j}e_k,\ (2\le j\le n-2)$.
15.	$\mathcal{L}_{e_i}([e_{n+1},e_k])$	$b_{k,i}:=-a_{k,i},\ (2\le i\le n-3,\ 1+i\le k\le n-2)$, where i is fixed $\implies$ $b_{n-1,n-1}:=b_{1,1}-a-\sum_{m=3}^{n-2}a_{m,2}$ and $[e_{n+1},e_{n-1}]=\left(b_{1,1}-a-\sum_{m=3}^{n-2}a_{m,2}\right)e_{n-1}$, $[e_{n+1},e_j]=-a_{1,2}e_1+\sum_{m=2}^{j-1}a_{j,m}e_m-ae_j-\sum_{m=j+1}^{n-2}a_{m,j}e_m+$ $\sum_{m=n-1}^{n}b_{m,j}e_m,\ (2\le j\le n-2)$.
16.	$\mathcal{L}_{e_k}([e_{n+1},e_i])$	$a_{k,i}=0,\ (2\le i\le n-3,\ 1+i\le k\le n-2)$, where i is fixed $\implies$ $[e_j,e_{n+1}]=a_{1,2}e_1+ae_j+\sum_{m=n-1}^{n}a_{m,j}e_m,\ [e_{n+1},e_j]=$ $-a_{1,2}e_1-ae_j+\sum_{m=n-1}^{n}b_{m,j}e_m,\ (2\le j\le n-2)$, $[e_{n-1},e_{n+1}]=\left(a_{1,1}+a\right)e_{n-1},\ [e_{n+1},e_{n-1}]=\left(b_{1,1}-a\right)e_{n-1}$.
17.	$\mathcal{L}_{e_1}([e_{n+1},e_2])$	$b_{1,1}:=-a_{1,1}\implies[e_{n+1},e_1]=-a_{1,1}e_1+$ $b_{n-1,1}e_{n-1}+b_{n,1}e_n,\ [e_{n+1},e_{n-1}]=\left(-a_{1,1}-a\right)e_{n-1}$.
18.	$\mathcal{L}[e_1,e_{n+1}]$	$b_{n-1,1}:=-a_{n-1,1},\ a_{n,1}:=\frac{a_{1,1}b_{n,1}}{2a-a_{1,1}}\implies$ $[e_{n+1},e_1]=-a_{1,1}e_1-a_{n-1,1}e_{n-1}+b_{n,1}e_n$, $[e_1,e_{n+1}]=a_{1,1}e_1+a_{n-1,1}e_{n-1}+\frac{a_{1,1}b_{n,1}}{2a-a_{1,1}}e_n$.
19.	$\mathcal{L}[e_{n+1},e_{n+1}]$	$a_{n-1,n+1}=0\implies[e_{n+1},e_{n+1}]=a_{n,n+1}e_n$.
20.	$\mathcal{L}_{e_i}([e_{n+1},e_{n+1}])$	$b_{n-1,i}:=-a_{n-1,i},\ b_{n,i}:=a_{n,i}-\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}}\implies[e_{n+1},e_i]=$ $-a_{1,2}e_1-ae_i-a_{n-1,i}e_{n-1}+\left(a_{n,i}-\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}}\right)e_n,\ (2\le i\le n-2)$.

(3) *If* $a = 0,\ a_{1,1} \neq 0$, *then*

$$\begin{cases} [e_1, e_{n+1}] = a_{1,1}e_1 + a_{n,1}e_n, [e_2, e_{n+1}] = a_{1,2}e_1 + a_{n-1,2}e_{n-1} + a_{n,2}e_n, [e_i, e_{n+1}] = a_{1,2}e_1 \\ \quad + a_{n-1,i}e_{n-1}, \quad (3 \leq i \leq n-3), \quad [e_{n-2}, e_{n+1}] = a_{1,2}e_1, [e_{n-1}, e_{n+1}] = a_{1,1}e_{n-1}, \\ [e_{n+1}, e_{n+1}] = a_{n,n+1}e_n, [e_{n+1}, e_1] = -a_{1,1}e_1 - a_{n,1}e_n, [e_{n+1}, e_j] = -a_{1,2}e_1 - a_{n-1,j}e_{n-1} \\ \quad + b_{n,j}e_n, (2 \leq j \leq n-3), [e_{n+1}, e_{n-2}] = -a_{1,2}e_1 + b_{n,n-2}e_n, [e_{n+1}, e_{n-1}] = -a_{1,1}e_{n-1}. \end{cases}$$

Remark 6.2. A continuous family of solvable Leibniz algebras is left and right at the same time, because it is the same as in case (3) in Theorem 5.2.

(4) *If* $a_{1,1} := 2a,\ a \neq 0$, *then the brackets for the solvable Leibniz algebra are*

$$\begin{cases} [e_1, e_{n+1}] = 2ae_1 + a_{n,1}e_n, [e_2, e_{n+1}] = a_{1,2}e_1 + ae_2 + a_{n-1,2}e_{n-1} + a_{n,2}e_n, \\ [e_i, e_{n+1}] = a_{1,2}e_1 + ae_i + a_{n-1,i}e_{n-1}, \quad [e_{n-2}, e_{n+1}] = a_{1,2}e_1 + ae_{n-2}, \quad [e_{n-1}, e_{n+1}] = 3ae_{n-1}, \\ [e_{n+1}, e_1] = -2ae_1, [e_{n+1}, e_2] = -a_{1,2}e_1 - ae_2 - a_{n-1,2}e_{n-1} + \left(a_{n,2} - \dfrac{a_{1,2}a_{n,1}}{a}\right) e_n, \\ [e_{n+1}, e_i] = -a_{1,2}e_1 - ae_i - a_{n-1,i}e_{n-1} - \dfrac{a_{1,2}a_{n,1}}{a} e_n, \quad (3 \leq i \leq n-3), \\ [e_{n+1}, e_{n-2}] = -a_{1,2}e_1 - ae_{n-2} - \dfrac{a_{1,2}a_{n,1}}{a} e_n, [e_{n+1}, e_{n-1}] = -3ae_{n-1}, [e_{n+1}, e_n] = -2ae_n. \end{cases}$$

Proof. As for the right Leibniz algebras, once we know how to prove a generic case, we could prove other cases by applying applicable transformations. It turned out that the proof repeats case (1) in Theorem 5.2, except for the last transformation, where we apply instead $e'_i = e_i,\ (1 \leq i \leq n),\ e'_{n+1} = e_{n+1} + \frac{a_{n,n+1}}{2a} e_n$ to remove $a_{n,n+1}$ in the bracket $[e_{n+1}, e_{n+1}]$. □

Theorem 6.3. *There are four solvable indecomposable left Leibniz algebras up to isomorphism with a codimension one nilradical* $N_{n,18}$, $(n \geq 4)$, *which are given below:*

(i)
$$l_{n+1,1} : [e_{n+1}, e_1] = e_1, [e_1, e_{n+1}] = -e_1, [e_{n+1}, e_i] = -e_i, [e_i, e_{n+1}] = e_i,$$
$$(2 \leq i \leq n-2), \quad [e_{n+1}, e_n] = -2e_n, [e_{n+1}, e_{n+1}] = e_{n-1},$$
$$DS = [n+1,\ n,\ 2,\ 0], \quad LS = [n+1,\ n,\ n, \ldots],$$

(ii)
$$\mathfrak{g}_{n+1,2} : [e_{n+1}, e_i] = -e_i, [e_i, e_{n+1}] = e_i, (i = 1,\ n-1), [e_2, e_{n+1}] = \epsilon e_n, (\epsilon = 0, 1),$$
$$[e_{n+1}, e_j] = a_{j-1}e_n, (2 \leq j \leq n-2), \quad [e_k, e_{n+1}] = be_n, \quad (3 \leq k \leq n-2),$$
$$[e_{n+1}, e_{n+1}] = ce_n,$$
$$|a_{m-1}| \leq |a_m|, \quad (3 \leq m \leq n-3),$$
$$DS = [n+1,\ 3,\ 0], \quad LS = [n+1,\ 3,\ 2,\ 2, \ldots].$$

Remark 6.3. To guarantee a continuous family of algebras is unique within its class, when $\epsilon = 0$ and $b = 0$, we may assume that $|a_1| \leq |a_{k-1}|, (3 \leq k \leq n-2)$. If $\epsilon = 1, b = 1, (n \geq 6)$, then we have as well that $|a_1| \leq |a_{k-1}|, (3 \leq k \leq n-2)$. If $\epsilon = 1, b \neq 0$ and $n = 5$, then either $|b| \leq 1$ or $|a_1| \leq |a_2|$.

(iii) $$l_{n+1,3}: [e_{n+1},e_1]=-ae_1, [e_1,e_{n+1}]=ae_1, \qquad [e_{n+1},e_i]=-e_i, [e_i,e_{n+1}]=e_i,$$
$$(2\le i\le n-2), \qquad [e_{n-1},e_{n+1}]=(a+1)e_{n-1},$$
$$[e_{n+1},e_{n-1}]=(-a-1)e_{n-1}, [e_{n+1},e_n]=-2e_n,$$
$$DS=[n+1,\ n,\ 2,\ 0], \qquad LS=[n+1,\ n,\ n,\dots],$$

(iv) $$l_{n+1,4}: [e_{n+1},e_1]=-e_1, [e_1,e_{n+1}]=e_1, \qquad [e_{n+1},e_i]=-e_1-e_i, \qquad [e_i,e_{n+1}]=e_1+e_i,$$
$$(2\le i\le n-2), \qquad [e_{n-1},e_{n+1}]=2e_{n-1}, \qquad [e_{n+1},e_j]=-2e_j, \qquad (n-1\le j\le n),$$
$$DS=[n+1,\ n,\ 2,\ 0], \qquad LS=[n+1,\ n,\ n,\dots].$$

Proof. The idea and the cases are the same as in Theorem 5.3. However there is a slight difference in the transformations. In order to have an insight of that difference, we only show a generic case (1)1. in details.

(1) Suppose $a\neq 0,\ a_{1,1}+a\neq 0,\ 2a-a_{1,1}\neq 0$. We have the left (derivation) and right (not a derivation) multiplication operators restricted to the nilradical

$$\mathcal{L}_{e_{n+1}}=\begin{bmatrix} -a_{1,1} & -a_{1,2} & -a_{1,2} & -a_{1,2} & \cdots & -a_{1,2} & -a_{1,2} & 0 & 0\\ 0 & -a & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdot & 0 & -a & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdot & 0 & 0 & -a & \cdots & 0 & 0 & 0 & 0\\ \cdot & \cdot & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & -a & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & -a & 0 & 0\\ 0 & -a_{n-1,2} & -a_{n-1,3} & -a_{n-1,4} & \cdots & -a_{n-1,n-3} & 0 & -a-a_{1,1} & 0\\ b_{n,1} & a_{n,2}-\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}} & -\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}} & -\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}} & \cdots & -\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}} & -\frac{2a_{1,2}b_{n,1}}{2a-a_{1,1}} & 0 & -2a \end{bmatrix},$$

$$\mathcal{R}_{e_{n+1}}=\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,2} & a_{1,2} & a_{1,2} & \cdots & a_{1,2} & a_{1,2} & 0 & 0\\ 0 & a & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdot & 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdot & 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0\\ \cdot & \cdot & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0\\ 0 & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & a_{n-1,5} & \cdots & a_{n-1,n-3} & 0 & a+a_{1,1} & 0\\ \frac{a_{1,1}b_{n,1}}{2a-a_{1,1}} & a_{n,2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Suppose $a_{1,1} \neq a$.
   - We consider the transformation $e_1' = e_1$, $e_i' = e_i + \frac{a_{1,2}}{a-a_{1,1}} e_1$, $(2 \leq i \leq n-2)$, $e_j' = e_j$, $(n-1 \leq j \leq n+1)$ to remove $a_{1,2}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{1,2}$ in $\mathcal{L}_{e_{n+1}}$ from the entries in the $(1,2)$, $(1,3), \dots, (1, n-2)$ positions, respectively, but other entries are affected as well. As the result we change the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ to $a_{n,2} + \frac{a_{1,2}a_{1,1}b_{n,1}}{(a-a_{1,1})(2a-a_{1,1})}$ and call the entries in the $(n,3)$, $(n,4), \dots, (n, n-2)$ positions in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ by $\frac{a_{1,2}a_{1,1}b_{n,1}}{(a-a_{1,1})(2a-a_{1,1})}$.
   - Applying the transformation $e_1' = e_1 + \frac{b_{n,1}}{2a-a_{1,1}} e_n$, $e_i' = e_i - \frac{a_{n-1,i}}{a_{1,1}} e_{n-1}$, $(2 \leq i \leq n-3)$, $e_j' = e_j$, $(n-2 \leq j \leq n+1)$, we remove $a_{n-1,2}, a_{n-1,3}, \dots, a_{n-1,n-3}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{n-1,2}, -a_{n-1,3}, \dots, -a_{n-1,n-3}$ in $\mathcal{L}_{e_{n+1}}$; $\frac{a_{1,1}b_{n,1}}{2a-a_{1,1}}$ from the $(n,1)^{st}$ entry in $\mathcal{R}_{e_{n+1}}$ and $b_{n,1}$ from the $(n,1)^{st}$ entry in $\mathcal{L}_{e_{n+1}}$ without affecting other entries.
   - We apply the transformation $e_1' = e_1$, $e_2' = e_2 + \left(\frac{a_{n,2}}{a} + \frac{a_{1,2}a_{1,1}b_{n,1}}{a(a-a_{1,1})(2a-a_{1,1})}\right) e_n$, $e_i' = e_i + \frac{a_{1,2}a_{1,1}b_{n,1}}{a(a-a_{1,1})(2a-a_{1,1})} e_n$, $(3 \leq i \leq n-2)$, $e_k' = e_k$, $(n-1 \leq k \leq n+1)$ to remove $a_{n,2} + \frac{a_{1,2}a_{1,1}b_{n,1}}{(a-a_{1,1})(2a-a_{1,1})}$ from the $(n,2)^{nd}$ entry in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$; $\frac{a_{1,2}a_{1,1}b_{n,1}}{(a-a_{1,1})(2a-a_{1,1})}$ from the $(n,i)^{th}$ entries in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$.
   - Finally we apply the transformation $e_i' = e_i$, $(1 \leq i \leq n)$, $e_{n+1}' = \frac{e_{n+1}}{a}$ to scale a to unity. We also rename $\frac{a_{1,1}}{a}$ by $a_{1,1}$ and obtain a Leibniz algebra

$$\begin{cases} [e_1, e_{n+1}] = a_{1,1}e_1, [e_i, e_{n+1}] = e_i, [e_{n-1}, e_{n+1}] = (a_{1,1}+1)e_{n-1}, \\ [e_{n+1}, e_1] = -a_{1,1}e_1, [e_{n+1}, e_i] = -e_i, [e_{n+1}, e_{n-1}] = (-a_{1,1}-1)e_{n-1}, \\ [e_{n+1}, e_n] = -2e_n, (2 \leq i \leq n-2, a_{1,1} \neq 1, a_{1,1} \neq -1, a_{1,1} \neq 2). \end{cases} \tag{6.1}$$

2. If $a_{1,1} = a$, $a \neq 0$, then we obtain

$$\begin{cases} [e_1, e_{n+1}] = e_1, \quad [e_i, e_{n+1}] = a_{1,2}e_1 + e_i, \\ [e_{n-1}, e_{n+1}] = 2e_{n-1}, \quad [e_{n+1}, e_1] = -e_1, \\ [e_{n+1}, e_i] = -a_{1,2}e_1 - e_i, (2 \leq i \leq n-2), \\ [e_{n+1}, e_{n-1}] = -2e_{n-1}, [e_{n+1}, e_n] = -2e_n. \end{cases}$$

   If $a_{1,2} = 0$, then we have a limiting case of (6.1) with $a_{1,1} = 1$. If $a_{1,2} \neq 0$, then we scale it to unity and obtain a Leibniz algebra $l_{n+1,4}$.

(2) Suppose $a_{1,1} := -a$, $a \neq 0$. In this case we have

$$\begin{cases} [e_1, e_{n+1}] = -e_1, \qquad [e_i, e_{n+1}] = e_i, \\ [e_{n+1}, e_{n+1}] = a_{n-1,n+1}e_{n-1}, \qquad [e_{n+1}, e_1] = e_1, \\ [e_{n+1}, e_i] = -e_i, [e_{n+1}, e_n] = -2e_n, \qquad (2 \leq i \leq n-2). \end{cases}$$

If $a_{n-1,n+1} = 0$, then we have a limiting case of (6.1) with $a_{1,1} = -1$. If $a_{n-1,n+1} \neq 0$, then we scale it to unity and it gives us a Leibniz algebra $l_{n+1,1}$.

(3) Suppose $a = 0$ and $a_{1,1} \neq 0$.

Remark 6.4. This case is identically the same as case (3) in Theorem 5.3 and gives us again a solvable Leibniz algebra $\mathfrak{g}_{n+1,2}$.

(4) Suppose $a_{1,1} := 2a$, $a \neq 0$. In this case we obtain a limiting case of (6.1) with $a_{1,1} = 2$. Altogether (6.1) and all its limiting cases after replacing $a_{1,1}$ with a give us a Leibniz algebra $l_{n+1,3}$.

□

6.2. Solvable indecomposable left Leibniz algebras with a codimension two nilradical $N_{n,18}$

The non-zero inner derivations of $N_{n,18}$, $(n \geq 4)$ are given by

$$\mathcal{L}_{e_1} = \sum_{i=2}^{n-2} E_{n-1,i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{L}_{e_i} = -E_{n-1,1} + E_{n,i}, \quad (2 \leq i \leq n-2).$$

Following the same argument as in Section 5.2, we have at most two outer derivations $\mathcal{L}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+2}}$ given in the proof of Theorem 6.3 with $\begin{pmatrix} a_{1,1} \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \alpha_{1,1} \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. We apply the same steps, but with the right Leibniz identity replaced by left and (or) the right multiplication operator replaced by left.

(i) One could find that

$$\begin{cases} \mathcal{L}_{[e_i,e_{n+1}]} = \mathcal{L}_{e_i}, (1 \leq i \leq n-2), \quad \mathcal{L}_{[e_j,e_{n+1}]} = 0, (n-1 \leq j \leq n+1), \\ \mathcal{L}_{[e_{n+1},e_{n+2}]} = (4-n)\alpha_{1,2}b_{n,1}\mathcal{L}_{e_2} + \alpha_{1,2}b_{n,1}\left(\sum_{j=3}^{n-2}\mathcal{L}_{e_j}\right), \\ \mathcal{L}_{[e_1,e_{n+2}]} = 0, \\ \mathcal{L}_{[e_k,e_{n+2}]} = \alpha_{1,2}\mathcal{L}_{e_1} + \mathcal{L}_{e_k}, (2 \leq k \leq n-2), \\ \mathcal{L}_{[e_m,e_{n+2}]} = 0, (n-1 \leq m \leq n), \\ \mathcal{L}_{[e_{n+2},e_{n+1}]} = (n-4)\alpha_{1,2}b_{n,1}\mathcal{L}_{e_2} - \alpha_{1,2}b_{n,1}\left(\sum_{j=3}^{n-2}\mathcal{L}_{e_j}\right), \\ \mathcal{L}_{[e_{n+2},e_{n+2}]} = 0. \end{cases}$$

At the same time we obtain that $a_{1,2} = 0, \beta_{n,1} := 2b_{n,1}, \alpha_{n,2} := a_{n,2} + (n-3)\alpha_{1,2}b_{n,1}, \alpha_{n-1,i} = 0, (2 \leq i \leq n-3)$ and the outer derivations become

$$\mathcal{L}_{e_{n+1}} = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & -a_{n-1,2} & -a_{n-1,3} & -a_{n-1,4} & \cdots & -a_{n-1,n-3} & 0 & -2 & 0 \\ b_{n,1} & a_{n,2} & 0 & 0 & \cdots & 0 & 0 & 0 & -2 \end{bmatrix},$$

$$\mathcal{L}_{e_{n+2}} = \begin{bmatrix} 0 & -\alpha_{1,2} & -\alpha_{1,2} & -\alpha_{1,2} & \cdots & -\alpha_{1,2} & -\alpha_{1,2} & 0 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 \\ 2b_{n,1} & a_{n,2}+(n-5)\alpha_{1,2}b_{n,1} & -2\alpha_{1,2}b_{n,1} & -2\alpha_{1,2}b_{n,1} & \cdots & -2\alpha_{1,2}b_{n,1} & -2\alpha_{1,2}b_{n,1} & 0 & -2 \end{bmatrix}.$$

(ii) We include a linear combination of e_{n-1} and e_n to have the following brackets:

$$\begin{cases} [e_i, e_{n+1}] = e_i + c_{n-1,i}e_{n-1} + c_{n,i}e_n, \quad (1 \le i \le n-2), \quad [e_k, e_{n+1}] = c_{n-1,k}e_{n-1} + c_{n,k}e_n, \\ \quad (n-1 \le k \le n+1), [e_{n+2}, e_{n+1}] = (n-4)\alpha_{1,2}b_{n,1}e_2 - \sum_{j=3}^{n-2}\alpha_{1,2}b_{n,1}e_j + c_{n-1,n+2}e_{n-1} \\ \quad + c_{n,n+2}e_n, [e_1, e_{n+2}] = d_{n-1,1}e_{n-1} + d_{n,1}e_n, [e_m, e_{n+2}] \\ \qquad = \alpha_{1,2}e_1 + e_m + d_{n-1,m}e_{n-1} + d_{n,m}e_n, (2 \le m \le n-2), \\ \quad [e_{n-1}, e_{n+2}] = d_{n-1,n-1}e_{n-1} + d_{n,n-1}e_n, \\ \quad [e_n, e_{n+2}] = d_{n-1,n}e_{n-1} + d_{n,n}e_n, \\ \quad [e_{n+1}, e_{n+2}] = (4-n)\alpha_{1,2}b_{n,1}e_2 + \sum_{j=3}^{n-2}\alpha_{1,2}b_{n,1}e_j + d_{n-1,n+1}e_{n-1} + d_{n,n+1}e_n, \\ [e_{n+2}, e_{n+2}] = d_{n-1,n+2}e_{n-1} + d_{n,n+2}e_n. \end{cases}$$

(iii) We notice that $\mathcal{L}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+2}}$ restricted to the nilradical do not change, but the remaining brackets are given below.

$$\begin{cases} [e_1, e_{n+1}] = e_1 + b_{n,1}e_n, [e_2, e_{n+1}] = e_2 + a_{n-1,2}e_{n-1} + a_{n,2}e_n, [e_i, e_{n+1}] = e_i + a_{n-1,i}e_{n-1}, \\ \qquad (3 \le i \le n-3), [e_{n-2}, e_{n+1}] = e_{n-2}, [e_{n-1}, e_{n+1}] = 2e_{n-1}, [e_{n+1}, e_{n+1}] = c_{n,n+1}e_n, \\ [e_{n+2}, e_{n+1}] = (n-4)\alpha_{1,2}b_{n,1}e_2 - \sum_{j=3}^{n-2}\alpha_{1,2}b_{n,1}e_j + c_{n-1,n+2}e_{n-1} + (c_{n,n+1} \\ \qquad + (n-4)\alpha_{1,2}b_{n,1}a_{n,2})e_n, [e_2, e_{n+2}] = \alpha_{1,2}e_1 + e_2 + (a_{n,2} + (n-3)\alpha_{1,2}b_{n,1})e_n, \\ [e_i, e_{n+2}] = \alpha_{1,2}e_1 + e_i, (3 \le i \le n-2), [e_{n-1}, e_{n+2}] = e_{n-1}, \\ [e_{n+1}, e_{n+2}] = (4-n)\alpha_{1,2}b_{n,1}e_2 + \sum_{j=3}^{n-2}\alpha_{1,2}b_{n,1}e_j - c_{n-1,n+2}e_{n-1} + d_{n,n+1}e_n, \\ [e_{n+2}, e_{n+2}] = \left(d_{n,n+1} + (n-4)\alpha_{1,2}b_{n,1}a_{n,2} + (n-4)(n-3)\left(\alpha_{1,2}b_{n,1}\right)^2\right)e_n. \end{cases}$$

(iv) We apply the following change of basis transformations:

- Applying the transformation $e_1' = e_1, e_i' = e_i + \alpha_{1,2}e_1, (2 \le i \le n-2), e_j' = e_j, (n-1 \le j \le n+2)$, we remove $\alpha_{1,2}$ in $\mathcal{R}_{e_{n+2}}$ and $-\alpha_{1,2}$ in $\mathcal{L}_{e_{n+2}}$ from the entries in the $(1,2)$, $(1,3), \ldots, (1, n-2)$ positions, respectively. In $\mathcal{L}_{e_{n+2}}$ the transformation removes $-2\alpha_{1,2}b_{n,1}$ from the entries in the $(n,3)$, $(n,4), \ldots, (n, n-2)$ positions, and we rename the $(n,2)^{nd}$ entry by $a_{n,2} + (n-3)\alpha_{1,2}b_{n,1}$ there. This transformation affects other entries as well.

In $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$ we call the entries in the $(n,3),\ (n,4),\dots,(n,n-2)$ positions by $\alpha_{1,2}b_{n,1}$ and rename the entry in the $(n,2)^{nd}$ position by $a_{n,2}+\alpha_{1,2}b_{n,1}$.

- Considering the transformation $e_1'=e_1,\ e_i'=e_i-a_{n-1,i}e_{n-1},\ (2\le i\le n-3),\ e_j'=e_j,\ (n-2\le j\le n+2)$, we remove $a_{n-1,2},\ a_{n-1,3},\dots,a_{n-1,n-3}$ in $\mathcal{R}_{e_{n+1}}$ and $-a_{n-1,2},\ -a_{n-1,3},\dots,-a_{n-1,n-3}$ in $\mathcal{L}_{e_{n+1}}$. This transformation affects the coefficients in front of e_{n-1} in $[e_{n+1},e_{n+2}]$ and $[e_{n+2},e_{n+1}]$, which we change back to $-c_{n-1,n+2}$ and $c_{n-1,n+2}$, respectively.
- The transformation $e_1'=e_1+b_{n,1}e_n,\ e_2'=e_2+\left(a_{n,2}+(n-3)\alpha_{1,2}b_{n,1}\right)e_n,\ e_i'=e_i+\alpha_{1,2}b_{n,1}e_n,\ (3\le i\le n-2),\ e_k'=e_k,\ (n-1\le k\le n+2)$ removes the element from the $(n,1)^{st}$ position in $\mathcal{R}_{e_{n+1}}$, $\mathcal{L}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+2}}$, removes $a_{n,2}+(n-3)\alpha_{1,2}b_{n,1}$ from the $(n,2)^{nd}$ position in $\mathcal{R}_{e_{n+2}}$ and $\mathcal{L}_{e_{n+2}}$. This transformation affects the $(n,2)^{nd}$ entries in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$, so we change them to $(4-n)\alpha_{1,2}b_{n,1}$. It also removes the elements from the $(n,i)^{th}$ positions in $\mathcal{R}_{e_{n+1}}$ and $\mathcal{L}_{e_{n+1}}$, but introduces $-\alpha_{1,2}b_{n,1}$ in $\mathcal{R}_{e_{n+2}}$ and $\mathcal{L}_{e_{n+2}}$ in those positions. It changes the coefficients in front of e_n in $[e_{n+1},e_{n+2}]$ and $[e_{n+2},e_{n+1}]$ to $d_{n,n+1}+(n-4)\alpha_{1,2}b_{n,1}a_{n,2}+(n-4)^2\left(\alpha_{1,2}b_{n,1}\right)^2$ and $c_{n,n+1}-(n-4)^2\left(\alpha_{1,2}b_{n,1}\right)^2$, respectively.
- We apply the transformation $e_1'=e_1,\ e_2'=e_2+(4-n)\alpha_{1,2}b_{n,1}e_n,\ e_i'=e_i,\ (3\le i\le n+2)$ to remove the element from the $(n,2)^{nd}$ position in $\mathcal{L}_{e_{n+1}}$ and $\mathcal{R}_{e_{n+1}}$, but it introduces $(n-4)\alpha_{1,2}b_{n,1}$ in the same position in $\mathcal{L}_{e_{n+2}}$ and $\mathcal{R}_{e_{n+2}}$. This transformation also changes the coefficient in front of e_n in $[e_{n+1},e_{n+2}]$ and $[e_{n+2},e_{n+1}]$ to $d_{n,n+1}+(n-4)\alpha_{1,2}b_{n,1}a_{n,2}$ and $c_{n,n+1}$, respectively.

  We finish with carrying out the technique of "absorption."
- Applying the transformation $e_i'=e_i,\ (1\le i\le n+1),\ e_{n+2}'=e_{n+2}+(4-n)\alpha_{1,2}b_{n,1}e_2+\sum_{j=3}^{n-2}\alpha_{1,2}b_{n,1}e_j$, we remove the elements from the $(n,2),(n,3),\dots,(n,n-2)$ positions in $\mathcal{L}_{e_{n+2}}$ and $\mathcal{R}_{e_{n+2}}$. It also removes the coefficients in front of $e_2,\ e_3,\ e_4,\dots,e_{n-2}$ in $[e_{n+2},e_{n+1}]$ and $[e_{n+1},e_{n+2}]$ as well. This transformation changes the coefficient in front of e_n in $[e_{n+2},e_{n+2}]$ to $d_{n,n+1}+(n-4)\alpha_{1,2}b_{n,1}a_{n,2}$.
- The transformation $e_i'=e_i,\ (1\le i\le n+1),\ e_{n+2}'=e_{n+2}-\frac{c_{n-1,n+2}}{2}e_{n-1}$ removes the coefficients $c_{n-1,n+2}$ and $-c_{n-1,n+2}$ from e_{n-1} in $[e_{n+2},e_{n+1}]$ and $[e_{n+1},e_{n+2}]$, respectively.
- Finally applying the transformation $e_i'=e_i,\ (1\le i\le n),\ e_{n+1}'=e_{n+1}+\frac{c_{n,n+1}}{2}e_n,\ e_{n+2}'=e_{n+2}+\frac{d_{n,n+1}+(n-4)\alpha_{1,2}b_{n,1}a_{n,2}}{2}e_n$, we remove the coefficient $c_{n,n+1}$ in front of e_n in $[e_{n+1},e_{n+1}]$ and $[e_{n+2},e_{n+1}]$, the coefficient $d_{n,n+1}+(n-4)\alpha_{1,2}b_{n,1}a_{n,2}$ in front of e_n in $[e_{n+1},e_{n+2}]$ and $[e_{n+2},e_{n+2}]$ and obtain the algebra $l_{n+2,1}$. This proves the following:

Theorem 6.4. *There is one solvable indecomposable left Leibniz algebra up to isomorphism with a codimension two nilradical $N_{n,18}$, $(n\ge 4)$, which is given below:*

$$
\begin{aligned}
&l_{n+2,1}:[e_i,e_{n+1}]=e_i,\quad [e_{n+1},e_i]=-e_i,\qquad (1\le i\le n-2),\\
&[e_{n-1},e_{n+1}]=2e_{n-1},\\
&[e_{n+1},e_j]=-2e_j,\qquad (n-1\le j\le n),\qquad [e_k,e_{n+2}]=e_k,\\
&[e_{n+2},e_k]=-e_k,\qquad (2\le k\le n-1),\\
&[e_{n+2},e_n]=-2e_n,\\
&DS=[n+2,\ n,\ 2,\ 0],\qquad LS=[n+2,\ n,\ n,\dots].
\end{aligned}
$$

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