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II. Non-Associative Structures

E. N. Kuz'min, I. P. Shestakov

Translated from the Russian
by R. M. Dimitrić

Contents

Both mathematics of the 20th century and theoretical physics include the methods of non-associative algebras in their arsenals more and more actively. It suffices to mention the Jordan algebras that had grown as an apparatus of quantum mechanics. On the other hand, Lie algebras, being non-associative, reflect fundamental properties of such associative objects as Lie groups. The present survey includes basic classes of non-associative algebras, close to a certain degree to the associative algebras: alternative, Jordan and Malcev algebras. We tried, as much as possible, to point out their applications in different areas of mathematics. A separate section is devoted to the survey of the theory of quasigroups and loops. Sections 1–4 have been written by I.P. Shestakov, whereas Sect. 5 and 6 have been written by E.N. Kuz'min. The authors are genuinely grateful to V.D. Belousov who has given them essential assistance in the work on Sect. 6. The authors are also grateful to A.I. Kostrikin and to I.R. Shafarevich for their constructive remarks directed towards the improvement of the manuscript of this survey.

Enumeration of formulas in different sections is independent; when refering to a formula from a different section, the section number is added in front of the formula number.

Note that the references cited with the results are not necessarily pointing to the first authors of those results.

§1. Introduction to Non-Associative Algebras

1.1. The Main Classes of Non-Associative Algebras. Let A be a vector space over a field F. Let us assume that a bilinear multiplication of vectors is defined on A, i.e. the mapping $(u, v) \mapsto uv$ from $A \times A$ into A is given, with the following conditions:

$$(\alpha u + \beta v)w = \alpha(uw) + \beta(vw); \quad u(\alpha v + \beta w) = \alpha(uv) + \beta(uw), \tag{1}$$

for all $\alpha, \beta \in F$; $u, v, w \in A$. In this case, the vector space A, together with the multiplication defined on it is called an *algebra* over the field F.

An algebra over an associative-commutative ring Φ, with unity is defined analogously. It is a left unitary Φ-module A with the product $uv \in A$, satisfying conditions (1), for $\alpha, \beta \in \Phi$. One of the advantages of the notion of an algebra over Φ (or a Φ-algebra) is that it allows for a study of algebras over fields and rings at the same time; the latter are obtained from a Φ-algebra when $\Phi = \mathbb{Z}$ is the ring of integers. We will be primarily interested in algebras over fields.

Every finite-dimensional algebra A over the field F may be defined by a "multiplication table" $e_i e_j = \sum_{k=1}^n \gamma_{ij}^k e_k$, where $e_1, \ldots, e_n$ is an arbitrary basis of A and $\gamma_{ij}^k \in F$ are the so-called *structural constants of the algebra*, corresponding to the given basis. Every collection γ_{ij}^k defines an algebra.

The just introduced notion of an algebra is too general to lead to interesting structural results (cf. examples in 1.2). In order to get such results, we need to impose some additional conditions on the operation of multiplication. Depending on the form of the imposed restrictions, different classes of algebras are obtained.

One of the most natural restrictions is that of *associativity* of multiplication

$$(xy)z = x(yz) \tag{2}$$

This is obviously satisfied when the elements of the algebra A are mappings of a set into itself and when the composition of mappings is taken as multiplication. One can show that every associative algebra is isomorphic to an algebra of linear transformations of an appropriate vector space. Thus, the condition of associativity of multiplication characterizes the algebras of linear transformations (with composition as multiplication).

The class of associative algebras assumes an important place in the theory of algebras and it is most thoroughly studied. In mathematics and its applications, however, other classes of algebras where condition (2) is not satisfied arise often. Such algebras are called *non-associative.*

The first class of non-associative algebras that was subject to serious and systematic study, was that of Lie algebras, that first arising in the theory of Lie groups. An algebra L is called a *Lie algebra,* if its operation of multiplication is *anticommutative,* i.e.

$$x^2 = 0 \tag{3}$$

and if it satisfies the *Jacobi identity*

$$J(x, y, z) \equiv (xy)z + (yz)x + (zx)y = 0. \tag{4}$$

If A is an associative algebra, then the algebra $A^{(-)}$ obtained by introducing a new multiplication on the vector space A, with the aid of the *commutator*

$$[x, y] \equiv xy - yx,$$

satisfies conditions (3) and (4) and consequently is a Lie algebra. This example is quite general, since the Poincaré-Birkhoff-Witt theorem implies that every Lie algebra over a field is isomorphic to a subalgebra of the algebra $A^{(-)}$, for a suitable associative algebra A.

Lie algebras have a rather developed theory, finding applications in different areas of mathematics. An extensive literature is devoted to them, and among them a sequence of surveys in this series. In our paper they will play a secondary role, appearing only marginally, basically as algebras of derivations of other algebras.

In analogy with the commutator or the Lie multiplication $[x, y]$ in an associative algebra A, we may introduce a symmetric *(Jordan) multiplication*

$$x \circ y = xy + yx.$$

Over the fields of characteristic $\neq 2$, however, it is more suitable to consider the operation

$$x \cdot y = \frac{1}{2}(xy + yx),$$

since in this case, the powers of an element x in the algebra A coincide with its powers with respect to the operation $(\cdot)$. The algebra obtained after introducing the multiplication $x \cdot y$ on the vector space A is denoted by $A^{(+)}$. We note that the mapping $x \mapsto \frac{1}{2}x$ establishes an isomorphism between the algebra $A^{(+)}$ with the corresponding algebra and the multiplication operation $x \circ y$.

The algebra $A^{(+)}$ is *commutative* i.e. satisfies the equality

$$xy = yx, \tag{5}$$

and generally speaking is not associative, although it satisfies the following weak associativity law

$$x^2(yx) = (x^2y)x. \tag{6}$$

The algebras satisfying identities (5) and (6) are called *Jordan algebras*.

Jordan algebras appeared first in 1934 in the joint paper by Jordan, von Neumann and Wigner (1934). In the ordinary interpretation of quantum mechanics the observables are Hermitian matrices or the Hermitian operators on a Hilbert space. The linear space of Hermitian matrices is not closed with respect to the ordinary product xy, but it is closed with respect to the symmetrized product $x \cdot y$. The program suggested by Jordan, consisted in at first singling out basic algebraic properties of Hermitian matrices in terms of the operation $x \cdot y$, and then in studying all the algebraic systems satisfying those properties. The authors hoped that in this process, new algebraic systems would be found, that would give a more suitable interpretation of quantum mechanics. They had chosen identities (5) and (6), satisfied by the operation $x \cdot y$, to be the basic properties. Although this path did not give any intrinsic generalizations of the matrix formalism of quantum mechanics, the class of algebras introduced by these authors had attracted attention of algebraists. The theory of Jordan algebras had started developing fast and soon thereafter its interesting applications in real and complex analysis, in the theory of symmetric spaces, in Lie groups and algebras had been found. In recent times the Jordan algebras again attract physicists in searching for models for explanations of properties of elementary particles. In relation to the physical theory of supersymmetry, Jordan superalgebras have appeared and studies have begun on them.

For an associative algebra A, the algebras of the form $A^{(+)}$ and their subalgebras are called *special Jordan algebras*. They are already not as such universal examples of Jordan algebras as the algebras $A^{(-)}$ and their subalgebras in the case of Lie algebras. There exist Jordan algebras, not isomorphic to subalgebras of the algebra $A^{(+)}$, for any associative algebra A. Such algebras are called *exceptional*.

The study of exceptional Jordan algebras intrinsically relies on the knowledge of properties of algebras of another class which is somewhat wider than the class of associative algebras. These are so-called *alternative algebras* defined by the identities

$$x^2y = x(xy), \tag{7}$$
$$yx^2 = (yx)x, \tag{8}$$

first of which is called the identity of *left alternativity* and the second – the identity of *right alternativity*. It is clear that every associative algebra is alternative. On the other hand, according to Artin's theorem (see 2.3 in the sequel), every two elements in an alternative algebra generate an associative subalgebra, thus alternative algebras are sufficiently close to the associative ones. A classical example of an alternative non-associative algebra is the famous algebra of Cayley numbers, that was constructed as far back as in 1845 by A. Cayley. This algebra and its generalizations – so-called Cayley-Dickson algebras – play an important role in the theory of alternative algebras and their applications in algebra and geometry.

If A is an alternative non-associative algebra, then the commutator algebra $A^{(-)}$ is not a Lie algebra. However, it is not difficult to show that, in this case, the algebra $A^{(-)}$ satisfies the following *Malcev identity*:

$$J(x, y, xz) = J(x, y, z)x, \tag{9}$$

where $J(x, y, z) \equiv (xy)z + (yz)x + (zx)y$ is the Jacobian of the elements x, y, z. An anticommutative algebra satisfying identity (9), is called a *Malcev algebra*. This class of algebras was first introduced by A. I. Malcev in 1955 (under the name of "Moufang-Lie algebras") in studies on analytic Moufang loops; the Malcev algebras are related to them in approximately the same way as the Lie algebras are to Lie groups. Every Lie algebra is a Malcev algebra; on the other hand, every two-generated Malcev algebra is a Lie algebra. The latter condition defines the class of *binary Lie algebras*, wider than the class of Malcev algebras. If the characteristic is not equal to 2, this class may be defined by the following identities

$$x^2 = 0, \quad J(xy, x, y) = 0.$$

Alternative algebras, Jordan algebras as well as Malcev algebras, along with Lie algebras are the main and the best researched classes of non-associative algebras. All of them are in one or another way closely related to associative algebras (the Malcev algebras, through the alternative algebras), and for this reason they are sometimes united under the general name of "almost associative algebras". The main portion of this survey is exactly devoted to these classes of algebras (except of Lie algebras). There are, after all, other classes of non-associative algebras with quite satisfactory structure theories. We will consider some of them in Sect. 4. Nonetheless, the almost associative algebras that arose on the meeting of ring theory with other mathematical areas remain still the richest, from the point of view of applications and relations. Besides, the methods of their research are fairly universal and may be applied (and are being applied successfully) in the studies of other classes of algebras.

1.2. General Properties of Non-Associative Algebras. Numerous notions and results in the theory of associative algebras in fact do not use the associativity property and carry over without changes to arbitrary algebras. Some notions of this kind are definitions of subalgebras, one- and two-sided ideals, simple algebras, direct sums of algebras, homomorphisms, quotient algebras etc. The fundamental homomorphism theorems remain valid for arbitrary algebras too.

At the same time, there is a series of important notions, whose definitions intrinsically use the associativity property, thus not allowing for automatic expansion to arbitrary algebras. For instance, the power a^n of an element a and the power A^n of an algebra A is not, in general, a uniquely definable notion because, in a non-associative algebra, the result of multiplication of n elements depends on the arrangement of the brackets in the product. In particular, this product may equal to zero with one arrangement of the brackets and non-zero with another (even if all the elements are equal). Thus there are several analogues of nilpotency in the theory of non-associative algebras. The most important of them all are solvability and nilpotency.

Let A be an arbitrary algebra. If B and C are subspaces of A, then BC will denote the linear subspace generated by all the products bc, where $b \in B$, $c \in C$. We set $A^1 = A^{(0)} = A$, and further by induction

$$A^{n+1} = \sum_{i+j=n+1} A^i \cdot A^j, \quad A^{(n+1)} = A^{(n)} \cdot A^{(n)}.$$

An algebra A is called *nilpotent*, if there is an n such that $A^n = 0$ and is called *solvable*, if $A^{(m)} = 0$, for some m. The smallest numbers n and m with these properties are respectively called the *nilpotency index* and the *solvability index* of the algebra A. It is easy to see that the algebra A is nilpotent of index n, if and only if the product of any n of its elements, with any arrangement of the brackets equals zero and if there exists a non-zero product of $n-1$ elements. Every nilpotent algebra is solvable, but the converse is not generally true.

Proposition. *The sum of two solvable (two-sided) ideals of the algebra A is again a solvable ideal. If A is finite-dimensional, then A contains the greatest solvable ideal $S = S(A)$. Moreover, the factor algebra A/S does not contain non-zero solvable ideals.*

The ideal $S(A)$ defined in the proposition, is called the *solvable radical* of the finite-dimensional algebra A. In general, an ideal I of a (not necessarily finite-dimensional) algebra A, with certain property $\mathcal{R}$, is called an *$\mathcal{R}$-radical* of the algebra A and is denoted by $I = \mathcal{R}(A)$, if I contains all the ideals of the algebra A with the property $\mathcal{R}$ and the quotient algebra A/I does not contain such non-zero ideals (i.e. $\mathcal{R}(A/I) = 0$). In addition, it is assumed that the property $\mathcal{R}$ is preserved under homomorphisms (the class of $\mathcal{R}$-algebras is homomorphically closed).

The notion of a radical is one of fundamental instruments in constructing the structure theory of various classes of algebras. After a successful choice of a radical, everything reduces to description of the *radical algebras* (i.e. algebras A for which $A = \mathcal{R}(A)$) and the *semisimple algebras* (i.e. algebras A for which $\mathcal{R}(A) = 0$); arbitrary algebras are then described as extensions of the semisimple ones, by the radical ones. At first this method was used by Molien and Wedderburn, the founders of the structure theory of finite-dimensional associative algebras. They have considered a maximal nilpotent ideal of an algebra A as the radical $\mathcal{R}(A)$; the semisimple algebras were described as the direct sums of the full matrix algebras over division rings.

In the non-associative case, the class of nilpotent algebras, unlike the solvable case, is not closed under extensions (i.e. an algebra A may contain a nilpotent ideal I with a nilpotent quotient algebra A/I, but not be nilpotent itself.). Hence, the *nilpotent radical* does not exist in all the finite-dimensional algebras (for instance it does not exist in Lie algebras). Moreover, a finite-dimensional algebra may in general contain several different maximal nilpotent ideals. In those cases, the solvable radical $S(A)$ comes out to play the major role. It lies in the essence of the structure theories of finite-dimensional Lie algebras and finite-dimensional Malcev algebras of zero characteristic; on the other hand, in the cases of finite-dimensional alternative and Jordan algebras, where the nilpotent radical exists, $S(A)$ coincides with this radical.

Let us consider a few examples showing that in general it is difficult to count on a satisfactory structure theory of finite-dimensional algebras.

Example 1. Let A be an algebra over a field F, with a basis e_1, e_2, a, b and the following non-zero products of basis elements: $ae_1 = \epsilon e_1 a = e_2, be_2 = \epsilon e_2 b = e_1$, where $0 \neq \epsilon \in F$. Then $I_1 = Fe_1 + Fe_2 + Fa, I_2 = Fe_1 + Fe_2 + Fb$ are different maximal nilpotent ideals in A. By choosing $\epsilon = 1$ or $\epsilon = -1$ we obtain a commutative or anticommutative algebra A.

Example 2. Let $A_1, \dots, A_k$ be simple algebras over a field F with bases $\{v_i^{(1)} \mid i \in I_1\}, \dots, \{v_i^{(k)} \mid i \in I_k\}$. Let us consider the algebra $A = Fe + A_1 + \dots + A_k$ with multiplication, defined by the following conditions: a) A_i are subalgebras of A; b) $A_i A_j = 0$, for $i \neq j$; c) $ev_i^{(j)} = v_i^{(j)} e = e$, for all i, j; d) $e^2 = e$. Then $I = Fe$ is the unique minimal ideal in A, and $I^2 = I$. In particular, $S(A) = 0$, but A does not decompose into a direct sum.

We point out that if all the algebras A_i in this example are commutative, then A is commutative too. If all the A_i are anticommutative, then we may consider $\tilde{A} = A + Ff$ and replace conditions c) and d) by the following: c') $ev_i^{(j)} = -v_i^{(j)} e = f$, $v_i^{(j)} f = -f v_i^{(j)} = e$; d') $ef = -fe = f$. Then $\tilde{A}$ is an anticommutative algebra with a unique minimal ideal $I = Fe + Ff$, $I^2 \neq 0$ and again $S(\tilde{A}) = 0$, but $\tilde{A}$ does not decompose into a direct sum.

One more approach to the notion of a radical of a non-associative algebra is possible: we can take the radical of an algebra A to be the smallest ideal $\tilde{N}$, for which the quotient algebra $A/\tilde{N}$ decomposes into the direct sum of simple algebras. Such a radical exists in every finite-dimensional algebra A and, satisfies the condition $\tilde{N}(A/\tilde{N}) = 0$; moreover $\tilde{N}(A) = S(A)$ in the algebras we have mentioned above, while in general $\tilde{N}(A) \supseteq S(A)$. However, this radical is not only necessarily nilpotent or solvable, but can even be a simple algebra. For example, for the algebra A in Example 2, $\tilde{N}(A) = Fe \cong F$.

The following example shows that the simple finite-dimensional algebras also form a rather big class, which implies that their complete description can hardly be done, even in the case of small dimensions and an algebraically closed field.

Example 3. Let us consider the algebra $A = A(\alpha_{ij})$ over the field F with a basis $e_1, \dots, e_n$ and the multiplication table of the form $e_i e_j = \alpha_{ij} e_j$, where $0 \neq \alpha_{ij} \in F, i, j = 1, \dots, n$ and all the columns in the matrix (α_{ij}) are different. A peculiarity of the algebra $A(\alpha_{ij})$ consists in the fact that in a given basis, the matrices of operators of left multiplications $L_x : y \mapsto xy$ have

a diagonal form and are therefore commuting. Consequently all the algebras $A(\alpha_{ij})$ satisfy the identity

$$x(yz) = y(xz).$$

It is not difficult to show that $A(\alpha_{ij})$ is a simple algebra: in addition $A(\alpha_{ij}) \cong A(\beta_{ij})$ if and only if, $\alpha_{ij} = \lambda_i \beta_{\sigma(i)\sigma(j)}$, where $0 \neq \lambda_i \in F, i = 1, \dots, n; \sigma \in S_n$. If, in addition, we set $\alpha_{i1} = 1, i = 1, \dots, n$, then to every matrix (α_{ij}) there corresponds only a finite number ($\leqslant n!$) of matrices (β_{ij}) of the same type, for which $A(\alpha_{ij}) \cong A(\beta_{ij})$. Consequently, the aforementioned simple algebras form a family that depends on $n^2 - n$ "independent" parameters.

An element a of an algebra A is called *nilpotent* if the algebra generated by it in A is nilpotent. If all the elements of an algebra (ideal) are nilpotent, then such an algebra (ideal) is called a *nilalgebra* (a *nilideal*). In general the class of nilalgebras is not closed with respect to extensions. On the other hand, this condition is satisfied under the additional conditions of associativity of powers or power-associativity, defined in the sequel.

An algebra A is called a *power-associative algebra*, if its every element lies in an associative subalgebra. It is not difficult to show that all the algebras considered in 1.1 are power-associative. Over a field of characteristic 0, the class of power-associative algebras may be defined by identities

$$(x^2)x = x(x^2), \quad (x^2x)x = x^2x^2.$$

The powers a^n ($n \geqslant 1$) of an element a are defined in a natural way in every power-associative algebra; in addition the equalities $(a^n)^m = a^{nm}, a^n a^m = a^{n+m}$ hold, and the element a is nilpotent if and only if $a^n = 0$, for some n.

Just as for the associative algebras, the following is proved in a standard way:

Proposition. *Every power-associative algebra A contains a unique maximal two-sided nilideal $Nil(A)$; moreover, the quotient algebra $A/Nil A$ does not contain two sided non-zero nilideals (i.e. it is a nilsemisimple algebra).*

The ideal $\mathrm{Nil}\, A$ is called the *nilradical* of the algebra A. If A is a finite dimensional power-associative algebra, then $S(A) \subseteq \mathrm{Nil}\, A$; the inclusion may be strict, as the example of a Lie algebra shows, where $\mathrm{Nil}\, A = A$. We will see in the sequel that, for finite-dimensional alternative and Jordan algebras the ideal $\mathrm{Nil}\, A$ is nilpotent. In particular, in these cases $\mathrm{Nil}\, A = S(A)$. In the case of finite-dimensional commutative power-associative algebras the question on equality of the radicals $S(A)$ and $\mathrm{Nil}\, A$ is open and is known as Albert's problem. The following example shows that in this case the ideal $\mathrm{Nil}\, A$ is not necessarily nilpotent.

Example 4 (Suttles, 1972). Let A be a commutative algebra over a field of characteristic $\neq 2$, with the basis $\{e_1, e_2, e_3, e_4, e_5\}$ and the following multiplication table:

$$e_1e_2 = e_2e_4 = -e_1e_5 = e_3, \quad e_1e_3 = e_4, \quad e_2e_3 = e_5;$$

and all the other products are zero. Then A is a solvable power-associative nilalgebra of index 4, which is not nilpotent.

It is not difficult to see that Albert's problem is equivalent to the following question: are there any simple finite-dimensional commutative power-associative nilalgebras? The answer to this question is not known even without the assumption on power-associativity. The structure of nilsemisimple finite-dimensional commutative power-associative algebras is known.

Theorem (Albert, 1950, Kokoris, 1956). *Every nilsemisimple finite-dimensional commutative power-associative algebra over a field of characteristic* $\neq 2, \neq 3, \neq 5$ *has a unity and decomposes into the direct sum of simple algebras, such that each of them is either a Jordan algebra or an algebra of degree 2, over a field of positive characteristic.*

We clarify that the *degree of an algebra* A over a field F is a maximal number of mutually orthogonal idempotents in the scalar extension $\overline{F} \otimes_F A$, where $\overline{F}$ is the algebraic closure of the field F. Exceptional algebras of degree 2 from the conclusion of the theorem have been described in (Oehmke, 1962); their construction is fairly complex and we will not state it here. A description of simple Jordan algebras will be given in Sect. 3.

In general, the structure of nilsemisimple finite-dimensional power-associative algebras remains unknown. It is possible to get the description of these algebras only under some additional restrictions (cf. Sect 4). An effective method of studying power-associative algebras is a passage to the associated commutative power-associative algebra $A^{(+)}$, since properties of the algebra $A^{(+)}$ often give an essential information about the properties of A.

Let A be an algebra and let $a \in A$. Let us denote by R_a and L_a respectively the operators of the right and left multiplication by the element a:

$$R_a : x \mapsto xa, \quad L_a : x \mapsto ax.$$

The subalgebra of the algebra $\mathrm{End}\, A$ of the endomorphisms of the linear space A, generated by all the operators R_a, where $a \in A$ is called the *algebra of right multiplications* of the algebra A and is denoted by $R(A)$. The *algebra of left multiplications* $L(A)$ of the algebra A is defined analogously. The subalgebra of $\mathrm{End}\, A$ generated by all the operators R_a, L_a, $a \in A$ is called the *multiplication algebra* of the algebra A and is denoted by $M(A)$. If B is a subalgebra of A, then $M^A(B)$ will denote a subalgebra of the algebra $M(A)$, generated by all the operators R_b, L_b, where $b \in B$.

Properties of an algebra A are reflected in a certain way in the properties of its multiplication algebra $M(A)$. For example, the algebra A is nilpotent if and only if its associative algebra $M(A)$ is nilpotent. If a finite-dimensional algebra A is semisimple (is a direct sum of simple algebras), then the algebra $M(A)$ has the same property; and if A is simple, then $M(A)$ is also simple and is isomorphic to the full matrix algebra over its center.

Along with associative multiplication algebras, it is sometimes suitable to consider also the *Lie multiplication algebra* $\mathrm{Lie}\,(A)$ defined as the subalgebra of the Lie algebra $(\mathrm{End}\, A)^{(-)}$, generated by all the operators R_a, L_a, where $a \in A$. It is clear that $\mathrm{Lie}\,(A) \subseteq (M(A))^{(-)}$. Another Lie algebra naturally connected with every algebra A is the *derivation algebra* $\mathrm{Der}\, A$.

Recall that the *derivation* of an algebra A is a linear operator $D \in \mathrm{End}\, A$ which satisfies the equality

$$(xy)D = (xD)y + x(yD), \text{ for all } x, y \in A. \tag{10}$$

The set $\mathrm{Der}\, A$ of all derivations of an algebra A is a subspace of the vector space $\mathrm{End}\, A$; moreover, it is not difficult to see that if $D_1, D_2 \in \mathrm{Der}\, A$, then the commutator $[D_1, D_2] \in \mathrm{Der}\, A$ too, thus $\mathrm{Der}\, A$ is also a subalgebra of the Lie algebra $(\mathrm{End}\, A)^{(-)}$.

Equality (10) may be rewritten in terms of right and left multiplications: the operator $D \in \mathrm{End}\, A$ is a derivation if and only if any of the following two equalities is satisfied:

$$[R_y, D] = R_{yD}, \text{ for every } y \in A, \tag{11}$$
$$[L_x, D] = L_{xD}, \text{ for every } x \in A, \tag{12}$$

A derivation D of an algebra A is called an *inner derivation*, if $D \in \mathrm{Lie}\,(A)$. Equalities (11) and (12) easily imply that the set $\mathrm{Inder}\, A$ of all inner derivations of the algebra A is an ideal of the algebra $\mathrm{Der}\, A$.

Inner derivations play an important role in the theory of associative and Lie algebras. It is well known that every derivation of a finite-dimensional semisimple associative or Lie algebra of zero characteristic is inner. More generally, the following hold:

Proposition (Schafer, 1966). *Let A be a finite-dimensional algebra, over a field of characteristic 0, which is the direct sum of simple algebras, and A has either right or left unity. Then every derivation of the algebra A is inner.*

This proposition is not valid without the assumption on the existence of an one-sided unity (Walcher, 1987).

We know from the theory of Lie groups that there is a close connection between derivations and automorphisms of finite-dimensional algebras over the field of real numbers. Namely, the algebra $\mathrm{Der}\, A$, in this case, is nothing else but the Lie algebra of the automorphism group of the algebra A. The correspondence between derivations and automorphisms is established by $D \mapsto \exp D = 1 + D + \frac{D^2}{2!} + \ldots$; in the core of the proof that $\exp D$ is an automorphism lies the well-known Leibniz' formula:

$$(xy)D^n = \sum_{i=0}^{n} \binom{n}{i} (xD^i)(yD^{n-i}).$$

If A is an arbitrary algebra over a field of characteristic 0 and if D is a nilpotent derivation of A, then the operator $G = \exp D$ makes sense too; moreover, it is not difficult to show, with the aid of the Leibniz formula, that G is an automorphism of the algebra A.

In considering non-associative algebras, the following notion of an *associator* turns out to be useful

$$(x, y, z) \equiv (xy)z - x(yz).$$

The ideal $D(A)$ of an algebra A, generated by all the associators is called an *associator ideal* of the algebra A. A dual to this notion is the notion of the *associative center* $N(A)$ of an algebra A:

$$N(A) = \{n \in A \mid (n, A, A) = (A, n, A) = (A, A, n) = 0\}.$$

An algebra A is associative if and only if $D(A) = 0$ (or $N(A) = A$). The *center* $Z(A)$ *of an algebra* A is the set

$$Z(A) = \{z \in N(A) \mid [z, A] = 0\}.$$

Proposition. *For every algebra A, the associative center as well as the center are subalgebras. Moreover,*

$$D(A) = (A, A, A) + (A, A, A)A = (A, A, A) + A(A, A, A).$$

The proof follows from the following two identities valid in every algebra:

$$x(y, z, t) + (x, y, z)t = (xy, z, t) - (x, yz, t) + (x, y, zt), \quad (13)$$
$$[xy, z] - x[y, z] - [x, z]y = (x, y, z) - (x, z, y) + (z, x, y), \quad (14)$$

The notions of a bimodule and a birepresentation play an important role in the theory of algebras.

Let $\mathfrak{M}$ be a class of algebras over a field F. Let us assume that, for an algebra A in $\mathfrak{M}$ and a vector space M over F, the bilinear compositions $A \times M \longrightarrow M$, $M \times A \longrightarrow M$, written as am and ma, for $a \in A$ and $m \in M$ have been defined. Then the direct sum $A \dotplus M$ of the vector spaces A and M may be made into an algebra by defining multiplication via the following rule:

$$(a_1 + m_1)(a_2 + m_2) = a_1a_2 + (m_1a_2 + a_1m_2),$$

where $a_i \in A$, $m_i \in M$. This algebra is called the *split null extension* of the algebra A, by M. If the algebra $A \dotplus M$ again belongs to the class $\mathfrak{M}$, then M is called a *bimodule over the algebra A* (or an *A-bimodule*) *in the class $\mathfrak{M}$*.

For instance, if $\mathfrak{M}$ is the class of all algebras over F, then, no conditions are required in the definition of a bimodule in the class $\mathfrak{M}$, except that the operations am and ma are bilinear. If $\mathfrak{M}$ is the class of all the associative algebras, then the bimodule operations must satisfy the following conditions:

$$(ma)b = m(ab), \quad (am)b = a(mb), \quad (ab)m = a(bm),$$

for all $a, b \in A, m \in M$; in other words, we arrive at the usual well-known definition of an associative bimodule.

In the class of Lie algebras, the corresponding conditions for bimodule operations are of the form

$$am = -ma, \quad m(ab) = (ma)b - (mb)a.$$

In general, if a class $\mathfrak{M}$ is defined by a set of multilinear identities $\{f_i(x_1, \ldots, x_{n_i}) = 0 \mid i \in I\}$, then it is not difficult to see that M is a bimodule over an algebra $A \in \mathfrak{M}$ in the class $\mathfrak{M}$, if and only if the following conditions are satisfied:

$$f_i(a_1, \ldots, a_{k-1}, m, a_{k+1}, \ldots, a_{n_i}) = 0, \quad k = 1, \ldots, n_i; \quad i \in I,$$

for every $a_j \in A, m \in M$. In case when the relations f_i are not multilinear, the corresponding conditions for bimodules may also be written down fairly simply, with the aid of the operators of *"partial linearizations"* (cf. for instance Jacobson, 1968); if the class $\mathfrak{M}$ is defined by a finite number of identities, then bimodules in the class $\mathfrak{M}$ are also defined by a finite number of relations. For a concrete class $\mathfrak{M}$, defined by identities of small degree, it is simpler to find the conditions for bimodules directly. Let us show this through the examples of the alternative and Jordan algebras.

1) Let $\mathfrak{M}$ be the class of alternative algebras.

In terms of the associators, the class $\mathfrak{M}$ is defined via the following identities:

$$(x, x, y) = 0, \quad (x, y, y) = 0. \quad (15)$$

Thus a *bimodule* M over an alternative algebra A is *alternative* if and only if the following relations hold in the split null extension $A \dotplus M$:

$$(a + m, a + m, b + n) = 0, \quad (a + m, b + n, b + n) = 0,$$

for all $a, b \in A$; $m, n \in M$. Because of $M^2 = 0$ (in the algebra $A \dotplus M$), these relations give us the following conditions for a bimodule M to be alternative:

$$\begin{aligned} &(a, a, m) = 0, \quad (a, m, b) + (m, a, b) = 0 \\ &(m, b, b) = 0, \quad (a, m, b) + (a, b, m) = 0. \end{aligned} \quad (16)$$

2) Let $\mathfrak{M}$ be the class of Jordan algebras.

The defining relations for the class $\mathfrak{M}$ are of the following form:

$$xy = yx, \quad (x^2, y, x) = 0.$$

If A is a Jordan algebra and M is a bimodule over A, then the algebra $A \dot{+} M$ is Jordan if and only if the following relations hold there:

$$(a+m)(b+n) = (b+n)(a+m), \quad ((a+m)^2, b+n, a+m) = 0,$$

for all $a, b \in A; m, n \in M$. It is easy to see that these relations are equivalent to the following:

$$am = ma, \quad (a^2, m, a) = 0, \quad (a^2, b, m) + 2(am, b, a) = 0, \tag{17}$$

for all $a, b \in A, m \in M$.

If M is an A-bimodule, then the mappings $\rho(a) : m \to ma$ and $\lambda(a) : m \to am$ are linear operators on M and the mappings $a \to \rho(a)$, $a \to \lambda(a)$ are linear mappings from A into the algebra $\operatorname{End} M$. The pair (ρ, λ) of linear mappings from A into the algebra $\operatorname{End} M$ of endomorphisms of some vector space M is called a *birepresentation of the algebra A in the class $\mathfrak{M}$*, if M, equipped with the compositions $ma = m\rho(a), am = m\lambda(a)$, is a bimodule over A in the class $\mathfrak{M}$. It is obvious that the notions of a bimodule and a birepresentation define each other. Using relations (16) and (17) which define alternative and Jordan bimodules, we can easily write down the conditions defining birepresentations in these classes. For instance the alternative birepresentations are defined by the following conditions

$$\begin{aligned} &\lambda(a^2) - \lambda(a)^2 = 0, \quad [\lambda(a), \rho(b)] + \rho(a)\rho(b) - \rho(ab) = 0, \\ &\rho(a^2) - \rho(a)^2 = 0, \quad [\lambda(a), \rho(b)] + \lambda(ab) - \lambda(b)\lambda(a) = 0. \end{aligned} \tag{18}$$

Every algebra A may be considered in a natural way to be a bimodule over itself, interpreting ma and am as multiplication in the algebra A. Bimodules of this kind along with the corresponding birepresentations $a \mapsto R_a$, $a \mapsto L_a$ are called *regular bimodules.* Note that subbimodules of a regular bimodule A are the two-sided ideals of the algebra A.

If a class $\mathfrak{M}$ is defined via a system of identities $\{f_i\}$, then the regular bimodule for an algebra $A \in \mathfrak{M}$, generally speaking, may be not a bimodule in the class $\mathfrak{M}$. Indeed, it is evident from the examples considered above that, for this property to hold, the algebra A must not only satisfy the identities $\{f_i\}$, but also some new identities (for instance, in case of Jordan algebras the identity $(a^2, b, c) + 2(ac, b, a) = 0$ should hold in A). These new identities, called partial linearizations of the identities $\{f_i\}$, do not in general follow from $\{f_i\}$. However, this is the case if all the f_i are homogeneous and the number of elements in the field F is not smaller than the degree of every f_i in its every participating variable. In particular, every regular bimodule over an alternative algebra is alternative. The same is valid for Jordan algebras over a field F of characteristic $\neq 2$.

In considerations on a family of linear transformations it is often useful to pass to the enveloping associative algebra of this family. For instance, the

enveloping algebra of the family $\{R_a, L_a \mid a \in A\}$ is the multiplication algebra $M(A)$. In the case of arbitrary birepresentations, a study of the enveloping algebra of the family $\{\rho(a), \lambda(a) \mid a \in A\}$ is largely facilitated by introducing the universal enveloping algebra. Let us show how to construct this algebra through the example of alternative algebras.

Let A be an alternative algebra and let $B = A \dot{+} A^0$ be the direct sum of vector spaces, where A^0 is a vector space isomorphic to A under the isomorphism $a \mapsto a^0$. Let us consider the tensor algebra $T(B) = F \dot{+} B \dot{+} B \otimes B \dot{+} B \otimes B \otimes B \dot{+} \dots$ For every pair (ρ, λ) of linear transformations from A to $\operatorname{End} M$, we can constract a linear transformation $\phi : B \longrightarrow \operatorname{End} M$, setting $\phi(a + b^0) = \rho(a) + \lambda(b)$. Because of the properties of the tensor algebra, ϕ is uniquely extendable to the homomorphism of associative algebras: $\tilde{\phi} : T(B) \longrightarrow \operatorname{End} M$. It is not difficult to see that the pair (ρ, λ) is an alternative birepresentation (i.e. satisfies identities (18)), if and only if $\operatorname{Ker} \tilde{\phi}$ contains the following set of elements:

$$\begin{aligned} &(a^2)^0 - a^0 \otimes a^0, \quad a^0 \otimes b - b \otimes a^0 + a \otimes b - ab, \\ &a^2 - a \otimes a, \quad a^0 \otimes b - b \otimes a^0 + (ab)^0 - b^0 \otimes a^0; a, b \in A. \end{aligned} \tag{19}$$

For instance, we have

$$\begin{aligned} \lambda(a^2) - \lambda(a)^2 = \phi((a^2)^0) - (\phi(a^0))^2 = \tilde{\phi}((a^2)^0) - (\tilde{\phi}(a^0))^2 = \\ \tilde{\phi}((a^2)^0) - \tilde{\phi}(a^0 \otimes a^0) = \tilde{\phi}((a^2)^0 - a^0 \otimes a^0). \end{aligned}$$

Denote by I the ideal of the algebra $T(B)$ generated by the set of elements (19), denote by $U(A)$ the quotient algebra $T(B)/I$ and let $\mathcal{R} : a \mapsto a + I, \mathcal{L} : a \mapsto a^0 + I$ be linear transformations from A to $U(A)$. It is clear that the pair $(\mathcal{R}, \mathcal{L})$ satisfies the equalities in (18); in addition, it is not difficult to see that, for every alternative birepresentation $(\rho, \lambda) : A \longrightarrow \operatorname{End} M$ there exists a unique homomorphism of associative algebras $\phi : U(A) \longrightarrow \operatorname{End} M$, such that $\rho = \mathcal{R} \circ \phi, \lambda = \mathcal{L} \circ \phi$. In this way, M may be considered as a right (associative) $U(A)$-module. Conversely, every right $U(A)$-module is an alternative A-bimodule with respect to the compositions $ma = m\mathcal{R}(a), am = m\mathcal{L}(a)$. The algebra $U(A)$ is called the *universal multiplicative enveloping algebra* of the algebra A (in the class of alternative algebras).

We can also construct, in an analogous way, multiplicative enveloping algebras for other classes of algebras. Note that, for a Lie algebra L, the algebra $U(A)$ is the ordinary universal (Birkhoff-Witt) enveloping algebra; if A is associative, then $U(A) \cong A^\sharp \otimes (A^\sharp)^0$, where $A^\sharp = F \cdot 1 + A$ and the algebra $(A^\sharp)^0$ is anti-isomorphic to $A^\sharp$.

If (ρ, λ) is an arbitrary birepresentation of the algebra A (in the class $\mathfrak{M}$), then the enveloping algebra of the family $\{\rho(a), \lambda(a) \mid a \in A\}$ is a homomorphic image of the algebra $U(A)$. In particular, if regular bimodules for algebras in $\mathfrak{M}$ are bimodules in the class $\mathfrak{M}$, then the multiplication algebra $M(A)$ and, more generally, the algebra $M^B(A)$, for every algebra $B \in$

$\mathfrak{M}$ containing the algebra A, are homomorphic images of the algebra $U(A)$. Generally, introduction of the algebra $U(A)$ reduces the problem of describing birepresentations of the algebra A to determination of the structure of $U(A)$ and to the description of the right (associative) representations of $U(A)$.

As the concluding remark, we point out that the notions of a right module and a right representation, which play a fundamental role in the theory of associative algebras, do not always yield to such simple and natural definitions in other classes of algebras (except the cases of commutative and anticommutative algebras, where the notions of representation and birepresentation coincide by default). It is still unknown whether a right alternative module may be defined by a finite number of relations. On this matter see (Slin'ko, Shestakov, 1974), where the notion of the right representation in an arbitrary class of algebras has been defined and studied, and the theory of right representations of alternative algebras has been constructed.

Non-associative superalgebras have been studied more and more actively in recent times. They first arose in physics and geometry and turned out to be rather useful in algebra. An algebra A is called a *$\mathbb{Z}_2$-graded algebra* or a *superalgebra*, if $A = A_0 \dotplus A_1$, where $A_iA_j \subseteq A_{i+j}, i, j \in \mathbb{Z}_2$. For example, the Grassmann algebra $G = G_0 \dotplus G_1$ is a superalgebra, where G_0 (G_1) denotes a subspace, generated by the words of even (odd) length, on the generators of the algebra G. Let $\mathfrak{M}$ be a class of algebras over an infinite field, defined by some system of identities. The superalgebra $A = A_0 \dotplus A_1$ is called an $\mathfrak{M}$-superalgebra if its *Grassmann envelope* $G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$ belongs to $\mathfrak{M}$. The $\mathfrak{M}$-superalgebra A, generally speaking, does not itself belong to the class $\mathfrak{M}$; its even part A_0 is a subalgebra contained in $\mathfrak{M}$ and its odd part A_1 is an $\mathfrak{M}$-bimodule over A_0. If the identities defining $\mathfrak{M}$ are known, then it is possible to write down "superidentities" defining $\mathfrak{M}$-superalgebras. For example, the superalgebra $A = A_0 \dotplus A_1$ is a *Lie superalgebra*, if the following identities hold there:

$$a_ia_j + (-1)^{ij}a_ja_i = 0,$$
$$(a_ia_j)a_k - a_i(a_ja_k) - (-1)^{jk}(a_ia_k)a_j = 0;$$

the *alternative superalgebras* are defined by the following identities:

$$(a_i, a_j, a_k) + (-1)^{jk}(a_i, a_k, a_j) = 0,$$
$$(a_i, a_j, a_k) + (-1)^{ij}(a_j, a_i, a_k);$$

and the *Jordan superalgebras* are defined by the following identities.

$$a_ia_j - (-1)^{ij} \cdot a_ja_i = 0,$$
$$(-1)^{l(i+k)}(a_ia_j, a_k, a_l) + (-1)^{i(j+k)}(a_ja_l, a_k, a_i) +$$
$$(-1)^{j(l+k)}(a_la_i, a_k, a_j) = 0,$$

where $a_s \in A_s$, $s = i, j, k, l \in \{0, 1\}$, everywhere.

§2. Alternative Algebras

2.1. Composition Algebras. We begin presentation of the theory of alternative algebras with consideration of the most important class of these algebras, namely the composition algebras. An algebra A with unity 1 over a field F of characteristic $\neq 2$ is called a *composition algebra* if a non-degenerate quadratic form $n(x)$ has been defined on the vector space A such that

$$n(xy) = n(x)n(y) \tag{1}$$

In this case, we also say that the form $n(x)$ allows composition on A. Typical representatives of composition algebras are fields of real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$, the quaternion skew-field $\mathbb{Q}$ as well as the algebra of Cayley numbers (octonions) $\mathbb{O}$, with the Euclidean norm $n(x) = (x, x) = |x|^2$. The first three among them are associative and the algebra $\mathbb{O}$ provides us with the first and most important example of an alternative non-associative algebra. Equality (1), written down in an orthonormal basis, for each of those algebras, gives an identity of the following form:

$$(x_1^2 + \cdots + x_k^2)(y_1^2 + \cdots + y_k^2) = z_1^2 + \cdots + z_k^2, \quad k = 1, 2, 4, 8,$$

where z_i are bilinearly expressible through x_r, y_s. The efforts of many mathematicians of the last century were devoted to finding all the k for which these identities were valid, and only in 1898, Hurwitz had shown that the values $k = 1, 2, 4, 8$ were the only possible. We will see in the sequel that this claim is a consequence of a general fact that the dimension of a composition algebra may only be equal to 1, 2, 4, 8.

Proposition. *Let A be a composition algebra. Then A is alternative and every element of the algebra A satisfies a quadratic equation with the coefficients in F (i.e. the* algebra A is quadratic *over F).*

Proof. Substituting $y + w$ for y in (1) we get $n(x)n(y + w) = n(xy + xw)$. Subtracting the identity (1) from this equality as well as subtracting the identity obtained from (1) by substituting y by w, we get

$$n(x)f(y, w) = f(xy, xw), \tag{2}$$

where $f(x, y) = n(x + y) - n(x) - n(y)$ is a non-singular symmetric bilinear form associated with the quadratic form $n(x)$. Running the same procedure with x, we obtain

$$f(x, z)f(y, w) = f(xy, zw) + f(zy, xw) \tag{3}$$

The procedure just performed is called the *linearization of identity* (1) in y and x respectively. The idea of this procedure is in lessening the degree of the identity in a given variable, through introduction of new variables, and in arriving after all to a multilinear identity. We will apply this procedure in the sequel, without detailed explanations. Now set $z = 1, y = xu$ in (3):

$$f(x,1)f(xu,w) = f(x \cdot xu, w) + f(xu, xw). \tag{4}$$

Since $f(xu, xw) = n(x)f(u,w)$, and because of (2), (4) may be rewritten in the following way

$$f(x \cdot xu + n(x)u - f(x,1)xu, w) = 0,$$

which implies that

$$x \cdot xu + n(x)u - f(x,1)xu = 0, \tag{5}$$

because the form $f(x,y)$ is non-degenerate and w is arbitrary. Setting here $u = 1$, we obtain

$$x^2 - f(x,1)x + n(x) = 0 \tag{6}$$

which proves the second half of the proposition. It remains to prove that the algebra A is alternative.

Multiplying (6) on the right by u and comparing with (5) we obtain $x^2 u = x(xu)$. The proof that $u \cdot x^2 = (ux)x$ is analogous. Thus, the algebra A is alternative and the proof is complete. □

Recall that an endomorphism ϕ of a vector space A is called an *involution of the algebra* A, if $\phi(\phi(x)) = x$ and $\phi(xy) = \phi(y)\phi(x)$, for all $x, y \in A$.

Proposition. *In the composition algebra A, the mapping $x \mapsto \overline{x} = f(1,x) - x$ is an involution, fixing the elements of the field $F = F \cdot 1$; in addition, the elements $t(x) = x + \overline{x}$ and $n(x) = x\overline{x}$ are in F, for all x in A.*

We prove only the equality $\overline{x}\,\overline{y} = \overline{yx}$, as the other claims are fairly obvious. Linearising relation (6) in x, we obtain

$$xy + yx - f(1,x)y - f(1,y)x + f(x,y) = 0. \tag{7}$$

Moreover, for $w = z = 1$ in (3), we obtain the following:

$$f(x,1)f(y,1) = f(xy,1) + f(y,x). \tag{8}$$

Substituting it in (7), we get

$$xy + yx - f(1,x)y - f(1,y)x + f(1,x)f(1,y) - f(1,xy) = 0.$$

therefore

$$(f(1,x) - x)(f(1,y) - y) = f(1,xy) - yx.$$

We infer from (8) that $f(1,xy) = f(1,yx)$. Thus $\overline{x}\,\overline{y} = \overline{yx}$. □

Let us show now that the condition of the algebra A being alternative is not only necessary but also sufficient for the relation (1) to hold.

Proposition. *Let A be an alternative algebra over a field F with unity 1 and involution $x \mapsto \overline{x}$, such that the elements $t(x) = x + \overline{x}$ and $n(x) = x\overline{x} \in F$, for all $x \in A$. Then the quadratic form $n(x)$ satisfies condition (1).*

Proof. Note first of all that the following equalities hold in A:

$$x(\overline{x}y) = (y\overline{x})x = n(x)y, \tag{9}$$

which is easily implied by the alternativity conditions. Furthermore, by linearizing the alternativity identities (1.15) in x and y respectively, we obtain

$$(x,y,z) + (y,x,z) = 0, \tag{10}$$
$$(x,y,z) + (x,z,y) = 0, \tag{11}$$

which imply that, in an alternative algebra, the associator (x,y,z) is an alternating function of its arguments. In particular, we have the identity

$$(x,y,z) = (z,x,y). \tag{12}$$

Finally, in view of (9) and (12) we obtain the following:

$$n(xy) = (xy)(\overline{xy}) = (xy)(\overline{y}\,\overline{x}) = (xy \cdot \overline{y})\overline{x} - (xy, \overline{y}, \overline{x}) =$$
$$n(x)n(y) - (\overline{x}, xy, \overline{y}) = n(x)n(y) - (\overline{x} \cdot xy)\overline{y} + \overline{x}(xy \cdot \overline{y}) =$$
$$n(x)n(y) - n(x)n(y) + n(x)n(y) = n(x)n(y).$$

□

Now let A be an algebra with the unity 1, over a field F and an involution $a \mapsto \overline{a}$, where $a + \overline{a}, a\overline{a} \in F$, for every $a \in A$. Let us fix $0 \neq \alpha \in F$ and let us define on the vector space $A \dotplus A$ the following operation of multiplication:

$$(a_1, a_2) \cdot (a_3, a_4) = (a_1 a_3 - \alpha a_4 \overline{a_2}, \overline{a_1} a_4 + a_3 a_2).$$

The resulting algebra (A, α) is called the algebra derived from the algebra A by the *Cayley-Dickson process.* It is clear that A is isomorphically embeddable into (A,α) and that $\dim(A,\alpha) = 2\dim A$. Let $v = (0,1)$; then $v^2 = -\alpha \cdot 1$ and $(A,\alpha) = A \dotplus vA$. For an arbitrary element $x = a_1 + va_2 \in (A,\alpha)$, set $\overline{x} = \overline{a_1} - va_2$. Then $x + \overline{x} = a_1 + \overline{a_1}, x\overline{x} = a_1\overline{a_1} + \alpha a_2 \overline{a_2} \in F$ and the mapping $x \mapsto \overline{x}$ is an involution of the algebra (A,α) extending the involution $a \mapsto \overline{a}$ of the algebra A. If the quadratic form $n(a) = a\overline{a}$ is non-degenerate on A, then the quadratic form $n(x) = x\overline{x}$ is non-degenerate on (A,α).

The Cayley-Dickson process may be applied to every composition algebra A; furthermore, the algebra (A,α) will again be a composition algebra, if and only if it is alternative. Let us clarify under what conditions this is the case. Let $x, y \in (A, \alpha), x = a + vb, y = c + vd$. We have

$$(x, x, y) = (a + vb, a + vb, c + vd) = \alpha(\bar{a}, d, \bar{b}) - v(a, c, b).$$

Since $\overline{(y, x, x)} = -(\bar{x}, \bar{x}, \bar{y})$, the algebra (A, α) is alternative if and only A is associative.

We can now give the following examples of composition algebras over F:

I. A field F of characteristic $\neq 2$.

II. $\mathbb{C}(\alpha) = (F, \alpha), \alpha \neq 0$. If the polynomial $x^2 + \alpha$ is irreducible over F, then $\mathbb{C}(\alpha)$ is a field; otherwise, $\mathbb{C}(\alpha) = F \oplus F$.

III. $\mathbb{H}(\alpha, \beta) = (\mathbb{C}(\alpha), \beta), \beta \neq 0$ – the *algebra of generalized quaternions.* It is easy to check that the algebra $\mathbb{H}(\alpha, \beta)$ is associative, but not commutative.

IV. $\mathbb{O}(\alpha, \beta, \gamma) = (\mathbb{H}(\alpha, \beta), \gamma), \gamma \neq 0$ – the *Cayley-Dickson algebra.* It is not difficult to get convinced that the algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is not associative, thus our inductive process of constructing composition algebras breaks up.

If $F = \mathbb{R}$ is the field of real numbers, then the construction described above gives, for $\alpha = \beta = \gamma = 1$, the classical algebras of complex numbers $\mathbb{C} = \mathbb{C}(1)$, the quaternions $\mathbb{H} = \mathbb{H}(1, 1)$ and the *Cayley numbers* $\mathbb{O} = \mathbb{O}(1, 1, 1)$.

Let us show now that the examples I–IV exhaust all the composition algebras.

Let A be a composition algebra with the quadratic form $n(x) = x\bar{x}$; $f(x, y) = x\bar{y} + y\bar{x}$ be the bilinear form associated with $n(x)$. If B is a subspace in A, then we will denote by $B^{\perp}$ the orthogonal complement of B with respect to the form $f(x, y)$.

Lemma. *Let B be a subalgebra of A containing the unity 1 of the algebra A. Then, if the restriction of the form f to B is non-degenerate and if $B \neq A$, then for a suitable $v \in B^{\perp}$, the subspace $B_1 = B + vB$ is a subalgebra of A, obtained from B by the Cayley-Dickson process.*

The following theorem is easy to prove with the aid of the lemma.

Theorem. *Every composition algebra is isomorphic to one of the algebras of types I–IV, given above.*

Indeed we may set $B = F$, since the subalgebra F is non-degenerate with respect to $f(x, y)$. If $F \neq A$, then A contains the subalgebra $B_1 = (F, \alpha)$ of type II. If $B_1 \neq A$, then A contains the subalgebra $B_2 = (B_1, \beta)$ of type III. If finaly, $B_2 \neq A$, then A contains the subalgebra $B_3 = (B_2, \gamma)$ of type IV. The process must stop here, since, in the opposite case, the algebra A would contain a non-alternative subalgebra $B_4 = (B_3, \delta)$, which is impossible. Thus, A coincides with one of its subalgebras F, B_1, B_2, B_3 which proves the theorem. □

Corollary. *A non-degenerate quadratic form $n(x)$, defined on a finite-dimensional vector space V over a field F of characteristic $\neq 2$, allows composition*

if and only if, $\dim_F V = 1, 2, 4, 8$ *and, in some basis of the space V, the form $n(x)$ is respectively of one of the following forms:*

1) $n(x) = x_0^2$;
2) $n(x) = x_0^2 + \alpha x_1^2$;
3) $n(x) = (x_0^2 + \alpha x_1^2) + \beta(x_2^2 + \alpha x_3^2)$;
4) $n(x) = [(x_0^2 + \alpha x_1^2) + \beta(x_2^2 + \alpha x_3^2)] + \gamma[(x_4^2 + \alpha x_5^2) + \beta(x_6^2 + \alpha x_7^2)]$, *where* $\alpha, \beta, \gamma \in F, \alpha\beta\gamma \neq 0$.

We can choose a canonical basis in every composition algebra for which the form $n(x)$ is of one of the forms 1)–4). Let $\mathbb{C}(\alpha) = F \dotplus Fv_1, \mathbb{H}(\alpha, \beta) = \mathbb{C}(\alpha) \dotplus \mathbb{C}(\alpha)v_2, \mathbb{O}(\alpha, \beta, \gamma) = \mathbb{H}(\alpha, \beta) \oplus \mathbb{H}(\alpha, \beta)v_3$; then $v_1^2 = -\alpha, v_2^2 = -\beta, v_3^2 = -\gamma, \overline{v_i} = -v_i, v_i v_j = -v_j v_i$, for $i \neq j$ and the elements $e_0 = 1, e_1 = v_1, e_2 = v_2, e_3 = v_3, e_4 = v_1 v_2, e_5 = v_2 v_3, e_6 = v_1(v_2 v_3), e_7 = v_1 v_3$ form a canonical basis of the algebra $\mathbb{O}(\alpha, \beta, \gamma)$. Note that $\overline{e_i} = -e_i, e_i e_j = -e_j e_i$, for $i, j \geqslant 1, i \neq j$. If $\mathbb{O} = \mathbb{O}(1, 1, 1)$ is the algebra of Cayley numbers, then $e_i^2 = -1$, for all $i \geqslant 1$ and $e_i e_j = \lambda e_k, \lambda = \pm 1$, for all $i, j \geqslant 1, i \neq j$ and a suitable $k \geqslant 1$; in addition, for every cyclic permutation σ of the symbols i, j, k, $e_{\sigma(i)} e_{\sigma(j)} = \lambda e_{\sigma(k)}$. With these properties in hand, the multiplication table in the algebra $\mathbb{O}$ is fully determined by the following conditions:

$$e_i e_{i+1} = e_{i+3}, \quad i = 1, \ldots, 7; \quad e_{7+j} = e_j, \text{ for } j > 0. \qquad (13)$$

In case of arbitrary $\alpha, \beta, \gamma \in F$, we set formally $e_1 = \sqrt{\alpha} e_1', e_2 = \sqrt{\beta} e_2', e_3 = \sqrt{\gamma} e_3', e_4 = \sqrt{\alpha\beta} e_4', e_5 = \sqrt{\beta\gamma} e_5', e_6 = \sqrt{\alpha\beta\gamma} e_6', e_7 = \sqrt{\alpha\gamma} e_7'$; then multiplication of the elements e_i' is according the formulas analogous to those in (13) and, the multiplication table for the elements e_i will contain only positive integer powers of the parameters α, β, γ:

$$e_1 e_2 = e_4, e_2 e_4 = \beta e_1, e_5 e_6 = \beta\gamma e_1, \ldots$$

The multiplication table of the algebra of Cayley numbers may be also defined with the aid of the scheme in Fig. 1 below. The enumeration of the vertices may be arbitrary since different enumerations give isomorphic algebras. The indicated enumeration is in accordance with the choice of a basis satisfying conditions (13).

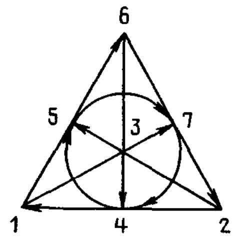

Fig. 1

A composition algebra A is called a *split composition algebra* if it satisfies one of the following equivalent conditions:

a) $n(x) = 0$, for some $x \neq 0$ in A;
b) $xy = 0$, for some $x \neq 0, y \neq 0$ in A;
c) A contains a non-trivial idempotent (i.e. an element $e \neq 0, 1$ such that $e^2 = e$).

Recall that an algebra A is called a *division algebra*, if, for every a, b $(a \neq 0)$ in A, the following equations are solvable in A:

$$ax = b, \quad ya = b.$$

If, for $a \neq 0$, each of these equations has a unique solution and A contains a unity, then A is called a *(skew) field*. It is easy to see that every finite-dimensional algebra without zero divisors is a division algebra, thus every composition algebra is either split or else is a division algebra (and therefore a skew-field).

Let us give examples of split composition algebras over a field F.

1. $n = \dim_F A = 2, A = F \oplus F$, with the involution $\overline{(\alpha, \beta)} = (\beta, \alpha)$.
2. $n = 4, A = F_2$ – the algebra of 2×2 matrices over F with the simplectic involution $\overline{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$.
3. $n = 8, A = \mathbb{O}(F)$ – the so-called *"Cayley-Dickson matrix algebra"* consists of all the matrices of the form $\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}$, where $\alpha, \beta \in F$ and u, v are vectors in the three-dimensional vector space F^3, with ordinary matrix operations of addition and multiplication by a scalar and the following multiplication:

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \begin{pmatrix} \gamma & z \\ w & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + (u, w) & \alpha z + \delta u - v \times w \\ \gamma v + \beta w + u \times z & \beta\delta + (v, z) \end{pmatrix},$$

where (x, y) denotes the scalar product of vectors $x, y \in F^3$ and $x \times y$ denotes their "vector" product. Involution in the algebra $\mathbb{O}(F)$ is defined in the same way as in the algebra F_2 and, for the element $a = \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix}$, we have $n(a) = a\bar{a} = \alpha\beta - (u, v), t(a) = a + \bar{a} = \alpha + \beta$.

Theorem (Jacobson, 1958, Zhevlakov, Slin'ko, Shestakov, Shirshov, 1978). *Every split composition algebra over a field F is isomorphic to one of these algebras: $F \oplus F$, $F_2, \mathbb{O}(F)$.*

Note that condition a) in the definition of a split algebra is always satisfied in a composition algebra over an algebraically closed field, thus the following holds:

Corollary. *There are only four non-isomorphic composition algebras over an algebraically closed field F.*

Classification of the composition algebras over the fields $\mathbb{Q}_p$ of p-adic numbers is the same, since every quadratic form in 5 and more variables over $\mathbb{Q}_p$ represents zero. Over the field $\mathbb{R}$ of real numbers, there exist only 7 non-isomorphic composition algebras: 3 split and 4 division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The latter 4 algebras are unique finite-dimensional alternative division algebras over $\mathbb{R}$. There are also non-alternative, finite-dimensional division algebras over $\mathbb{R}$ (Kuz'min, 1966). In general, they have not been described, but the following fundamental result holds:

Theorem (Bott, Milnor, 1958). *Finite-dimensional division algebras over $\mathbb{R}$ exist only in dimensions 1, 2, 4, 8.*

An algebraic proof of this result is not yet known. The known proof is topological and is based on investigations of topological properties of the mapping of the sphere S^{n-1} into itself, induced by multiplication in an n-dimensional division algebra. With the aid of the methods of mathematical logic, using the completeness of the elementary theory of real closed fields, it can be shown that an analogous result holds for finite-dimensional division algebras over an arbitrary real closed field.

In the conclusion we note that the classification of the composition algebras over the fields of algebraic numbers is also known (Jacobson, 1958).

2.2. Projective Planes and Alternative Skew-Fields. We have seen that the alternative algebras have naturally risen from the study of quadratic forms, admitting composition. Another factor that stimulated the development of alternative algebras was their relation with the theory of projective planes, established at the beginning of the thirties in papers by Moufang.

We note that the ordered pair of sets $\pi = (\pi_0, \pi^0)$ is called a *projective plane* with the set of points π_0 and the set of lines π^0, if a relation of incidence (i.e. belonging of a point P to a line l) exists between these two sets, subject to the following conditions:

1. If $P_1, P_2 \in \pi_0, P_1 \neq P_2$, then there exists a unique line $l \in \pi^0$, containing P_1 and P_2 (denoted by $l = P_1P_2$).
2. If $l_1, l_2 \in \pi^0, l_1 \neq l_2$, then there exists a unique point $P \in \pi_0$ that belongs both to l_1 and to l_2 $(P = l_1 \cap l_2)$.
3. There exist 4 point in a general position, i.e. a position such that no three of these points belong to one line.

A classical example of a projective plane is a two-dimensional projective space PF^2, over a field F, whose points are one-dimensional subspaces of the linear space F^3 and whose lines are the two-dimensional subspaces. If $F = F_q$ is a field with q elements, then PF^2 is a finite plane, containing $q^2 + q + 1$ points and lines. In particular, for $q = 2$ we get the smallest projective plane, the so-called *Fano plane*. It may be pictured as in Fig. 1, if the arrows there are removed and if the "lines" are considered to be the sides and the altitudes of the triangle as well as its inscribed circle.

Let P be a point and l be a line in the projective plane π. The plane π is called a *(P,l)-Desargues plane*, if for any of its different points A, B, C, A', B', C', such that (1) $AA' \cap BB' \cap CC' = P$, (2) $AB \neq A'B'$, $AC \neq A'C'$, $BC \neq B'C'$, (3) $AB \cap A'B' \in l$, $AC \cap A'C' \in l$, the intersection $BC \cap B'C'$ also belongs to l.

The two resulting configurations (depending on whether the point P belongs to the line l or not) are pictured in Fig. 2 and 3:

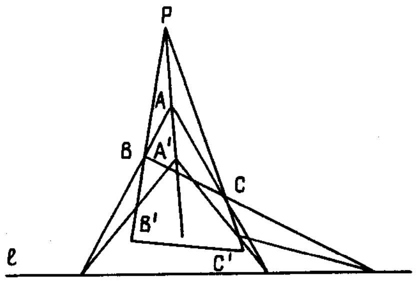

Fig. 2

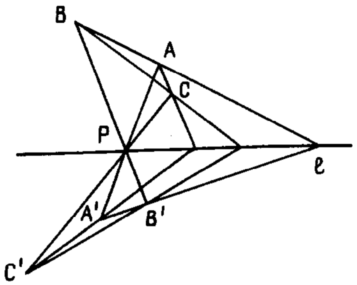

Fig. 3

If a plane π is (P,l)-Desargues, for every point $P \in l$, then π is called *l-Desargues* (or else we speak of π as of the translation plane, with respect to the line l). A plane π is called a *Desargues plane*, if it is a (P,l)-Desargues plane, for every P and l. In this case we say that the *Desargues theorem* holds in π. An example of a Desargues projective plane is the aforementioned plane PF^2. If the plane π is l-Desargues, for every line l, then we say that the *little Desargues theorem* holds in π and π is called a *Moufang plane* in this case.

Coordinates may be introduced into every projective plane in the following way. Let X, Y, O, I be four points in a general position. Let us call the line XY – the line at infinity l_∞ and, call the line OI – the line $y = x$. On the line OI, assign the coordinates $(0,0)$ to the point O, the coordinates $(1,1)$ to the point I, and assign the single coordinate (1) to the point Z of the intersection of the lines OI and XY. We assign the coordinates (b,b) to other points of the line OI, where b are symbols different for different points. Let now $P \notin l_\infty$ and $XP \cap OI = (b,b)$, $YP \cap OI = (a,a)$. Then we assign the coordinates (a,b) to the point P. By this rule, the previous coordinates are assigned to the points of the line OI. Let the line connecting $(0,0)$ and $(1,m)$ intersect l_∞ in a point M. Assign a unique coordinate (m) to the point M; it may be interpreted as a characterization of the slope of the line OM. Finally, assign the symbol (∞) as the coordinate of the point Y (cf. Fig. 4).

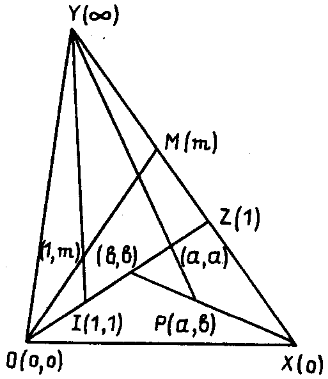

Fig. 4

We point out that by this very construction we may obtain the Cartesian coordinates in the ordinary real plane. In doing this, the point X should be considered to be the point at infinity of the axis x, and Y – the point at infinity of the axis y, O is the coordinate origin and I is the point $(1,1)$.

Let us define now algebraic operations of addition and multiplication on the set R of the coordinate symbols b. Set $a + b = c$, for some elements $a, b, c \in R$, if the point (a,c) belongs to the line connecting the points $(0,b)$ and (1) and $a \cdot b = c$, if the point (a,c) lies on the line connecting the point $(0,0)$ with (b). It is easy to see that the following equalities are satisfied:

$$a + 0 = 0 + a = a,$$
$$0 \cdot a = a \cdot 0 = 0,$$
$$1 \cdot a = a \cdot 1 = a.$$

Moreover, the following equations in R are uniquely solvable in x, y:

$$y + a = b, \quad a + x = b, \quad c \cdot x = a, \quad y \cdot c = a,$$

for every a, b, c $(c \neq 0)$ in R.

The algebraic system $\langle R, +, \cdot \rangle$ does not in general satisfy the axioms of a (non-associative) ring and, generaly speaking, does not define the plane π. Moreover, in choosing different coordinates, one plane π may be assigned several different coordinatizing systems $\langle R, +, \cdot \rangle$, which may be non-isomorphic. The situation changes if we impose some Desargues conditions on π.

Theorem 1 (Skornyakov, 1951), (Pickert, 1955). *A projective plane π is an l-Desargues plane, for two different lines l, containing the point $Y = (\infty)$, if and only if the corresponding coordinatizing system $\langle R, +, \cdot \rangle$ is a left alternative skew-field. A plane π is a Moufang (Desargues) plane if and only if every of its coordinatizing systems is an alternative (respectively, associative) skew-field. Every two skew-fields coordinatizing a Desargues plane are mutually isomorphic.*

Conversely, given a (non-associative) skew-field R, then we can always construct a projective plane π by it, whose one of the coordinatizing skew-fields is R. In addition, if R is alternative (associative), then π is Moufang (respectively, Desargues).

In the case of an arbitrary projective plane it is more suitable to consider one ternary operation, instead of the binary operations of addition and multiplication, on the set of the coordinate symbols R. If $a, b, c, d \in R$, then set $d = a \cdot b \circ c$, if the point (a, d) lies on the line connecting the points (b) and $(0, c)$. It is easy to see, that $a + b = a \cdot 1 \circ b$, $a \cdot b = a \cdot b \circ 0$, i.e. the former binary operations are expressible in terms of the ternary ones. The set R, together with the introduced operation $a \cdot b \circ c$ is called *ternary of the plane* π. The advantage of a ternary is in the fact that the plane it coordinatizes may be uniquely restored by the ternary. However, in this case too, several non-isomorphic ternaries may correspond to one plane.

Relation of the theory of projective planes with the alternative rings has initiated a series of algebraic questions on their structure. First of all, the question of description of alternative skew-fields had risen. A study on them was initiated by Zorn and Moufang. One of the results obtained by Zorn was the following:

Theorem 2. *A finite alternative skew-field is associative and is the Galois field* F_q.

This theorem easily follows from Artin's theorem on associativity of two-generated alternative ring (cf. 2.3 in the sequel) and the classical Wedderburn theorem on finite associative skew-fields, which states that every such skew-field is a field and is generated by one element.

Because of Theorem 1, Theorem 2 implies the following

Corollary. *Every finite Moufang plane is a Desargues plane.*

A final description of alternative skew-fields was obtained at the beginning of the fifties by Bruck and Kleinfeld and independently by L.A Skornyakov, who proved that every alternative, non-associative skew-field is a Cayley-Dickson algebra over its center. This result had enabled them to prove, in particular, that every two alternative skew-fields coordinatizing the same Moufang plane π are mutually isomorphic. Somewhat later, L.A. Skornyakov had proved that every right alternative (or left alternative) skew-field is alternative. In view of Theorem 1, the latter means that if a projective plane π is an l-Desargues plane, for two different lines l, then π is a Moufang plane.

Thus, the Moufang (non-Desargues) planes are exactly the planes that can be coordinatized by the Cayley-Dickson division algebras (Cayley-Dickson skew-fields). By the corollary of Theorem 2, they are all infinite. We will give another realization of these planes in 3.5.

In the conclusion we add a few words about finite planes. It is not difficult to show that, in a finite projective plane π, every line contains exactly as many points as the number of lines passing through an arbitrary point. If this number equals $n + 1$, then we say that π is of order n. In this case, the number of all the points in π equals $n^2 + n + 1$. For instance, PF_q^2 is of order $q = p^r$ (where p is prime). It turns out that, not for every n are there planes of order n. At this time, no finite plane is known of order different from p^r. It has been proved for instance that there are no planes of orders 6 and 14. The question for $n = 10$ remains open. It is known that there exist non-Desargues planes of orders p^r, for all $r \geqslant 2$ and all $p \neq 2$, and also of orders 2^{2r}, where $r \geqslant 2$. For $n = p^r < 9$, there exist only Desargues planes.

2.3. Moufang's Identities and Artin's Theorem. Let A be an alternative algebra (a. a.) over a field F. We have already observed that the associator $(x, y, z) = (xy)z - x(yz)$ in the algebra A is an alternating function of its arguments. In particular, the following identity holds in A:

$$(x, y, x) = 0. \tag{14}$$

The algebras satisfying (14) are called *flexible algebras*. It is easy to see that, for instance, every commutative or anticommutative algebra is flexible.

Let us prove that the following identities are satisfied in an a. a. A:

$$(x, y, yz) = (x, y, z)y \tag{15}$$

$$(x, y, zy) = y(x, y, z). \tag{16}$$

By (1.13), we have

$$(x, y, xy) = -(x, xy, y) = x(x, y, y) + (x, x, y)y - (x^2, y, y) - (x, x, y^2) = 0.$$

Linearizing this identity in x, we get

$$(xy, z, y) + (zy, x, y) = 0,$$

which implies, by (1.13) and (14), the following identity:

$$0 = (xy, z, y) + (x, y, zy) = (x, y, z)y + (x, yz, y).$$

This proves (15). Identity (16) is proved analogously.

Well known *Moufang identities* are easily provable using (15) and (16):

$$(xy \cdot z)y = x(y \cdot zy) - \textit{the right Moufang identity},$$

$$(yz \cdot y)x = y(z \cdot yx) - \textit{the left Moufang identity},$$

$$(xy)(zx) = x(yz)x - \textit{the central Moufang identity}.$$

For instance, $(xy \cdot z)y - x(yzy) = (x \cdot yz)y + (x, y, z)y - (x \cdot yz)y + (x, yz, y) = 0$.

We can now prove the following theorem we mentioned earlier:

Artin's Theorem. *In an a. a. A, any two elements generate an associative subalgebra.*

Proof. Let A_0 be the subalgebra generated by elements $a, b \in A$. In order to prove its associativity, it is enough to prove, because of the distributivity of multiplication, that, for arbitrary finite products u_1, u_2, u_3, of the elements a, b, the identity $(u_1, u_2, u_3) = 0$ holds. We will prove this claim by induction on the total number of factors in the products u_1, u_2, u_3. The basis of induction consists in the alternativity and flexibility conditions. By the inductive hypothesis, we may assume that u_1, u_2, u_3 are associative products of the elements a, b. In this case, in two of them, the rightmost factors must coincide. Let, for instance, $u_1 = v_1 a$, $u_2 = v_2 a$. If either v_1 or v_2 are absent then, by the inductive hypothesis and (15) and (16), $(u_1, u_2, u_3) = 0$. We can therefore assume that v_1 and v_2 are non-empty. Let us linearize identity (16) in y:

$$(x, y, zw) + (x, w, zy) = y(x, w, z) + w(x, y, z).$$

Setting here $x = u_1, y = u_3, z = v_2, w = a$, we get the following, by the inductive hypothesis:

$$(u_1, u_2, u_3) = -(v_1 a, u_3, v_2 a) = (v_1 a, a, v_2 u_3) - a(v_1 a, u_3, v_2) - u_3(v_1 a, a, v_2) = (v_1 a, a, v_2 u_3) = a(v_1, a, v_2 u_3) = 0.$$

The theorem has been proved. □

The following more general claim may be proved by similar arguments: any three elements a, b, c in an a. a. A that satisfy the relation $(ab)c = a(bc)$, generate in A an associative subalgebra (compare with Moufang's theorem in 6.2).

Corollary. *Every a. a. is power-associative.*

In particular, in every a. a. A, there is a uniquely defined nilradical $\operatorname{Nil} A$.

2.4. Finite-Dimensional Alternative Algebras (Schafer, 1966). Let A be an a. a., let M be an alternative A-bimodule and let (ρ, λ) be the corresponding birepresentation of the algebra A (cf. 1.2). It is said that the algebra A acts nilpotently on M, if the algebra $\langle \rho(A) \cup \lambda(A) \rangle$, generated in $\operatorname{End} M$ by the set $\rho(A) \cup \lambda(A)$ is nilpotent. If $x \in A$, then it is said that x acts nilpotently on M, if the subalgebra $\langle \rho(x), \lambda(x) \rangle$ is nilpotent. (1.18) implies that the algebra $\langle \rho(x), \lambda(x) \rangle$ is commutative and that $\rho(x^k), \lambda(x^k) \in \langle \rho(x), \lambda(x) \rangle$, for every $k \geqslant 1$.

Theorem. *Let A be an a. a. over the field F and let M be a finite-dimensional alternative A-bimodule. Let, in addition, C be a multiplicatively closed subset in A generating the algebra A. Then, if every element $c \in C$ acts nilpotently on M, A too acts nilpotently on M.*

Proof. Note that there are multiplicatively closed subsets $B \subseteq C$, such that the algebra $\langle B \rangle$, generated by B acts nilpotently on M. For instance the set $\{x^k \mid k \geqslant 1\}$, for every $x \in C$. Moreover, the set $\{x \in A \mid Mx = xM = 0\}$ is an ideal in A (it easily follows from the fact that the split null extension $A \dotplus M$ is alternative). Consequently, we may assume, without loss of generality, that the birepresentation (ρ, λ) is exact (i.e. $\operatorname{Ker} \lambda \cap \operatorname{Ker} \rho = 0$) as well as that the algebra A is finite-dimensional over F. Let D be a maximal multiplicatively closed subset in C, such that the subalgebra $\langle D \rangle$ acts nilpotently on M. We may assume that $D \subsetneqq C$. By assumption, there is an n such that $M\sigma_1 \ldots \sigma_n = 0$, for all $\sigma_1, \ldots, \sigma_n$ from the set $\{\rho(x), \lambda(x) \mid x \in D\}$. Let $M_i = \{m \in M \mid m\sigma_1 \ldots \sigma_i = 0$, for all $\sigma_1, \ldots, \sigma_i\}$; then $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$.

Let us consider an arbitrary element w of the form

$$w = \sigma_1 \ldots \sigma_{i-1} \tau \sigma_{i+1} \ldots \sigma_{2n}, \qquad (*)$$

where σ_i are same as above, $\tau = \rho(a)$ or $\tau = \lambda(a)$, $i = 1, \ldots, 2n-1$. It is clear that $w = 0$. Let now y be an arbitrary element in A, which is a product of $2n+1$ factors, $2n$ of which belong to D. It follows easily from (1.18) that $\rho(y)$ and $\lambda(y)$ are linear combinations of elements of the form (*), thus $\rho(y) = \lambda(y) = 0$ and $y = 0$. Hence, $\langle D \rangle$ acts nilpotently on A (and, in particular, on C), thus there exists an element $z \in C$, such that $z \notin D, zD \cup Dz \subseteq D$. Clearly $z^k D \cup D z^k \subseteq D$, for all $k \geqslant 1$, thus the set $E = D \cup \{z^k \mid k \geqslant 1\}$ is multiplicatively closed and properly contains D.

It only remains to prove that $\langle E \rangle$ acts nilpotently on M. (1.18) implies that every M_i is invariant with respect to $\rho(z)$ and $\lambda(z)$, hence M_i is a $\langle E \rangle$-bimodule. The algebra $\langle E \rangle$ acts nilpotently on every quotient M_i/M_{i-1}, hence it acts nilpotently on M too. This contradiction finishes the proof. □

Note that the explicit form of the alternativity condition has not been used in the proof of the theorem. The same proof is applicable in case when A is, for instance, a Lie algebra and M a Lie bimodule. Moreover, the reasoning used is still valid in every class $\mathfrak{M}$ of algebras, defined by the homogeneous identities of the third degree, such that, for every algebra A and its every ideal I, the set I^2 is again an ideal of A. The necessary changes, related to the possible non-associativity of powers in A are fairly obvious (Stitzinger, 1983).

The right and left Moufang identities imply that every alternative birepresentation (ρ, λ) satisfies the following relations

$$\rho(xyx) = \rho(x)\rho(y)\rho(x), \quad \lambda(xyx) = \lambda(x)\lambda(y)\lambda(x).$$

This and the relations $\rho(x^2) = \rho(x)^2, \lambda(x^2) = \lambda(x)^2$ imply easily that $\rho(x^k) = \rho(x)^k, \lambda(x^k) = \lambda(x)^k$, for all $k \geqslant 1$. In particular, if the element x is nilpotent, then it acts nilpotently on every alternative bimodule.

Using a regular birepresentation, we obtain the following

Corollary 1. *Let A be a finite-dimensional a. a. and C be a multiplicatively closed nilsubset of A. Then the subalgebra $\langle C\rangle$ acts nilpotently on A and, in particular, is itself nilpotent.*

Corollary 2. *The nilradical $\mathrm{Nil}\,A$ of a finite-dimensional a. a. A is nilpotent.*

The quotient algebra $A/\mathrm{Nil}\,A$ does not contain non-zero nilideals, i.e. it is semisimple. The structure of semisimple a. a. is described by the following

Theorem. *A finite-dimensional semisimple a. a. is isomorphic to the direct sum of simple algebras, each of which is either associative and is a matrix algebra over a skew-field or is a Cayley-Dickson algebra over its center.*

A finite-dimensional a. a. A over the field F is called a *separable algebra* if, for every extension K of the field F, the algebra $A_K = K \otimes_F A$ is semisimple. As in the case of the associative algebras, this is equivalent to the property that the algebra A is semisimple and the center of its every simple component is a separable extension of the field F.

The following theorem generalizes the classical Wedderburn-Malcev theorem in the theory of associative algebras to the a. a.

Theorem. *Let A be a finite-dimensional a. a. over the field F and let $N = \mathrm{Nil}\,A$ be its nilradical. If the quotient algebra A/N is separable over F, then $A = B \dotplus N$ (the direct sum of vector spaces), where B is a subalgebra of the algebra A, isomorphic to A/N. If F is a field of characteristic $\neq 2, \neq 3$ and B_1 is another subalgebra of A, isomorphic to A/N, then there exists an inner automorphism ϕ of the algebra A, such that $B_1 = B^\phi$.*

We will not specify the form of these inner automorphisms of the a. a., since they look fairly complicated in general. We only point out that, over a field of characteristic zero, ϕ may be chosen to be in the subgroup generated by the automorphisms of the form $\exp(D)$, where D is a nilpotent inner derivation of the algebra A lying in the radical of its multiplication algebra $M(A)$.

Let us clarify now how the inner derivations of the a. a. look, over the fields of characteristics $\neq 2, \neq 3$.

First of all it is not difficult to see that the mapping $R_a - L_a$ is a derivation of an a. a. A, for some $a \in A$, if and only if the element a is in the center $N(A)$ of the algebra A. Furthermore, the fact that the associator (x,y,z) is skew-symmetric implies the following relations:

$$-R_{xy}+R_xR_y = L_{xy}-L_yL_x = [L_y, R_x] = [R_y, L_x] = L_xL_y-L_{yx} = R_{yx}-R_yR_x$$

for all $x, y \in A$. This, in particular implies that

$$[R_x, R_y] = R_{[x,y]} - 2[L_x, R_y]. \tag{17}$$

Furthermore, in view of the relation $R_x \circ R_y = R_{x\circ y}$, we get

$$\begin{aligned}[R_y,[R_x,R_z]] &= (R_x\circ R_y)\circ R_z - (R_z\circ R_y)\circ R_x = \\ &R_{(x\circ y)\circ z-(z\circ y)\circ x} = R_{[y,[x,z]]-2(x,y,z)}\end{aligned} \tag{18}$$

Let us consider the following mapping, for arbitrary $x, y \in A$:

$$D_{x,y} = R_{[x,y]} - L_{[x,y]} - 3[L_x, R_y]. \tag{19}$$

For every $z \in A$, by (17) and (18) we have

$$\begin{aligned}2[R_z, D_{x,y}] &= 3[R_z, R_{[x,y]} - 2[L_x,R_y]] - [R_z, R_{[x,y]}] - 2[R_z, L_{[x,y]}] = \\ &3[R_z,[R_x,R_y]] - R_{[z,[x,y]]} = R_{2[z,[x,y]]-6(x,z,y)} = 2R_{zD_{x,y}},\end{aligned}$$

which implies that $D_{x,y}$ is a derivation of the algebra A. Clearly the derivations $R_a - L_a$ ($a \in N(A)$) and $D_{z,y}$ ($x, y \in A$) are inner. Let us show now that if A contains 1, then every inner derivation D of the algebra A is of the form

$$D = R_a - L_a + \sum_i D_{x_i,y_i}, \quad a \in N(A), x_i, y_i \in A. \tag{20}$$

Indeed relations (17) and (18) and their left analogues easily imply that the Lie algebra $\mathrm{Lie}\,(A)$ of multiplications of the algebra A consists of elements of the form $R_x + L_y + \sum_i [L_{x_i}, R_{y_i}]$, $x, y, x_i, y_i \in A$. It is clear that every such an element may be represented in the form $T = R_g + L_h + \sum_i D_{u_i,v_i}$. Now, if T is a derivation, then $0 = 1T = g + h$, hence $R_g - L_g = T - \sum_i D_{u_i,v_i}$ is a derivation of the algebra A, and $g \in N(A)$. This finishes the proof.

We have already pointed out in Sect. 1 that every derivation of a finite-dimensional central simple associative algebra is inner. Let us clarify now what does the derivation algebra $\mathrm{Der}\,\mathbb{O}$ of the Cayley-Dickson algebra $\mathbb{O}$ look like.

Let $\mathbb{O}$ be a Cayley-Dickson algebra over a field F of characteristic $\neq 2, \neq 3$, let $n(x)$ be the norm of an element $x \in \mathbb{O}$ and let $f(x,y)$ be the bilinear form on $\mathbb{O}$, associated with $n(x)$. Let us represent $\mathbb{O}$ in the form $\mathbb{O} = F\cdot 1 \dotplus \mathbb{O}_0$, where $\mathbb{O}_0$ is the set of elements in $\mathbb{O}$ with the zero trace. It is not difficult to show that every element in $\mathbb{O}_0$ is the sum of commutators, thus $\mathbb{O}D \subseteq \mathbb{O}_0$, for every $D \in \mathrm{Der}\,\mathbb{O}$. In addition, for every $x \in \mathbb{O}$ we have $(x + \overline{x})D = (t(x)\cdot 1)D = 0$, hence $\overline{x}D = -xD = \overline{xD}$. Now

$$\begin{aligned}f(xD,y) + f(x,yD) &= xD\cdot\overline{y} + y\cdot\overline{xD} + x\cdot\overline{yD} + yD\cdot\overline{x} = \\ &t(xD\cdot\overline{y} + x\cdot\overline{yD}) = t(xD\cdot\overline{y} + x\cdot\overline{y}D) = t((x\overline{y})D) = 0,\end{aligned}$$

i.e. D is skew-symmetric, with respect to the form f. It is easy to see, that the operators of multiplication by elements in $\mathbb{O}_0$ are skew-symmetric with respect to f. The set $o(8,f)$ of all the skew-symmetric linear transformations of $\mathbb{O}$ with respect to f, form a subalgebra of the Lie algebra $(\mathrm{End}\,\mathbb{O})^{(-)}$ – an orthogonal Lie algebra of dimension $\frac{1}{2}\cdot 8\cdot(8-1) = 28$ over F. The elements of the form $D + R_x + L_y$, where $D \in \mathrm{Der}\,\mathbb{O}$, $x, y \in \mathbb{O}_0$ form a subspace of

$o(8, f)$, which is the direct sum of the spaces $\operatorname{Der}\mathbb{O}$, $R_{\mathbb{O}_0} = \{R_x \mid x \in \mathbb{O}_0\}$ and $L_{\mathbb{O}_0} = \{L_x \mid x \in \mathbb{O}_0\}$. In fact the following equality holds:

$$\operatorname{Der}\mathbb{O} \dotplus R_{\mathbb{O}_0} \dotplus L_{\mathbb{O}_0} = o(8, f),$$

thus, $\dim \operatorname{Der}\mathbb{O} = 28 - 7 - 7 = 14$. In addition, the algebra $\operatorname{Der}\mathbb{O}$ is simple. Thus the following theorem holds:

Theorem. *Let $\mathbb{O}$ be the Cayley-Dickson algebra, over a field of characteristic $\neq 2, \neq 3$. Then the derivation algebra $\operatorname{Der}\mathbb{O}$ is a 14-dimensional central simple Lie algebra.*

According to the classification of finite-dimensional simple Lie algebras over an algebraically closed field K of characteristic 0, only one of them is of dimension 14 – it is the exceptional algebra $\mathbb{G}_2$. The theorem implies that $\mathbb{G}_2 \cong \operatorname{Der}\mathbb{O}$, where $\mathbb{O}$ is the (split) Cayley-Dickson algebra over K. Let $\overline{F}$ be the algebraic closure of the field F. A central simple Lie algebra L over F is called an algebra of type $\mathbb{G}_2$, if $\overline{F} \otimes_F L \cong \operatorname{Der}\mathbb{O}$, where $\mathbb{O}$ is the (split) Cayley-Dickson algebra over $\overline{F}$. It is clear that for every Cayley-Dickson algebra $\mathbb{O}$, the algebra $\operatorname{Der}\mathbb{O}$ is an algebra of type $\mathbb{G}_2$. Conversely, every central simple Lie algebra of type $\mathbb{G}_2$ over the field of characteristic $\neq 2, \neq 3$, is isomorphic to the algebra $\operatorname{Der}\mathbb{O}$, for an appropriate Cayley-Dickson algebra $\mathbb{O}$; in addition, $\operatorname{Der}\mathbb{O}_1 \cong \operatorname{Der}\mathbb{O}_2$, if and only if $\mathbb{O}_1 \cong \mathbb{O}_2$.

Corollary. *Every derivation D of a Cayley-Dickson algebra $\mathbb{O}$ of characteristic $\neq 2, \neq 3$ is inner and is of the form $D = \sum_i D_{x_i,y_i}$, where $x_i, y_i \in \mathbb{O}$.*

Indeed, for every a. a. A the derivations of the form $\sum_i D_{x_i,y_i}$ form an ideal in the algebra $\operatorname{Der} A$, and since this ideal is non-zero in the algebra $\operatorname{Der}\mathbb{O}$, everything follows from the fact that $\operatorname{Der}\mathbb{O}$ is simple.

Combining this result with known facts on derivations of central simple associative algebras, we arrive, in a standard way, to the fact that every derivation D of a finite-dimensional separable a. a. A of characteristic $\neq 2, \neq 3$, is inner and is of the form (20). Furthermore, in case of characteristic 0, we may choose $a = 0$.

We also point out that a finite-dimensional a. a. A of characteristic 0 is semisimple if and only if the Lie algebra $\operatorname{Der} A$ is semisimple.

At the conclusion let us shortly look into the structure of the bimodules over finite-dimensional a. a. Just as in the case of the associative algebras, every alternative bimodule over a separable a. a. is completely reducible. The structure of irreducible bimodules is described in the following

Theorem (Schafer, 1952). *Let A be a finite-dimensional a. a., let M be a faithful irreducible alternative A-bimodule and let (ρ, λ) be the corresponding birepresentation of the algebra A. Then either M is an associative bimodule over the (associative) algebra A or one of the following cases holds:*

1) A is the algebra of generalized quaternions, λ is a (right) associative irreducible representation of A and $\rho(a) = \lambda(\overline{a})$, for every $a \in A$;

2) A is the Cayley-Dickson algebra and $M = A$ is a regular A-bimodule.

A few words about the methods of study of finite-dimensional a. a. These methods have a lot in common for different classes of algebras that are nearly associative (except the Lie algebras and their generalizations). In investigating simple and semisimple algebras the methods consist in passing to an algebraically closed field, in finding a sufficient number of orthogonal idempotents, and using the properties of Pierce decomposition of algebras. In studying solvable and nilalgebras the methods consist in the passage to the associative enveloping algebras. Apart from these, all the cases explore the traditional methods of finite-dimensional linear algebra: eigen vectors, minimal polynomials, the trace bilinear form etc.

2.5. Structure of Infinite-Dimensional Alternative Algebras (Zhevlakov, Slin'ko, Shestakov, Shirshov, 1978). We have already mentioned in Sect.1 that the fundamental instrument in building a structure theory of one or another class of algebras is the notion of a radical. In the case of infinite-dimensional a. a., the most important role is played by the quasi-regular and prime radicals.

The quasi-regular radical $\operatorname{Rad} A$ of an a. a. A is a direct generalization of the corresponding notion from the theory of associative algebras and allows several equivalent characterizations:

1) $\operatorname{Rad} A$ is the largest right (left) quasi-regular ideal of the algebra A;
2) $\operatorname{Rad} A$ is the intersection of all maximal modular right (left) ideals of the algebra A;
3) $\operatorname{Rad} A$ is the intersection of the kernels of all of the irreducible right (left) representations of the algebra A.

Here, just as in the case of associative algebras, an ideal I is called a *quasi-regular ideal* if every element $x \in I$ is *quasi-invertible* (i.e. the element $1-x$ is invertible in the algebra $A^\sharp$, obtained from A by adjoining an external unity); a right ideal I is a *modular right ideal*, if there exists an element $e \in A$ such that $x - ex \in I$, for every $x \in A$.

An algebra A is called *semisimple*, if $\operatorname{Rad} A = 0$ and is called (right) *primitive*, if A contains a maximal modular right ideal that does not contain non-zero two-sided ideals.

Construction of semisimple a. a. is described in the following

Theorem. *Every semisimple a. a. is isomorphic to a subdirect sum of primitive algebras, each of which is either associative or is the Cayley-Dickson algebra.*

The *prime radical* of an a. a. A is defined as the smallest ideal $P(A)$ for which the quotient algebra $A/P(A)$ is *semiprime* (i.e. does not contain non-zero nilpotent ideals). The ideal $P(A)$ may be not nilpotent in general, although it is a nilideal. Every semiprime algebra is isomorphic to a subdirect sum of *prime algebras* (i.e. algebras where the product of every two non-zero two-sided ideals is always different from 0). Simple algebras and algebras without zero divisors are examples of prime algebras.

The problem of describing the simple algebras is one of the central questions in studying every class of algebras. In a difference from the associative algebras, where this problem is practicably invisible (since every associative algebra may be embedded into a simple one), simple a. a. have an exhaustive description, modulo associative algebras.

Theorem. *Every simple non-associative a. a. is the Cayley-Dickson algebra over its center.*

This theorem generalizes, in particular, the Bruck-Kleinfeld-Skornyakov theorem on the structure of alternative skew-fields that was mentioned in 2.2.

One can relate a series of prime algebras with every central simple algebra. Let A be a central algebra over a field F. A subring $B \subseteq A$ is called a *central order* in A, if its center Z is contained in F and the ring of fractions $Z^{-1}B = \{z^{-1}b \mid 0 \neq z \in Z, b \in B\}$ coincides with A. It is easy to see that every central order of a simple algebra is a prime algebra. The central orders in Cayley-Dickson algebras are called the *Cayley-Dickson rings.* Prime a. a. are exhausted by these rings, with the exception of some "pathological" cases.

Theorem. *Every prime non-associative a. a. of characteristic* $\neq 3$ *is a Cayley-Dickson ring.*

The restriction on the characteristic is, generally speaking, essential. But it may be replaced, for instance, by the condition of A not having *absolute zero divisors* (i.e. elements a for which $aAa = 0$) or by the condition of A not having non-zero locally nilpotent ideals. Recall that an algebra A is called a *locally nilpotent algebra,* if every finitely generated subalgebra of A is nilpotent. Every a. a. A contains the largest locally nilpotent ideal $LN(A)$, which is called the *locally nilpotent radical* of the algebra A. The radical $LN(A)$ contains all the one-sided locally nilpotent ideals of the algebra A and all the absolute zero divisors; the quotient algebra $A/LN(A)$ is LN-semisimple and is isomorphic to a subdirect sum of prime LN-semisimple algebras.

The radicals $\operatorname{Nil} A$ (cf. 1.2), $\operatorname{Rad} A$, $LN(A)$ and $P(A)$ are related by the following inclusions:

$$\operatorname{Rad} A \supseteq \operatorname{Nil} A \supseteq LN(A) \supseteq P(A), \tag{21}$$

which are strict, in general, already in the case of the associative algebras. In finite-dimensional a. a. all these radicals coincide with the ordinary nilpotent radical (see 2.4). Moreover, they coincide within the class of *Artinian* a. a. (i.e. the algebras satisfying the minimality condition for the right ideals), for which the generalization of the classical associative theory is valid.

Theorem. *The* $\operatorname{Rad} A$ *is nilpotent in every Artinian a. a.* A. *An algebra* A *is Artinian semisimple if and only if it is a finite direct sum of full matrix algebras over skew-fields and the Cayley-Dickson algebras.*

It is not difficult to derive from here that, in an Artinian a. a., every nil-subalgebra is nilpotent. In particular, the properties of solvability and nilpotency are equivalent within the class of Artinian algebras. With some constraints on the characteristic, these properties are equivalent for finitely generated algebras and their subalgebras too. This is not the case in general – there exist examples of solvable, non-nilpotent a. a. over every field. Nevertheless, solvability and nilpotency are closely related within the class of a. a., as the following result shows:

Theorem (Pchelintsev, 1984, Shestakov, 1989). *Let* A *be a solvable alternative* Φ*-algebra. Then the subalgebra* A^2 *is nilpotent and, if* $\frac{1}{6} \in \Phi$, *then* $(A^n)^3 = 0$, *for some* n.

We also point out that over a field of characteristic 0, every alternative nilalgebra of bounded index is solvable.

Free algebras play an important role in the theory of every class of algebras: free non-associative algebra, free alternative algebra, free Lie algebra etc. Recall that the algebra $F_{\mathfrak{M}}[X]$ from the class $\mathfrak{M}$, with the set of generators X is called a *free algebra in the class* $\mathfrak{M}$ (or $\mathfrak{M}$-free), if every mapping from the set X into an arbitrary algebra A in $\mathfrak{M}$ is uniquely extendable to a homomorphism $F_{\mathfrak{M}}[X]$ to A.

The set of all the non-associative words made up out of elements of the set X forms the *basis of the free non-associative algebra* $F\{X\}$, over a field F; its elements may be viewed as non-associative, non-commutative polynomials in variables from X. If the class $\mathfrak{M}$ is defined by a system of identities $\{f_\alpha\}$, then the $\mathfrak{M}$-free algebra $F_{\mathfrak{M}}[X]$ is isomorphic to the quotient algebra $F\{X\}/I_{\mathfrak{M}}$, where $I_{\mathfrak{M}}$ is the ideal of the algebra $F\{X\}$, generated by the set $\{f_\alpha(y_1, \ldots, y_{n_\alpha})/y_i \in F\{X\}\}$. Thus, the *free alternative algebra* $F_{\mathrm{Alt}}[X]$ is isomorphic to the quotient algebra $F\{X\}$, mod the ideal I_{Alt}, generated by all the elements of the form (f_1, f_1, f_2), (f_1, f_2, f_2), where $f_1, f_2 \in F\{X\}$.

Many questions in the theory of a. a. are reduced to the study of the structure of free and PI-*algebras*, i.e. the algebras satisfying *essential polynomial identities*, namely identities which are not consequences of associativity. A general scheme of this reduction is as follows: In the free algebra, one looks for fully invariant (i.e. stable under endomorphisms) ideals, subalgebras or subspaces with certain nice properties (for instance being contained in some center). If, in an algebra A, the value of the elements from this ideal (or the subalgebra) are not all equal to zero, then A has a series of nice properties; otherwise, A is a PI-algebra.

Some basic properties of a free a. a. are described in the following

Theorem (Zhevlakov, Slin'ko, Shestakov, Shirshov, 1978; Il'tyakov, 1984; Filippov, 1984; Shestakov, 1976, 1977, 1983; Zel'manov, Shestakov, 1990).

Let $A = Alt[X]$ be the free a. a. over a field F of characteristic $\neq 2,3$ on the set of free generators X; let $Z(A)$ be its center and $N(A)$ its associative center. Then $[x,y]^4 \in N(A), (x,y,z)^4 \in Z(A)$, for all $x,y,z \in A$; for $|X| > 2$ the algebra A is not prime and for $|X| > 3$ it is not semiprime; $Rad\,A = \{x \in A \mid x^{n(x)} = 0\} = T(\mathbb{O}) \cap D(A)$, where $T(\mathbb{O})$ is the ideal of identities of the split Cayley-Dickson algebra over F and $D(A)$ is the associator ideal of A; if either $|X| < \infty$ or F is a field of characteristic 0, then $Rad\,A$ is nilpotent, and if $|X| \leqslant 3$, then $Rad\,A = 0$; if $|X| < |Y|$, then there is an identity in $Alt[X]$ which does not hold in $Alt[Y]$.

The study of structure of PI-algebras goes mainly by the pattern of the associative PI-theory. At this time, many principal results of that theory have been transfered to a. a. One of effective methods of investigation of alternative PI-algebras is a passage to algebras from other classes which are in one way or another connected with the given PI-algebra A. For instance, it is not difficult to check that, for an a. a. A, the algebra $A^{(+)}$ is a special Jordan algebra (see 3.1), where, if A is a PI-algebra, then $A^{(+)}$ is a Jordan PI-algebra. The most brilliant example of utilization of this connection are the results of A. I. Shirshov, devoted to solving the well known *Kurosh problem* inside the class of alternative PI-algebras. This problem, which is a typical example of a problem of the "Burnside" type, is formulated as follows: if in an algebra A, every singly generated subalgebra is finite-dimensional, then is every finitely generated subalgebra of A finite-dimensional? In general the answer is negative, already for the associative algebras (although for skew-fields the answer is not known), but if A is a PI-algebra, the answer is affirmative both for the associative as well as for the alternative and Jordan algebras. Furthermore, the solution of the problem for the a. a. relies essentially on the case of special Jordan algebras. Let us state an important partial case of this result.

Theorem. *An a. a. with the identity $x^n = 0$ is locally nilpotent.*

Along with the Jordan algebra $A^{(+)}$, it is useful to draw the algebra of right multiplications $R(A)$ into the study of properties of an a. a. A; this algebra inherits many properties of A. For instance, if A is a finitely generated PI-algebra, then the algebra $R(A)$ is of the same kind, and in this case we may apply the well developed associative PI-theory for studying A. On this path, we for example prove the following

Theorem (Shestakov, 1983). *The radical $Rad\,A$ of a finitely generated alternative PI-algebra A over a field is nilpotent.*

In investigating properties of free a. a., alternative superalgebras (cf. 1.2) proved to be useful; they satisfy the following classification theorem which has been proved recently (E. I. Zel'manov, I. P. Shestakov, 1990): every prime alternative superalgebra $A = A_0 \dotplus A_1$ of a characteristic $\neq 2,3$ is either associative or $A_1 = 0$ and A_0 is the Cayley-Dickson ring.

§3. Jordan Algebras

3.1. Examples of Jordan Algebras. Recall that an algebra is called a *Jordan algebra* (J. a.), if it satisfies the following identities:

$$xy = yx$$
$$(x^2y)x = x^2(yx).$$

In this section we assume that the base field F is of characteristic not equal to 2.

Example 1. Let A be an associative algebra. Then, the algebra $A^{(+)}$ is a J. a., as noted in 1.1. Every subspace J in A, closed with respect to the operation $x \cdot y = \frac{1}{2}(xy + yx)$, forms a subalgebra of the algebra $A^{(+)}$ and is consequently a J. a. Such a J. a. J is called a *special Jordan algebra* and the subalgebra A_0 in A generated by J is called the *associative enveloping algebra* of J. Properties of the algebras A and $A^{(+)}$ are closely related: A is simple (prime, nilpotent), if and only if $A^{(+)}$ has the corresponding properties. The algebra $A^{(+)}$ may happen to be a Jordan algebra for non-associative A too. For instance, if A is a right alternative (in particular, alternative) algebra, then $A^{(+)}$ is a special J. a. (cf. 4.2).

Example 2. Let X be a vector space of dimension greater than 1 over F, with a symmetric nondegenerate bilinear form $f(x,y)$. Let us consider the direct sum of vector spaces $J(X,f) = F \dotplus X$ and let us define multiplication there in the following way:

$$(\alpha + x)(\beta + y) = (\alpha\beta + f(x,y)) + (\alpha y + \beta x).$$

Then $J(X,f)$ is a simple special J. a.; its enveloping algebra is the Clifford algebra $\mathrm{Cl}\,(X,f)$ of the bilinear form f. In case when $F = \mathbb{R}$ and $f(x,y)$ is the ordinary scalar product on X, the algebra $J(X,f)$ is called the *spin-factor* and is denoted by V_n, where $n - 1 = \dim X$.

Example 3. Let A be an associative algebra with involution $*$. The set $H(A,*) = \{h \in A \mid h^* = h\}$ of $*$-symmetric elements is closed with respect to Jordan multiplication $x \cdot y$, and therefore, is a special J. a. For instance, if D is a composition associative algebra over F, with involution $d \to \bar{d}$ (cf. 2.1) and if D_n is the algebra of $n \times n$ matrices over D, then the mapping $S : (a_{ij}) \mapsto (\overline{a_{ji}})$ is an involution in D_n and the set of the D-Hermitian matrices $H(D_n) = H(D_n, S)$ is a special J. a. If the algebra A is $*$-simple (i.e. does not contain proper ideals I, such that $I^* \subseteq I$), then $H(A,*)$ is simple; if A is $*$-prime, then $H(A,*)$ is prime. In particular, all the algebras $H(D_n)$ are simple. Every algebra of the form $A^{(+)}$ is isomorphic to the algebra $H(B,*)$, where $B = A \oplus A^0$, the algebra A^0 is anti-isomorphic to A, and $(a_1,a_2)^* = (a_2,a_1)$.

Example 4. If $D = \mathbb{O}$ is the Cayley-Dickson algebra, then the corresponding algebra $H(\mathbb{O}_n)$ of Hermitian matrices is a J. a., only for $n \leqslant 3$. In cases $n = 1, 2$, the algebras obtained in this way are isomorphic to some algebras in Example 2, and thus are special. The algebra $H(\mathbb{O}_3)$ is not special and gives us an example of a simple exceptional J. a. Albert was the first to prove that the algebra $H(\mathbb{O}_3)$ is exceptional. We will call a J. a. J the *Albert algebra*, if $J \otimes_F K \cong H(\mathbb{O}_3)$, for some extension K of the field F. Every Albert algebra is simple, exceptional and is of dimension 27 over its center.

We will see in the sequel, in 3.7 that the stated examples exhaust all the simple J. a.

3.2. Finite-Dimensional Jordan Algebras (Braun, Koecher, 1966), (Jacobson, 1968). Let J be a J. a. and let $a, b, c \in J$. Consider a the following regular birepresentation of the algebra J: $a \mapsto L_a, a \mapsto R_a$ (cf. 1.2). In view of (1. 17), we have

$$L_a = R_a, \quad [R_{a^2}, R_a] = 0, \qquad R_{a^2 b} - R_b R_{a^2} + 2R_a R_b R_a - 2R_a R_{ba} = 0. \tag{1}$$

Linearizing the last relation in a, we get the following:

$$R_{ac \cdot b} - R_b R_{ac} + R_a R_b R_c + R_c R_b R_a - R_a R_{bc} - R_c R_{ba} = 0. \tag{2}$$

It easily follows from (2) that, for every $k \geqslant 1$, the operator R_{a^k} belongs to the subalgebra $A \subseteq \operatorname{End} J$ generated by the operators R_a, R_{a^2}. In view of (1), A is commutative, thus we have $[R_{a^k}, R_{a^n}] = 0$, for all $k, n \geqslant 1$, or $(a^k, J, a^n) = 0$. In particular, every J. a. J is power-associative and the nilradical $\operatorname{Nil} J$ is uniquely defined. Just as in the case of the alternative algebras, the following theorem holds:

Theorem. *Let J be a finite-dimensional J. a. Then the radical $\operatorname{Nil} J$ is nilpotent and the quotient algebra $J/\operatorname{Nil} J$ is isomorphic to the direct sum of simple algebras.*

An important example of semisimple J. a. over $\mathbb{R}$ are so-called *formally real J. a.*, i.e. algebras where the equality $x^2 + y^2 = 0$ implies $x = y = 0$. In the foundational paper (Jordan, von Neumann, Wigner, 1934), the finite-dimensional formally real J. a. were characterized as the direct sums of simple algebras of one of the following forms: $\mathbb{R}, V_n, H(\mathbb{R}_n), H(\mathbb{C}_n), H(\mathbb{H}_n), H(\mathbb{O}_3)$, where $\mathbb{C}$ is the field of complex numbers, $\mathbb{H}$ is the quaternion skew-field and $\mathbb{O}$ is the algebra of Cayley numbers and $n \geqslant 3$. Simple finite-dimensional algebras over an algebraically closed field F are described in a similar fashion:

Theorem. *Every simple finite-dimensional J. a. over an algebraically closed field F is isomorphic either to F or to the algebra $J(X, f)$, or to the algebra of Hermitian matrices $H(D_n), n \geqslant 3$, over a composition algebra D, associative for $n > 3$.*

Recall (see 1.2) that every finite-dimensional commutative power-associative nilsemisimple algebra over a field of characteristic 0 is a Jordan algebra, thus this theorem at the same time gives a description of simple finite-dimensional commutative power-associative algebras of characteristic 0, which are not nilalgebras.

In a finite-dimensional J. a. J with the separable quotient algebra $J/\operatorname{Nil} J$ an analogue of the classical Wedderburn-Malcev theorem on splitting off of the radical and conjugation of semisimple factors holds.

Structure of Jordan bimodules over a J. a. J is defined by its *universal multiplicative enveloping algebra* $U(J)$, which is defined as the quotient algebra of the tensor algebra $T(J)$ over the ideal generated by the set of the elements of the forms

$$a^2 \otimes a - a \otimes a^2, a^2 b - b \otimes a^2 - 2a \otimes ba + 2a \otimes b \otimes a, \text{ where } a, b \in J$$

(see 1.2). A linear mapping $\rho : J \longrightarrow \operatorname{End} M$ is a representation of a J. a. J (or, equivalently, the pair (ρ, ρ) is a birepresentation of J) if and only if there exists a homomorphism of associative algebras $\phi : U(J) \longrightarrow \operatorname{End} M$, coinciding with ρ on the elements in J, identified with its cannonical images in $U(J)$. Thus descriptions of Jordan J-bimodules reduce to determining the structure of the algebra $U(J)$ and to a study of its associative representations. A finite-dimensional J. a. J is separable if and only if $U(J)$ is separable. This implies that every Jordan bimodule over a separable finite-dimensional J. a. is completely reducible. The construction of the algebra $U(J)$ is known for all central simple finite-dimensional J. a. J. This also determines irreducible J-bimodules (they correspond to simple components of the algebra $U(J)$).

Example 1. Let J be the Albert algebra. Then $U(J) = F \oplus F_{27}$. The component F corresponds to the trivial one-dimensional J-bimodule, and F_{27} to a regular J-bimodule. Since we usually do not consider trivial bimodules to be irreducible, every irreducible J-bimodule is isomorphic to a regular bimodule.

Example 2. $J = J(X, f)$. Then $U(J) = F \oplus \operatorname{Cl}(X, f) \oplus D(X, f)$, where $D(X, f)$, is the so-called "meson algebra", defined as the quotient algebra $T(X)/I$, where $T(X)$ is the tensor algebra of the space X and I is the ideal in $T(X)$ generated by all the elements of the form $x \otimes y \otimes x - f(x, y)x$, where $x, y \in X$. One can show that $D(X, f)$ is isomorphic to the subalgebra of the algebra $\operatorname{Cl}(X, f) \otimes \operatorname{Cl}(X, f)$, generated by elements of the form $x \otimes 1 + 1 \otimes x$, $x \in J$. We will not give the decomposition of $D(X, f)$ into simple components, since it is fairly complicated: it depends on the parity of the dimension of X and the discriminant of the form f.

3.3. Derivations of Jordan Algebras and Relations with Lie Algebras (Braun, Koecher, 1966; Schafer, 1966; Jacobson, 1966). With every J. a. J one can associate several interesting Lie algebras. We already know some of them: It is the derivation algebra Der J, the Lie multiplication algebra Lie (J) and the algebra of inner derivations Inder J = Der $J \cap$ Lie (J). We introduce two more algebras below: the structure algebra Strl J and the superstructure algebra (or the Tits-Kantor-Koecher construction) $K(J)$.

Let J be a J. a. with unity 1 and $a, b, c \in J$. By skewsymmetrizing (2) in a and b we get the following:

$$R_{(a,c,b)} = [R_c, [R_a, R_b]]. \tag{3}$$

Since $(a, c, b) = c[R_a, R_b]$, then, in view of (1. 11), (3) implies that the operator $[R_a, R_b]$ is a derivation of the algebra J. Moreover, (3) implies that Lie $(J) = R_J + [R_J, R_J]$, where $R_J = \{R_a \mid a \in J\}$. Since Der $J \cap R_J = 0$ (if $R_a \in$ Der J, then $0 = 1R_a = a$), we have Inder J = Der $J \cap$ Lie (J) = Der $J \cap (R_J + [R_J, R_J])$ = (Der $J \cap R_J) + [R_J, R_J] = [R_J, R_J]$, i.e. every inner derivation in J is of the form $D = \sum_i [R_{a_i}, R_{b_i}]$, $a_i b_i \in J$. At the same time, it has been proved that Lie $(J) = R_J \dotplus$ Inder J is the direct sum of vector spaces with multiplication

$$[R_a + D, R_b + D'] = R_{aD' - bD} + ([R_a, R_b] + [D, D']), \tag{4}$$

where $a, b \in J; D, D' \in$ Inder J. If D and D' in (4) are taken from the algebra Der J, then this formula will define multiplication on the vector space $R_J \dotplus$ Der J. The resulting algebra is again a Lie algebra, called the *structure algebra* of the algebra J and is denoted by Strl J.

Theorem. *Let J be a finite-dimensional semisimple J. a. over a field of characteristic 0. Then Der J = Inder J is a completely reducible Lie algebra. If J does not contain simple summands of dimension 3 over the center, then the algebra Der J is semisimple.*

The stated restriction is essential since Der V_3 is an one-dimensional Lie algebra. For the central simple J. a. J the algebra Der J is simple with the exception of the algebra $J(X, f)$, dim $X = 2, 3, 5$, and the algebra $H(F_4)$. If $J = H(D_n)$, where $n \geqslant 3$ and if D is a composition algebra of dimension d over F, then the dimension of the algebra Der J is given by the following table:

d	1	2	4	8
dim(Der J)	$\frac{n(n-1)}{2}$	$n^2 - 1$	$n(2n+1)$	52 $(n = 3)$

In particular, the algebra Der $(H(\mathbb{O}_3))$ is a simple 52-dimensional Lie algebra. According to the classification of finite-dimensional simple Lie algebras, over an algebraically closed field K of characteristic 0, only one of them has dimension 52, namely the exceptional algebra $\mathbb{F}_4$. Thus, over the field K, we have $\mathbb{F}_4$ = Der $(H(\mathbb{O}_3))$. For every Albert algebra J, the algebra Der J is

an algebra of type $\mathbb{F}_4$ and, conversely, every Lie algebra of type $\mathbb{F}_4$ is isomorphic to the algebra Der J, for a suitable Albert algebra J; furthermore, Der $J \cong$ Der J_1, if and only if $J \cong J_1$.

For a semisimple J. a. J, the structure algebra Strl J is not semisimple since its center contains the element $R_1 = \mathrm{Id}_J$, where 1 is the unity of J. Let $J_0 = \{x \in J \mid \mathrm{tr}(R_x) = 0\}$; then $J = F \cdot 1 + J_0$ and Strl $J = F \cdot R_1 \dotplus R_{J_0} \dotplus$ Der J. It is easy to see that (Strl $J)' = R_{J_0} \dotplus$ Der J. The subalgebra Strl$_0 J = R_{J_0} \dotplus$Der J of codimension 1 in Strl J is called the *reduced structure algebra* of the algebra J.

Theorem. *Let J be a central simple J. a. of finite dimension $n > 1$, over a field of characteristic 0. Then Strl$_0 J$ is a semisimple Lie algebra.*

If $J = H(D_n)$, where $n \geqslant 3$ and D is a composition algebra of dimension d over F, then the algebra Strl$_0 J$ is simple for $d = 1, 4, 8$, and for $d = 2$ it is the direct sum of two isomorphic simple algebras.

Example. The algebra Strl$_0(H(\mathbb{O}_3))$ is a simple Lie algebra of dimension (27-1)+52=78 and it acts irreducibly on $H(\mathbb{O}_3)$. There are three non-isomorphic simple Lie algebras of dimension 78, over an algebraically closed field K of characteristic 0: the algebra $\mathbb{E}_6$, the orthogonal Lie algebra $o(13)$ and the simplectic Lie algebra sp (12). However, the two latter algebras do not have irreducible representations of dimension 27. Thus, over the field K, we have $\mathbb{E}_6$ = Strl$_0(H(\mathbb{O}_3))$. If now J is an arbitrary Albert's algebra, then the algebra Strl$_0 J$ is a simple Lie algebra of type $\mathbb{E}_6$; in addition, the algebras Strl$_0 J$ and Strl$_0 J_1$, for Albert's algebras J and J_1, are isomorphic if and only if J and J_1 are isotopic (cf. 3.4 in the sequel).

We need a notion of the *triple Jordan product*, which plays an important role in the theory of J. a., in order to define the superstructure algebra $K(J)$. Let $a, b, c \in J$; set $\{abc\} = (a \cdot b) \cdot c + (c \cdot b) \cdot a - b \cdot (a \cdot c)$. Let us also define the linear operators $U_{a,b} : x \mapsto \{axb\}, V_{a,b} : x \mapsto \{abx\}, U_a = U_{a,a}$. If $J = A^{(+)}$, where A is an associative algebra with multiplication ab, then $\{aba\} = aba$. There are many arguments showing that the ternary operation $\{abc\}$ is more natural for J. a., then the ordinary multiplication. We will return later to properties of this operation. Let us point out now only that, if J has the unity, then the ordinary multiplication is expressible through the ternary by $V_{1,a} = R_a$.

Let us again consider the structure algebra Strl $J = R_J \dotplus$Der J. We define a mapping $*$ of the algebra Strl J into itself, by setting $(R_a + D)^* = -R_a + D$. It is easy to see that $*$ is an automorphism of order 2 of the algebra Strl J. We form the vector space $K(J) = J \dotplus$ Strl $J \dotplus \overline{J}$, where $\overline{J}$ is isomorphic to J under the isomorphism $a \mapsto \overline{a}$. Let us now define multiplication on $K(J)$, by setting

$$[a_1 + T_1 + \overline{b_1}, a_2 + T_2 + \overline{b_2}] = (a_1 T_2 - a_2 T_1) + (V_{a_1, b_2} - V_{a_2, b_1} + [T_1, T_2]) + (\overline{b_1 T_2^*} - \overline{b_2 T_1^*}).$$

The resulting algebra $K(J)$ is a Lie algebra, called the *superstructure algebra* or the *Tits-Kantor-Koecher construction* for the algebra J.

The correspondence $J \mapsto K(J)$ is functorial and there is a close relation between the properties of the algebras J and $K(J)$. The algebra J is simple (semisimple, solvable), if and only if $K(J)$ is of the same kind.

The algebra $K(J)$ has a less formal interpretation too. Let $\mathrm{Pol}\,(J)$ be the vector space of the polynomial transformations J into J. It is known that $\mathrm{Pol}\,(J)$ forms a Lie algebra with respect to the brackets

$$[p,q](x) \equiv \frac{\partial p(x)}{\partial x}q(x) - \frac{\partial q(x)}{\partial x}p(x),$$

the so-called "Lie algebra of polynomial vector fields on J". It is not difficult to ascertain that $K(J)$ is isomorphic to the subalgebra of the algebra $\mathrm{Pol}\,(J)$, which consists of quadratic polynomials of the form $a + Tx + \{xbx\}$, where $a, b \in J, T \in \mathrm{Strl}\, J$.

Example 1. Let $J = \mathbb{R}$. Then $K(\mathbb{R})$ consists of the polynomials of the form $p(x) = p_1 + p_2x + p_3x^2$, where $p_i \in \mathbb{R}$; furthermore

$$[p,q](x) = p'(x)q(x) - q'(x)p(x) =$$
$$(p_2q_1 - q_2p_1) + 2(p_3q_1 - q_3p_1)x + (p_3q_2 - q_3p_2)x^2.$$

Thus, $K(\mathbb{R}) \cong \mathrm{sl}\,(2,\mathbb{R})$.

Example 2. Let J be the Albert algebra. Then $K(J)$ is a simple Lie algebra of dimension 27+79+27=133, i.e. $K(J)$ is an algebra of type $\mathbb{E}_7$.

The Tits-Kantor-Koecher construction results in fact not only in one Lie algebra, but the whole series. Namely, let H be an arbitrary subalgebra of the algebra $\mathrm{Strl}\, J$ which contains the subalgebra $R_J \dotplus \mathrm{Inder}\, J$; then the vector space $K_H(J) = J \dotplus H \dotplus \overline{J}$ is a subalgebra of $K(J)$. All the algebras $K_H(J)$ are 3-graduated, i.e. are of the form $L = L_{-1} \dotplus L_0 \dotplus L_1$, where $L_iL_j \subseteq L_{i+j}, L_i = 0$, for $|i| > 1$; in addition $L_{-1} = J, L_1 = J, L_0 = H$.

In the conclusion of this part we give the *Tits construction* which brings about exceptional simple Lie algebras of every type, with the aid of the composition algebras and Jordan matrix algebras of order 3.

Let F be a field of characteristic $\neq 2, \neq 3$ and let A be a composition algebra over F; J is either F or is the algebra $H(D_3)$, where D is the composition algebra over F. Let us denote by $t(a)$ the trace of the element a in the algebra A and by $\mathrm{tr}\,(x)$ denote the ordinary trace of the matrix x in the algebra $H(D_3)$. Let $A_0 = \{a \in A \mid t(a) = 0\}$, $J_0 = \{x \in J \mid \mathrm{tr}\,(x) = 0\}$ ($J_0 = 0$, for $J = F$). For $a, b \in A$ and $x, y \in J$, set

$$a * b = ab - \frac{1}{2}t(ab), \quad x * y = xy - \frac{1}{3}\mathrm{tr}\,(xy);$$

then $a * b \in A_0, x * y \in J_0$. Let us define an anticommutative multiplication on the direct sum of vector spaces $\mathrm{Der}\, A \dotplus A_0 \otimes J_0 \dotplus \mathrm{Der}\, J$, according to the following rules:

1) $\mathrm{Der}\, A$ and $\mathrm{Der}\, J$ are commuting subalgebras in L;
2) $[a\otimes x, D] = aD\otimes x, [a\otimes x, E] = a\otimes xE$, for all $D \in \mathrm{Der}\, A, E \in \mathrm{Der}\, J, a \in A_0, x \in J_0$;
3) $[a \otimes x, b \otimes y] = \frac{1}{12}\mathrm{tr}\,(xy)D_{a,b} + (a * b) \otimes (x * y) + \frac{1}{2}t\,(ab)[R_x, R_y]$, for all $a, b \in A_0, x, y \in J_0$, where $D_{a,b} \in \mathrm{Der}\, A$ is of the form (2.19).

The resulting algebra L is a Lie algebra. For brevity, we denote $\mathbb{C}(\alpha) = \mathbb{C}, \mathbb{H}(\alpha, \beta) = \mathbb{H}$; then the type of the algebra L depending on the form of the algebras A and J is determined by the following table:

A \ J	F	$H(F_3)$	$H(\mathbb{C}_3)$	$H(\mathbb{H}_3)$	$H(\mathbb{O}_3)$
$\mathbb{F}$	0	A_1	A_2	C_3	F_4
$\mathbb{C}$	0	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	A_1	C_3	A_5	A_6	E_7
$\mathbb{O}$	G_2	F_4	E_6	E_7	E_8

For instance, if $A = \mathbb{O}, J = H(\mathbb{O}_3)$, then the algebra $L = \mathrm{Der}\, A \dotplus A_0 \otimes J_0 \dotplus \mathrm{Der}\, J$ is of dimension 14+7·26+52=248 and is a simple Lie algebra of type $\mathbb{E}_8$.

3.4. Isotopies of Jordan Algebras, Jordan Structures (Braun, Koecher, 1966; Jacobson, 1968; Meyberg, 1972; Springer, 1973; Loos, 1975). An element a in a J. a. J with unity 1 is called *invertible*, if the operator $U_a = 2R_a^2 - R_{a^2}$ is invertible in $\mathrm{End}\, J$. It is easy to see that a is invertible if and only if 1 is in the image of U_a. Set $a^{-1} = aU_a^{-1}$; then a^{-1} is also invertible and $(a^{-1})^{-1} = a$. If A is an associative algebra, then a is invertible in A if and only if it is invertible in $A^{(+)}$ and a^{-1} is same in A and in $A^{(+)}$. If every element in a J. a. J is invertible, then J is called a *division J. a.* We point out that the invertibility of an element a in a J. a., does not, generally speaking imply invertibility of the operator R_a, thus the equations $ax = b$ ($a \neq 0$) are not necessarily solvable in a division J. a.

The notion of isotopy, whose sources lie in the associative theory, plays an important role in the theory of J. a. If an invertible element c is fixed in an associative algebra A, and if a new c-multiplication $a_cb = ac^{-1}b$ is defined, then the resulting algebra $A^{(c)}$ will again be associative, where the element c will be the unity in $A^{(c)}$. Analogously, for a J. a. J with an invertible element c, the algebra $J^{(c)}$ obtained from J by introduction of a c-multiplication $a_c \cdot b = \{ac^{-1}b\}$, will be a J. a. with the unity c. The algebra $J^{(c)}$ is called the *c-isotope* of the algebra J. Two J. a. are called *isotopic*, if one of them is isomorphic to an isotope of the other; the corresponding isomorphism is called an *isotopy*. In the associative case, the algebras A and $A^{(c)}$ are always isomorphic: the mapping $x \mapsto xc$ is an isomorphism of A and $A^{(c)}$, thus the notion of isotopy does not play a special role here. In the Jordan case, the situation is different: the algebra $J^{(c)}$ may be non-isomorphic to J. For instance, the algebra $J = H(\mathbb{R}_2)$ does not contain nilpotent elements, while its isotope $J^{(c)}$, for $c = e_{12}+e_{21}$ contains the nilpotent element e_{11}: $e_{11}\cdot e_{11} = e_{11}(e_{12}+e_{21})^{-1}e_{11} = 0$. Nevertheless, many important properties of J. a., such as the properties of being simple or special, are invariant with respect to isotopy; isotopic J. a have isomorphic structure and superstructure Lie algebras.

An isotopy of a J. a. J with itself is called an *autotopy* (in other words, an autotopy is an isomorphism of an algebra with its isotop). The family of all the autotopies of J. a. J form a group, which is called the *structure group* of the algebra J and is denoted by $\operatorname{Str} J$. The group $\operatorname{Str} J$ is an algebraic group and its Lie algebra is the structure algebra $\operatorname{Strl} J$. The automorphisms of J are autotopies which fix the unity 1 of the algebra J. More generally, two isotopes $J^{(a)}$ and $J^{(b)}$ are isomorphic if and only if there is an autotopy carrying a to b.

A series of important theorems in the theory of J. a. hold "up to an isotopy", i.e. their conclusions do not apply to the algebras in question, but only to some of their isotopies. In this respect the group $\operatorname{Str} J$ often turns out to be more useful than the automorphism group $\operatorname{Aut} J$. For instance, there is no natural notion of an inner automorphism for a J. a., while at the same time, for every invertible $a \in J$, the operator U_a is an "inner autotopy" from J to $J^{(a^2)}$. All this suggests a thought about the existence of some algebraic object which unifies the J. a. J and all its isotopies and which has the group $\operatorname{Str} J$ as its automorphism group. Such an object exists indeed and it is the Jordan pair (J, J).

The *Jordan pair* (J. p.) over a field F of characteristic $\neq 2, \neq 3$ is the pair $V = (V^+, V^-)$ of vector spaces, with two trilinear mappings $V^\sigma \times V^{-\sigma} \times V^\sigma \mapsto V^\sigma, \sigma = \pm$, written as $(x, y, z) \mapsto \{xyz\}$ and which satisfying the following identities

$$\{xyz\} = \{zyx\}, \tag{5}$$

$$\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\}. \tag{6}$$

Examples of J. p.:

Example 1. $V(J) = (J, J)$, where J is a J. a. and $\{xyz\}$ is the triple Jordan product in J. For this J. p., $\operatorname{Aut} V(J) \cong \operatorname{Str} J$. If $V = (V^+, V^-)$ is an arbitrary J. p., then for every $a \in V^\sigma$, the space $V^{-\sigma}$, with the multiplication operation $x_a \cdot y = \{xay\}$ is a J. a. J_a. If, in addition, the element a is invertible, then $V \cong V(J_a)$. Thus the (unital) J. a. may be seen, "up to an isotopy", to be a J. p. with invertible elements.

Example 2. A vector space T with a trilinear operation $\{xyz\}$, satisfying (5) and (6), is called a *Jordan triple system* (J. t. s.). For instance, every J. a. is a J. t. s. with respect to the triple Jordan product; the rectangular $p \times q$ matrices $M_{p,q}(F)$ form a J. t. s. with respect to the operation $\{xyx\} = xy^tx$ (in view of (5), the trilinear operation $\{xyz\}$ in J. p. and J. t. s. is obtained by linearization of the quadratic operation $\{xyx\}$, thus it suffices to give the latter). With every J. t. s. T, one can relate in a natural way the J. p. $V(T) = (T, T)$. Not every J. p. is obtainable in this fashion, since there exist J. p. for which $\dim V^+ \neq \dim V^-$. The pair $V(T)$ has the involution $(t_1, t_2) \mapsto (t_2, t_1)$; conversely, every J. p. with involution is of the form $V(T)$, for an appropriate J. t. s. T. Thus J. t. s. may be viewed as a J. p. with involution.

Example 3. $V = (M_{p,q}(F), M_{q,p}(F)), \{xyx\} = xyx$. It is easy to see that $V \cong V(T)$, for the J. t. s. $T = M_{p,q}(F)$ from Example 2.

Example 4. $V(R) = (R_{-1}, R_1)$, where $R = R_{-1} \dotplus R_0 \dotplus R_1$ is an associative 3-graded algebra ($R_iR_j \subseteq R_{i+j}$, $R_i = 0$, for $|i| > 1$); $\{xyx\} = xyx$. If, in addition, R is a simple algebra and $R_{-1} + R_1 \neq 0$, then $V(R)$ is a simple J. p.

Example 5. $V(L) = (L_{-1}, L_1)$, where $L = L_{-1} \dotplus L_0 \dotplus L_1$ is a 3-graded Lie algebra; $\{xyz\} = [[x, y], z]$. An example of such a Lie algebra is the superstructure algebra $L = K(J)$, for a J. a. J; furthermore $V(L) \cong V(J)$. If L_0 acts faithfully on $L_{-1} + L_1$, then we may assume that $\operatorname{Inder} V(L) \subseteq L_0 \subseteq \operatorname{Der} V(L)$. The algebra L will be called faithful in this case. A subalgebra H of derivations of a J. p. V will be called large, if $\operatorname{Inder} V \subseteq H$.

The Tits-Kantor-Koecher construction has a generalization to J. p., by assigning, to every J. p. $V = (V^+, V^-)$ with a large subalgebra of derivations H, the faithful 3-graded Lie algebra $K_H(V) = V^- \dotplus H \dotplus V^+$; furthermore, there is a bijective correspondence between faithful 3-graded Lie algebras and J. p. with fixed large derivation subalgebras.

The simple Lie algebras $A_n, B_n, C_n, D_n, E_6, E_7$ have a non-trivial faithful 3-grading, thus they allow construction and study with the aid of J. p. The algebras G_2, F_4, E_8 do not have such a grading, but have a grading of the form $L = L_{-2} \dotplus L_{-1} \dotplus L_0 \dotplus L_1 \dotplus L_2$. Jordan methods are effectively applicable in studying these algebras, as well as the Lie algebras with an arbitrary finite $\mathbb{Z}$-grading (Kantor, 1974; Allison, 1976, 1978; Zel'manov, 1984).

Another important kind of Jordan structures has been studied in recent times, namely *Jordan superalgebras* (cf. 1.2). Just as in the case of ordinary algebras, there is a close tie between Jordan and Lie superalgebras; in particular, the Tits-Kantor-Koecher construction generalizes to Jordan superalgebras. With help of this connection, on the basis of a known classification of simple finite-dimensional Lie superalgebras over an algebraically closed field of characteristic 0, a classification of simple Jordan superalgebras with the same conditions had been obtained in (Kac, 1977). I. L. Kantor and E. I. Zel'manov have pointed out recently that the classification given in (Kac, 1977) has a gap: a series of simple Jordan superalgebras, connected with gradings of Hamiltonian Lie superalgebras has been omitted.

A typical example of a Jordan superalgebra is the algebra $A^{(+)s}$, obtained by introducing *Jordan supermultiplication* $x \cdot^s y = \frac{1}{2}(xy + (-1)^{ij}yx)$, $x \in A_i, y \in A_j$, on the vector space of the associative superalgebra $A = A_0 \dotplus A_1$. The superalgebra J is called special if it is embeddable in a suitable algebra

$A^{(+)s}$. For instance, if A has a superinvolution $*$ (i.e. $A_i^* \subseteq A_i$ and $(xy)^* = (-1)^{ij}y^*x^*$, for $x \in A_i, y \in A_j$), then the set of supersymmetric elements $H(A,*) = \{x \in A \mid x^* = x\}$ forms a subalgebra in $A^{(+)s}$. If $X = X_0 \dot{+} X_1$ is a vector space with a supersymmetric bilinear form f (f is symmetric on X_0 and skewsymmetric on X_1, $f(X_i, X_j) = 0$, for $i \neq j$), then the algebra of the bilinear form $J(X,f) = F \dot{+} X$ is a Jordan superalgebra with $J_0 = F + X_0, J_1 = X_1$. For every $\mathbb{Z}_2$-graded J. a. $J = J_0 + J_1$, its Grassmann envelope $G(J) = G_0 \otimes J_0 + G_1 \otimes J_1$ is a Jordan superalgebra.

An important class of Jordan superalgebras is connected with the algebras of Poison brackets. Let A be an associative and commutative algebra with a skew symmetric bilinear operation $\{x,y\}$ (Poison brackets), such that A is a Lie algebra under this operation and such that, for every $a \in A$, the mapping $x \to \{a,x\}$ is a derivation of the algebra A. Then the superalgebra $J = J_0 \dot{+} J_1$, where $J_0 = J_1 = A$ with the multiplication

$$(a_0 + b_1)(c_0 + d_1) = (ac + \{b,d\})_0 + (ad + bc)_1, \text{where } a,b,c,d \in A,$$

is a Jordan superalgebra. This construction has a generalization to the case when A is a commutative superalgebra.

There are two more types of Jordan structures that we have not mentioned. These are the *quadratic J. a.* (q. J. a.) and so-called *J-structures*. In the definition of a q. J. a., the bilinear operation of multiplication $x \cdot y$ is substituted by the quadratic operation $yU_x = \{xyx\}$, with the following axioms:

1) $U_1 = \mathrm{Id}$,
2) $U_x V_{y,x} = V_{x,y} U_x$,
3) $U_{yU_x} = U_x U_y U_x$,

where $zV_{x,y} \equiv \{zyx\} \equiv y(U_{x+z} - U_x - U_z)$. The advantage of the q. J. a. is in the fact that they cover the case of characteristic 2; in the case of characteristic $\neq 2$, they are equivalent to ordinary J. a. At present, almost all the fundamental theorems of J. a. have been carried over to q. J. a (McCrimmon, 1966; Jacobson, 1981; Zelmanov, McCrimmon, 1988)

The notion of a J-structure is based on the operation of inversion $x \mapsto x^{-1}$. The Hua identity $\{xyx\} = x - (x^{-1} - (x - y^{-1})^{-1})^{-1}$, which holds for any elements of an arbitrary J. a., for which the right-hand-side is defined, shows that, if a J. a. has "many" invertible elements, then the operation of inversion contains all the information about the algebra. In particular, this is so in the finite-dimensional case. A finite-dimensional space V, with a fixed element 1 and a birational mapping $x \mapsto x^{-1}$ is called a *J-structure* if 1) $1^{-1} = 1, (x^{-1})^{-1} = x, (\lambda x)^{-1} = \lambda^{-1}x^{-1}$, for $\lambda \in F$; 2) $(1+x)^{-1} + (1+x^{-1})^{-1} = 1$; 3) the orbit of 1, under the action of the structure group $G = \{g \in \mathrm{GL}(V) \mid (xg)^{-1} = (x^{-1})h$, for some $h \in \mathrm{GL}(V)\}$ is a Zariski open set in V. Over the fields of characteristic $\neq 2$, the J-structures are categorically equivalent to finite-dimensional J. a. with unity. This approach was used in (Springer, 1973), for a classification of simple finite-dimensional J. a., on the basis of the Cartan-Shevalley theory of semisimple linear algebraic groups.

3.5. Jordan Algebras in Projective Geometry. We have established in 2.2 that every Moufang plane may be coordinatized by a Cayley-Dickson skew-field, uniquely determined up to isomorphism. This coordinatization is still insufficient for describing isomorphisms (collineations) of Moufang planes in algebraic language – in the spirit of "geometric algebra". The latter is achievable with the aid of representations of Moufang planes in simple exceptional J. a.

Let $J = (H(D_3))^{(\gamma)}$, where D is the Cayley-Dickson skew-field and γ is an invertible diagonal element in J. Let us denote by P the set of all the elements of "rank 1" in J: $P = \{0 \neq x \in J \mid JU_x = Fx\}$. If $a \in P$, then either $a^2 = 0$, or $a = \alpha e$, where e is a primitive idempotent. Let $[a] = aF^*$ be the "ray" spanned over the element a. Denote by a^* and a_* two samples of the set $[a]$. We define the plane $\pi(J)$ with the set of points $\pi_0 = \{a_* \mid a \in P\}$ and the set of lines $\pi^0 = \{a^* \mid a \in P\}$, regarding a_* to be incident to b^*, if $\mathrm{tr}\,(ab) = 0$, where $\mathrm{tr}\,(x)$ is the trace of the matrix x. Then $\pi(J)$ is a projective Moufang plane, coordinatized by the skew-field D.

Fundamental theorem of projective geometry for Moufang planes $\pi(J)$ (Jacobson, 1968; Faulkner, 1970). *Every collineation of the projective Moufang plane $\pi(J)$ is induced by a semilinear autotopy of the algebra J, defined uniquely up to some factors in F^*. The planes $\pi(J)$ and $\pi(J_1)$ are isomorphic if and only if J and J_1 are isotopic (as rings).*

Another interesting application of J. a. in projective geometry concerns so-called Moufang polygonal geometries, where J. a. arise as the coordinatizing algebras (Faulkner, 1977).

3.6. Jordan Algebras in Analysis. J. a. have various and deep applications in differential geometry, in real, complex and functional analysis, in theory of automorphic functions (Koecher, 1962, 1971; Loos, 1969; McCrimmon, 1978; Iordanesku, 1979; Hanche-Olsen, Stormer, 1984; Ayupov, 1985; Upmeier, 1985). The essence of the majority of them is in close relations among formally real J. a., self-dual convex cones and the Hermitian symmetric spaces.

An analytic Riemannian manifold M is called a *Riemannian symmetric space*, if every point $p \in M$ is an isolated fixed point of some involutive isometry (a geodesic symmetry with respect to p). Important examples of such spaces are *self-dual convex cones*, i.e. open subsets Y of an Euclidean space X such that: 1) $(x,y) > 0$, for every $x,y \in Y$; 2) if $(x,y) > 0$, for all $0 \neq y \in \overline{Y}$, then $x \in Y$. Let $\mathrm{Aut}\,Y = \{A \in \mathrm{GL}(X) \mid A(Y) = Y\}$; the cone Y is called homogeneous, if the group $\mathrm{Aut}\,Y$ acts transitively on it. It is not difficult to see that, for every formally real J. a. J, the set $C(J) = \{x^2 \mid$

$0 \neq x \in J\} = \{\exp x \mid x \in J\}$ is a homogeneous self-dual convex cone (with respect to the form $(x,y) = \operatorname{tr} R_{xy}$). Conversely, the following holds:

Theorem (Vinberg, 1965; Iochum, 1984; Koecher, 1962). *Every homogeneous self-dual convex cone Y is of the form $Y = C(J)$, for some formally real J. a. J.*

Geometric structure of the cone $C(J)$ is quite compatible with the algebraic structure of J: the geodesic symmetry at a point p is the operation of inversion in the isotope $J^{(p)}$; the coefficients $\Gamma_{ij}^{k}(p)$ of affine connectedness coincide with the structural constants γ_{ij}^{k} of the algebra $J^{(p)}$. If J is simple, then $\operatorname{Str} J = \{\pm A \mid A \in \operatorname{Aut} C(J)\}$.

A complex analogue of a Riemann symmetric space is a *Hermitian symmetric space*, defined as a real Riemann symmetric space with complex structure, invariant with respect to geodesic symmetry. Examples of Hermitian symmetric spaces are *bounded symmetric domains*, i.e. bounded domains in $\mathbb{C}^n$ such that their every point is isolated fixed point of some involutive automorphism. A metric, the so-called Bergman metric, may be introduced in every such a domain, and the following theorem holds:

Theorem (Helgason, 1962; Loos, 1969). *A bounded symmetric domain with Bergman metric is a Hermitian symmetric space of a non-compact type. Conversely, every Hermitian symmetric space of a non-compact type is isomorphic to a bounded symmetric domain.*

The simplest examples of a Hermitian symmetric space of non-compact type and a bounded symmetric domain are respectively the upper half-plane and the unit disc in $\mathbb{C}$. An isomorphism between them is realized through the Cayley transformation $z \mapsto \frac{z-i}{z+i}$. Let now J be a formal real J. a. and let $C(J)$ be its related convex cone. Consider the set $H(J) = \{x+iy \mid x \in J, y \in C(J)\}$ in the complexification J_C of the algebra J and call it the half-space associated with the algebra J.

Theorem (Koecher, 1962). *The half-space $H = H(J)$ is a Hermitian symmetric space of a non-compact type, and the mapping $\phi : z \mapsto (z - i \cdot 1)(z + i \cdot 1)^{-1}$ defines an isomorphism of H with the bounded symmetric domain $D = \phi(H) = \{z \in J_C \mid 1 - z\overline{z} \in C(J)\}$.*

Example 1. $J = \mathbb{R}$, H is the upper half-plane and D – the unit disc.

Example 2. $J = H(\mathbb{R}_n)$, $H = \{A + iB \mid A, B \in J$ and B is positive definite$\}$ is the Siegel generalized upper half-plane (Helgason, 1962) and $D = \{z \in J_C \mid 1 - z\overline{z} \in C(J)\}$ is a generalized unit disc.

The geometry of the half-space $H(J)$ is described well in Jordan terms: for instance, the group $\operatorname{Aut}(H(J))$ consists of linear-fractional transformations, generated by the inversion $z \mapsto -z^{-1}$, by the translations $z \mapsto z + a$, $a \in J$, and the transformations in $\operatorname{Aut} C(J)$.

The bounded symmetric domains allow for another reduction to "nicer" objects – to so-called *bounded homogeneous circular domains*. These are defined as homogeneous bounded domains in $\mathbb{C}^n$ which contain the origin, and for which the transformation $x \mapsto e^{it}x$ is an automorphism for every $t \in \mathbb{R}$. These domains are symmetric: at the origin, the symmetry is given by the automorphism $x \mapsto -x = e^{i\pi}x$, and the symmetry exists in other points because of homogeneity. Examples of such domains are the domains $D = \phi(H)$, for the half-planes H considered above.

Theorem (Koecher, 1969; Upmeier, 1985). *Every bounded symmetric domain is biholomorphically equivalent to a bounded homogeneous circular domain.*

The circular domains of the stated form are in turn categorically equivalent to the *Hermitian Jordan triple systems*, i.e. to the real J. t. s. with a complex structure such that the triple product $\{xyz\}$ is $\mathbb{C}$-linear in x, z and is $\mathbb{C}$-antilinear in y, and the bilinear form $\langle x, y\rangle = \operatorname{tr} V_{x,y}$ is Hermitian and positive definite.

Theorem (Koecher, 1969; Upmeier, 1985). *There is a bijective correspondence between bounded homogeneous circular domains and the Hermitian J. t. s. If D is a domain with Bergman kernel $K(z,w)$, then the operation of multiplication in the J. t. s. $J = J(D)$ is defined by the following equality:*

$$\{uvw\} = \sum c_{ijkl} u_i \overline{v_j} w_k \overrightarrow{e_l}, \text{ where } c_{ijkl} = \frac{\partial^4 \ln K(z,z)}{\partial z_i \partial \overline{z_j} \partial z_k \partial \overline{z_l}} \Big|_{z=0}.$$

Conversely, for a given J. t. s J, the domain is obtained as follows:

$$D(J) = \{x \in J \mid 2Id - V_{x,x} > 0\}.$$

Relation between the geometric and the algebraic structures can be seen in further results: the Bergman kernel is

$$K(z,w) = \frac{1}{\mu(D)} \cdot \det{}^{-1}(\mathrm{Id} - 2V_{x,y} + U_x U_y);$$

the Bergman metric at the origin is $\langle u, v\rangle = \operatorname{tr} V_{u,v}$; the Shilov boundary of the closure $\overline{D}$ coincides with the set of maximal idempotents of J; a decomposition of J into the direct sum of simple J. t. s. corresponds to a decomposition of D into irreducible domains. We may easily get a classification of irreducible domains by using the description of simple Hermitian J. t. s.

The aforementioned approach is also successfully applicable in studying infinite-dimensional (Banach) symmetric domains (Kaup, 1981; Upmeier, 1985), where the role of formal real J. a. is played by the so-called *JB*-algebras, defined as real J. a. with complete norm which satisfy the following conditions: 1) $\|ab\| \leqslant \|a\| \cdot \|b\|$, 2) $\|a^2\| = \|a\|^2$, 3) $\|a^2\| \leqslant \|a^2 + b^2\|$. Finite-dimensional JB-algebras are exactly the formal real J. a. Examples of infinite-dimensional JB-algebras are the algebra $B(H)_{sa}$ of selfadjoint bounded operators in a Hilbert space H (with Jordan multiplication) and the algebra $C(S, H(\mathbb{O}_3))$ of all the continuous functions on a compact S, with values in $H(\mathbb{O}_3)$, where $\mathbb{O}$ is the algebra of Cayley numbers. These examples are fairly general, as is seen in the following theorem, which is analogous to a Gel'fand-Najmark theorem for C^*-algebras:

Theorem (Alfsen, Shultz, Stormer, 1978). *For every JB-algebra J, there exist a complex Hilbert space H and a compact topological space S, such that J is isometrically isomorphic to a closed subalgebra of the algebra $B(H)_{sa} \oplus C(S, H(\mathbb{O}_3))$.*

3.7. Structure of Infinite-Dimensional Jordan Algebras. The basic notions of the modern structure theory of J. a. are that of an absolute zero divisor, a non-degenerate algebra and a non-degenerate radical.

An element $a \neq 0$ of a J. a. J is called an *absolute zero divisor*, if $JU_a = 0$; if the algebra J does not contain absolute zero divisors, then it is called a *non-degenerate algebra*. The smallest ideal I of an algebra J such that the quotient algebra J/I is non-degenerate is called the *non-degenerate radical* of the algebra J. We will denote it by $\operatorname{rad} J$.

Example 1. Let $J = A^{(+)}$, where A is associative. Then the absolute zero divisors in $A^{(+)}$ are elements a such that $aAa = 0$; $A^{(+)}$ is non-degenerate if and only if A is semiprime; $\operatorname{rad} A^{(+)}$ coincides with prime radical of the algebra A.

Example 2. Let J be a finite-dimensional J. a. Then $\operatorname{rad} J = \operatorname{Nil} J$ is the greatest nilpotent ideal in J (cf. 3.2) and J is non-degenerate if and only if it is semisimple.

Every non-degenerate J. a. is isomorphic to a subdirect sum of prime non-degenerate algebras. The structure of the latter is described in

Theorem (Zel'manov, 1983a). *A J. a. J is prime non-degenerate, if and only if one of the following cases holds:*

1) *J is a central order* (cf. p. 203) *in the J. a. of the bilinear form $J(X, f)$;*
2) *$A^{(+)} \lhd J \subseteq (Q(A))^{(+)}$, where A is a prime associative algebra and $Q(A)$ is its Martindale's quotient ring* (Bokut', L'vov, Kharchenko, 1988)*;*
3) *$H(A, *) \lhd J \subseteq H(Q(A), *)$, where A is an associative prime algebra with involution $*$;*
4) *J is Albert's ring (a central order in Albert's algebra).*

Corollary. *Every prime non-degenerate J. a. is either special or is the Albert ring.*

The non-degeneracy condition in the theorem is essential: An example has been constructed in (Pchelintsev, 1986) of a (special) prime J. a. with a basis of absolute zero divisors and, naturally, not being an algebra of any of the types 1)–4). The question on validity of the corollary of the theorem, without the non-degeneracy condition, remains open.

Every simple J. a. is non-degenerate (see below), thus description of simple J. a. follows from the following description of prime non-degenerate J. a.:

Theorem (Zel'manov, 1983a). *Simple J. a. are exactly the algebras of one of the following types:*

1) *$J = J(X, f)$;*
2) *$J = A^{(+)}$, where A is an associative simple algebra;*
3) *$J = H(A, *)$, where A is a simple associative algebra, with involution $*$;*
4) *J is the Albert algebra.*

Division J. a. are described analogously: it is necessary only to assume that A is a skew-field, in cases 2) and 3) and, in cases 1) and 4), to impose natural restrictions on J.

The notions of absolute zero divisors and non-degeneracy carry over to J. p. and J. t. s., naturally. Prime, non-degenerate and simple J. p. and J. t. s. are also described in (Zel'manov, 1983b). We now give classification of simple J. p.:

Theorem (Zel'manov, 1983b). *A Jordan pair V is simple if and only if it is of one of the following forms: 1) $V = (R_{-1}, R_1)$, where $R = R_{-1} \dotplus R_0 \dotplus R_1$ is a simple 3-graded associative algebra with $R_{-1} + R_1 \neq 0$; 2) $V = (H(R_{-1}, *), H(R_1, *))$, where R is same as in 1), with the involution $*$ that preserves graduation; 3) $V = (J(X, f), J(X, f))$; 4) $V = (M_{1,2}(\mathbb{O}), M_{1,2}(\mathbb{O}^0))$ – a pair of 1×2 matrices over the Cayley-Dickson algebra $\mathbb{O}$, with the multiplication $\{xyx\} = x(y^tx)$, and $\mathbb{O}^0$ is anti-isomorphic to $\mathbb{O}$; 5) $V = (J, J)$, where J is Albert's algebra.*

We now turn to studying the properties of the radical $\operatorname{rad} J$. In every J. a. J, just as in the case of alternative algebras, the quasiregular radical $\operatorname{Rad} J$ (the greatest quasiregular ideal), the nilradical $\operatorname{Nil} J$, the locally nilpotent $LN(J)$ and prime radicals $P(J)$ are defined, and they are related by inclusions (2. 21). (However, it is still unclear, whether the ideal $P(J)$ is a "real radical", since it is not known whether a semiprime J. a. can contain P-radical ideals.) The place of the radical $\operatorname{rad} J$, among those radicals is shown in the following theorem:

Theorem (Zel'manov, 1982; Pchelintsev, 1986). *In every J. a. J the inclusions $LN(J) \supseteq \operatorname{rad} J \supseteq P(J)$, hold, and in general, each of them may be strict.*

Corollary 1. *In every J. a. J, every set of absolute zero divisors generates a locally nilpotent ideal.*

Since simple locally nilpotent algebras do not exist, the following holds:

Corollary 2. *Every simple J. a. is non-degenerate.*

For finite-dimensional J. a., all the aforementioned radicals coincide. Moreover, they all coincide in the class of J. a. with the minimal condition for so-called *inner* (or *quadratic*) *ideals*. The latter are analogues of one-sided ideals of associative algebras and are defined as a subspaces K of J. a. J such that $\{kak\} \in K$, for all $k \in K$ and $a \in J$.

Example 1. Let $J = A^{(+)}$, where A is associative. Then every one-sided ideal of the algebra A is an inner ideal in $A^{(+)}$.

Example 2. For every $a \in J$, the set $JU_a = \{xU_a \mid x \in J\}$ is an inner ideal of a J. a. J.

Example 3. Let $J = H(\mathbb{O}_3)$. Then, for every element a of rank 1 (cf. 3.5), the subspace $F \cdot a$ is a quadratic ideal of J.

For a J. a., the following analogue of the classical Wedderburn-Artin theorem holds:

Theorem (Jacobson, 1968; Zhevlakov, Slin'ko, Shestakov, Shirshov, 1978). *Let a J. a. J satisfy the minimal condition for inner ideals. Then $\mathrm{Rad}\, J$ is nilpotent and finite-dimensional, and the quotient algebra $J/\mathrm{Rad}\, J$ decomposes into a finite direct sum of simple J. a. of one of the following forms:* 1) *a division J. a.;* 2) $H(A,*)$, *for an associative artinian $*$-simple algebra A, with involution $*$;* 3) $J(X,f)$; 4) *The Albert algebra.*

Many results from the theory of alternative algebras on relations between solvability and nilpotence are valid for J. a. too. For instance, every finitely generated solvable J. a. is nilpotent; if J is solvable, then J^2 is nilpotent; over a field of characteristic 0, a Jordan nilalgebra of bounded index is solvable. At the same time, in contrast to alternative algebras, finitely generated J. a. may contain solvable, but not nilpotent subalgebras.

Free J. a. have been studied relatively poorly. One of the deepest results about their structure is a theorem by Shirshov, which ascertains that a free J. a. with two generators is special. The free J. a. $J[X]$, for $|X| \geqslant 3$ is not special and contains zero divisors and, for a sufficiently large number of generators, $\mathrm{rad}\, J[X] \neq 0$ (Medvedev, 1985). For special J. a., the role of a free algebra is played by the so-called *free special J. a.* $SJ[X]$, which is defined as the smallest subspace in the free associative algebra $\mathrm{Ass}\,[X]$, containing X and closed with respect to the Jordan multiplication. The elements of the J. a. $SJ[X]$ are called the *Jordan elements* of the algebra $\mathrm{Ass}\,[X]$. It is easy to see that $SJ[X] \subseteq H(\mathrm{Ass}\,[X],*)$, where $*$ is the involution of the algebra $\mathrm{Ass}\,[X]$, which is identity on X: $(x_1x_2\ldots x_n)^* = x_n \ldots x_2x_1$. The J. a. $H(\mathrm{Ass}\,[X],*)$ is generated by the set X and by all the possible "tetrades" $\{x_ix_jx_kx_l\} = x_ix_jx_kx_l + x_lx_kx_jx_i$; for $|X| \leqslant 3$, it coincides with the J. a. $SJ[X]$, and for $|X| > 3$ it properly contains it (since the tetrades are not Jordan elements). For $|X| > 3$, no criteria for elements in $\mathrm{Ass}\,[X]$ to be Jordan have been found, up to now. Every special J. a. is a homomorphic image of the algebra $SJ[X]$, but the converse is not always true. For instance, the quotient algebra $SJ[x,y,z]/I$, where I is an ideal generated by the element $x^2 - y^2$, is exceptional. This implies that it is impossible to define the class SJord of all special J. a. by identities. Let $\pi : J[X] \longrightarrow SJ[X]$ be a canonical epimorphism; then $\mathrm{Ker}\,\pi \neq 0$, for $|X| \geqslant 3$. The elements in $\mathrm{Ker}\,\pi$ are called *s-identities*; they are satisfied in all special J. a., but are not identities in the class of all J. a. An example of such an identity is a well-known Glennie s-identity $G(x,y,z) = K(x,y,z) - K(y,x,z)$, where

$$K(x,y,z) = 2\{\{y\{xzx\}y\}z(xy)\} - \{y\{x\{z(xy)z\}x\}y\}.$$

Let us denote by $\overline{\mathrm{SJord}}$ the class of all J. a. satisfying all the s-identities, and by Jord – the class of all the J. a.; then the following proper inclusions hold: $\mathrm{SJord} \subset \overline{\mathrm{SJord}} \subset \mathrm{Jord}$ (Albert's algebra does not satisfy the Glennie identity, thus it does not belong to $\overline{\mathrm{SJord}}$). The question of describing all the s-identities is still open. It is not clear even whether they all follow from a finite number of s-identities.

Note that the class SJord may be defined by quasi-identities, i.e. by the expressions of the form $(f(x) = 0 \Rightarrow g(x) = 0)$. This is however impossible to achieve with finite number of quasi-identities. Moreover, any number of quasi-identities in a bounded collection of variables does not suffice (Sverchkov, 1983).

A J. a. is called a *Jordan* PI-*algebra*, if it satisfies an identity, which is not an s-identity. For Jordan PI-algebras analogues of main structure theorems from the theory of associative PI-algebras hold:

Theorem (Zel'manov, 1983a; Medvedev, 1988). *Let J be a Jordan PI-algebra over a field F. Then* 1) $\mathrm{Nil}\, J = LN(J) = \mathrm{rad}\, J$; 2) *if J is prime and non-degenerate, then it is a central order in a simple J. a. with the same identity;* 3) *if J is simple, then either J is finite-dimensional over the center or $J = J(X,f)$;* 4) *if J is finitely generated, then $\mathrm{Rad}\, J$ is nilpotent.*

We point out that the non-degeneracy condition on J in 2) and the condition of J being finitely generated in 4) are essential.

As in the case of alternative algebras, an effective method of studying Jordan PI-algebras is a passage to different enveloping algebras. With regard to this, we mention the following result:

Theorem (Shestakov, 1983; Medvedev, 1988). *Let J be a finitely generated Jordan PI-algebra over a field F. Then* 1) *the universal multiplicative enveloping algebra $U(J)$ is an associative PI-algebra;* 2) *if J is special, then its associative enveloping algebra is also a PI-algebra.*

§4. Generalizations of Jordan and Alternative Algebras and Other Classes of Algebras

Just as in the previous section, F will be a field of characteristic $\neq 2$, in the sequel.

4.1. Non-Commutative Jordan Algebras (Schafer, 1966). A natural generalization of the class of Jordan algebras to the non-commutative case is a class of algebras satisfying the following Jordan identity:

$$(x^2y)x = x^2(yx). \tag{1}$$

If the algebra has a unity, then the identity (1) easily implies the following flexibility identity:

$$(xy)x = x(yx). \tag{2}$$

Thus, if we want the class of algebras we are introducing to be stable with respect to adjoining a unity to an algebra, then we need to add the flexibility identity (2) to the identity (1). Algebras satisfying identities (1) and (2) are called *non-commutative Jordan algebras* (n. J.).

It is not difficult to see that identity (1) in the definition of a n. J. algebra may be replaced by any of the following identities:

$$x^2(xy) = x(x^2y), \quad (yx)x^2 = (yx^2)x, \quad (xy)x^2 = (x^2y)x.$$

We have seen in 3.1 that, in case of a Jordan algebra J, the operators $R_{x^k}, k = 1, 2, \ldots$, for every $x \in J$ are in the commutative subalgebra, generated by the operators R_x and R_{x^2}. For a n. J. algebra the following analogue of this result holds:

Proposition. *Let A be a n. J. algebra and $a \in A$. Then the operators R_a, L_a, L_{a^2} generate a commutative subalgebra of the multiplication algebra $M(A)$, containing all the operators $R_{a^k}, L_{a^m}; k, m = 1, 2, \ldots$*

Corollary. *Every n. J. algebra is power-associative.*

The condition of commuting of multiplication operators with the powers of an element fully characterizes n. J. algebras, since the identities (1) and (2) are just special cases of this condition ($[L_{x^2}, R_x] = [L_x, R_x] = 0$).

Another characterization of n. J. algebras is this: they are flexible algebras A, such that the associatied algebra $A^{(+)}$ (see 1.1 and 3.1) is a Jordan algebra.

The class of n. J. is rather large. Apart from Jordan algebras, it contains all the alternative algebras, as well as arbitrary anti-commutative algebras. Let us give additional examples of n. J. algebras.

Example 1. Let A be an algebra over a field $F, \lambda \in F, \lambda \neq \frac{1}{2}$. Let us define new multiplication on the vector space A:

$$a_\lambda \cdot b = \lambda ab + (1 - \lambda)ba.$$

We denote the resulting algebra by $A^{(\lambda)}$. The passage from the algebra A to $A^{(\lambda)}$ is reversible: $A = (A^{(\lambda)})^{(\mu)}$, for $\mu = \frac{\lambda}{2\lambda - 1}$. Properties of algebras A and $A^{(\lambda)}$ are fairly closely related: the ideals (subalgebras) of the algebra A are ideals (subalgebras) of $A^{(\lambda)}$; the algebra $A^{(\lambda)}$ is nilpotent, solvable, simple if and only if A has the corresponding property. If A is an associative algebra, then it is easy to check that $A^{(\lambda)}$ is a n. J. algebra; furthermore, if the identity $[[x, y], z] = 0$ does not hold in A, then $A^{(\lambda)}$ is non-associative. In particular, if A is simple non-commutative associative algebra, then $A^{(\lambda)}$ gives us an example of a simple non-associative n. J. algebra. The algebras of the form $A^{(\lambda)}$, for an associative algebra A, are called the split quasi-associative algebras. More generally, an algebra A is called a *quasi-associative algebra*, if it has a scalar extensions which is a split quasi-associative algebra. Clearly, every quasi-associative algebra is also a n. J. algebra.

Example 2. Let $0 \neq \alpha_1, \ldots, 0 \neq \alpha_n \in F$ and let $A(\alpha_1) = (F, \alpha_1), \ldots, A(\alpha_1, \ldots, \alpha_n) = (A(\alpha_1, \ldots, \alpha_{n-1}), \alpha_n)$ be the algebras obtained from F by sequential application of the Cayley-Dickson process (cf. 2.1). Then $A(\alpha_1, \ldots, \alpha_n)$ is a simple central quadratic n. J. algebra of dimension 2^n.

In general, every quadratic flexible algebra is a n. J. algebra. The following theorem shows that, under some restrictions, the nilsemisimple (i.e. with zero nilradical) finite-dimensional, power-associative algebras are also n. J. algebras.

Theorem 1. *Let A be a finite-dimensional power-associative algebra, with a bilinear symmetric form (x, y), satisfying the following conditions:* 1) $(xy, z) = (x, yz)$, *for all* $x, y, z \in A$; 2) $(e, e) \neq 0$, *if* $0 \neq e = e^2$; 3) $(x, y) = 0$, *if xy is a nilpotent element. Then $Nil A = Nil A^{(+)} = \{x \in A \mid (x, A) = 0\}$, and, if characteristic of the field F is not equal to 5, then the quotient algebra $A/Nil A$ is a n. J. algebra.*

The following theorem describes the structure of the nilsemisimple, n. J. algebras:

Theorem 2. *Let A be a finite-dimensional nilsemisimple n. J. algebra over F. Then A has a unity and decomposes into a direct sum of simple algebras; in addition, if characteristic of the field F equals to zero, then each of the simple summands is an algebra of one of the following forms: a (commutative) Jordan algebra; a quasi-associative algebra; a quadratic flexible algebra.*

Over the field of positive characteristic, there is another type of simple n. J. algebra, the so-called nodal algebras. An algebra A with unity 1 is called a *nodal algebra*, if every element in A is representable in the form $\alpha \cdot 1 + n$, where $\alpha \in F$ and n is nilpotent, and if furthermore, the nilpotent elements do not form a subalgebra in A. Nodal algebras do not exist either in the classes of alternative and Jordan algebras, or in the class of n. J. algebras of characteristic 0. Every nodal algebra maps homomorphically onto some simple nodal algebra.

Theorem 3. *Let A be a simple finite-dimensional n. J. algebra over F. Then A is either anti-commutative, or satisfies the conclusion of Theorem 2, or is a nodal algebra. In the latter case, $\operatorname{char} F = p > 0$, the algebra $A^{(+)}$ is isomorphic to the p^n-dimensional associative-commutative algebra of truncated polynomials $F[x_1, \ldots, x_n], x_i^p = 0$, and multiplication in A is defined by the following formula:*

$$fg = f \cdot g + \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial x_j} \cdot c_{ij},$$

where $(\cdot)$ is the multiplication in $A^{(+)}$, and $c_{ij} = -c_{ji}$ are arbitrary elements in A, among which at least one is invertible in $A^{(+)}$.

The construction described in Theorem 3 does not always result in a simple algebra. However, all such algebras of dimension p^2 are simple and, for every even n, there exist simple algebras of that form of dimension p^n. The derivation algebras of the nodal n. J. algebras are related to the simple (non-classical) modular Lie algebras (Schafer, 1966; Strade, 1972).

In a difference from the case of alternative and Jordan algebras, an analogue of the main Wedderburn theorem on splitting of the nilradical, does not in general hold in the class of n. J. algebras.

The flexible power-associative algebras occupy an intermediate position between arbitrary power-associative algebras and n. J. algebras. Let A be a finite-dimensional algebra of that form, over a field of characteristic 0. The mapping $x \mapsto [a, x]$ is a derivation of the associated commutative power-associative algebra $A^{(+)}$, and, since the nilradical of a power-associative algebra of characteristic 0 is stable with respect to the derivations (Slin'ko, 1972), the inclusion $[A, \operatorname{Nil}(A^{(+)})] \subseteq \operatorname{Nil}(A^{(+)})$ holds, therefore $\operatorname{Nil}(A^{(+)}) \triangleleft A$ and $\operatorname{Nil}(A^{(+)}) = \operatorname{Nil} A$. This means that $(A/\operatorname{Nil} A)^{(+)} = A^{(+)}/\operatorname{Nil}(A^{(+)})$ is a nilsemisimple finite-dimensional commutative power-associative algebra. Over a field of characteristic 0, all such algebras are Jordan algebras (cf. 1.2), thus $A/\operatorname{Nil} A$ is a n. J. algebra. Thus, every finite-dimensional nilsemisimple flexible power-associative algebra over a field of characteristic 0 is a n. J. algebra. In the general case, the following holds:

Theorem 4 (Oehmke, 1958, 1962). *Let A be a finite-dimensional nilsemisimple flexible power-associative algebra over an infinite field of characteristic $\neq 2, 3$. Then A has unity and decomposes into the direct sum of simple algebras, each of which is either a n. J. algebra, or is an algebra of degree 2* (cf. p. 179) *over a field of positive characteristic.*

The algebras of the latter form have not been described up to date. A method of their description was given in (Block, 1968), where it was proved that, for every such algebra A, the algebra $A^{(+)}$ is also a simple algebra (of a known structure). An example of an flexible power-associative simple algebra of degree 2 which is neither commutative nor a n. J. algebra was given in (Mayne, 1973).

In the conclusion we point out that structure of arbitrary finite-dimensional nilsemisimple power-associative algebras is still unclear. It is known that in this case new simple algebras arise, among them the nodal algebras, even over an algebraically closed field of characteristic 0.

Example 3. Let V be a vector space of dimension $2n$ over a field F with a non-degenerate skew-symmetric bilinear form (x, y). Let us define multiplication on the vector space $A = F \dotplus V$, by the rule $(\alpha + v)(\beta + u) = (\alpha\beta + (v, u)) + (\alpha u + \beta v)$. Then A is a quadratic algebra over F, (hence, it is also power-associative), and it is simple and nodal. It is interesting that A turns out to be a Jordan superalgebra (cf. 3.4), if we set $A_0 = F, A_1 = V$.

4.2. Right-Alternative Algebras (Bakhturin, Slin'ko, Shestakov, 1981; Zhevlakov, Slin'ko, Shestakov, Shirshov, 1982; Skosyrskij, 1984). Among the algebras that do not satisfy the flexibility identity (2), right alternative algebras are the most well researched. Recall that an algebra is called *a right alternative algebra* (r. a.), if it satisfies the identity

$$(xy)y = x(yy). \tag{3}$$

Let A be a r. a. algebra over F; then (3) implies that the following relations hold in the algebra of its right multiplications $R(A)$, for all $a, b \in A$:

$$R_{a^2} = R_a^2, \quad R_{a \cdot b} = R_a \cdot R_b, \tag{4}$$

where $a \cdot b = \frac{1}{2}(ab + ba)$ is the multiplication in the associated algebra $A^{(+)}$. Thus, the transformation $a \mapsto R_a$ is a homomorphism of the algebra $A^{(+)}$ into the special Jordan algebra $(R(A))^{(+)}$ (cf. 3.1). If A has unity, then this transformation is injective. Since the class of r. a. algebras is closed with respect to adjoining a unity to the algebra, we have the following:

Proposition. *For every r. a. algebra A, the associated algebra $A^{(+)}$ is a special Jordan algebra.*

This proposition allows for application of the well-developed apparatus of the Jordan algebras to studies of r. a. algebras. The strongest results in the theory of r. a. algebras have been obtained exactly along this road.

Relations (4) easily imply that

$$R_{a^k} R_{a^n} = R_{a^{k+n}},$$

for every $a \in A, k, n = 1, 2, \ldots$ In particular, every r. a. algebra A is power-associative and it has a uniquely defined nilradical $\operatorname{Nil} A$

Theorem (Skosyrskij, 1984). *Let A be an arbitrary r. a. algebra. Then the quotient algebra $A/Nil A$ is alternative.*

Corollary 1. *Every simple r. a. algebra that is not a nilalgebra is alternative (and consequently, is either associative or a Cayley-Dickson algebra).*

Corollary 2. *Every r. a. algebra without nilpotent elements is alternative.*

Corollary 2 implies, in particular, Skornyakov's theorem on right alternative skew-fields, that we mentioned in 2.2.

Corollary 3. *Let A be a r. a. algebra and $a, b \in A$. Then the element (a, a, b) generates a nilideal in A.*

In general, the ideal generated by the associator (a, a, b) is not solvable. It is not clear however, whether it is a nilalgebra of fixed finite index. It is known only that $(a, a, b)^4 = 0$.

A subspace L of an algebra A is called right nilpotent, if $L^{\langle n\rangle} = 0$, for some n, where $L^{\langle 1\rangle} = L$, $L^{\langle i+1\rangle} = L^{\langle i\rangle}L$.

Theorem. *Let A be a r. a. nilalgebra of bounded index. Then every finite-dimensional subspace in A is right nilpotent.*

Corollary 4. *A finite-dimensional r. a. nilalgebra is right nilpotent.*

In particular, every finite-dimensional r. a. nilalgebra is solvable and therefore cannot be a simple algebra. At the same time, it can be non-nilpotent.

The "non-nil" restriction in Corollary 1 is essential in general: an example of a non-alternative, simple r. a. nilalgebra (with the identity $x^3 = 0$) was constructed in (Miheev, 1977). There are no examples of that kind among right artinian r. a. algebras, as is shown in the following

Theorem (Skosyrskij, 1985). *Let A be a r. a. algebra satisfying the minimal condition for right ideals. Then the ideal $N = Nil A$ is right nilpotent and the quotient algebra A/N is a semisimple artinian alternative algebra.*

The right artinian condition cannot be replaced by the left artinian: the simple algebra constructed in (Miheev, 1977) does not even contain proper left ideals.

Just as in the case of non-commutative Jordan algebras, the main Wedderburn theorem on splitting off of the radical, is not satisfied in general, in the class of r. a. algebras (Thedy, 1978).

4.3. Algebras of (γ, δ)-Type (Albert, 1949; Nikitin, 1974; Markovichev, 1978; Ng Seong Nam, 1984). Besides the alternative algebras, an important example of non-flexible power-associative algebras is given by the so-called algebras of (γ, δ)-type that arise in studies of classes of algebras A with the following structural property: (*) if I is an ideal of the algebra A, then I^2 is also an ideal of A. Despite its generality, this property enables proofs of fairly

meaningful structural results (cf. for instance, 2.4). At the same time, there are not so many generalizations of the associative algebras with property (*).

Proposition. *Let a class $\mathcal{K}$ of power-associative algebras, defined by a system of identities over an infinite field, is such that all of its algebras satisfy condition (*), and let $\mathcal{K}$ contain all the associative algebras. Then $\mathcal{K}$ is either the class of alternative algebras, or $\mathcal{K}$ is defined by the following identities*

$$(x, x, x) = 0, \tag{5}$$

$$S(x, y, z) \equiv (x, y, z) + (y, z, x) + (z, x, y) = 0, \tag{6}$$

$$(x, y, z) + \gamma(y, x, z) + \delta(z, x, y) = 0, \tag{7}$$

where γ and δ are fixed scalars such that $\gamma^2 - \delta^2 + \delta = 1$.

The algebras satisfying identities (5)–(7) are called the *algebras of (γ, δ)-type*. Their structure in the finite-dimensional case is described by the following

Theorem. *Let A be a finite-dimensional algebra of type (γ, δ), over a field F of characteristic $\neq 2, 3, 5$. Then the radical $Nil A$ is nilpotent and the quotient algebra $A/Nil A = \overline{A}$ is associative. If, in addition, the algebra $\overline{A}$ is separable over F, then $A = B \dotplus Nil A$, where B is a subalgebra of A isomorphic to $\overline{A}$.*

The class of algebras of (γ, δ)-type does not give new examples of simple algebras.

Theorem (Markovichev, 1978; Ng Seong Nam, 1984). *Every simple (not necessarily finite-dimensional) algebra of type (γ, δ) and characteristic $\neq 2, 3, 5$ is associative.*

At the same time, there exist prime non-associative algebras of type (γ, δ) of arbitrary characteristic (Pchelintsev, 1984). These algebras, just as in general, every non-associative algebra of type (γ, δ), contain non-zero locally nilpotent ideals.

4.4. Lie-Admissible Algebras (Myung, 1982, 1986). Studies of one more class of algebras have been fairly intensively initiated in recent times, under the influence of papers by a physicist Santilli (1978, 1982); these are the so-called *Lie-admissible algebras*, i.e. algebras A such that their commutator algebra $A^{(-)}$ is a Lie algebra. Apart from the associative and Lie algebras, this class for instance contains quasi-associative algebras and algebras of (γ, δ)-type. If L is a Lie algebra with multiplication $[x, y]$, then after defining an arbitrary commutative multiplication $x \circ y$ on L and setting $x * y = \frac{1}{2}([x, y] + x \circ y)$, we arrive at a Lie-admissible algebra $\tilde{L}$ (with respect to the multiplication $x * y$), for which $L^{(-)} \cong L$. It is clear that every Lie-admissible algebra allows such a realization; furthermore, if L is a simple algebra, then $\tilde{L}$ is also simple. Since the commutative multiplication $x \circ y$ was arbitrary, it is also clear that, in general, the problem of describing simple Lie-admissible algebras, even modulo Lie algebras, is hardly feasible. However, under some additional restrictions, it is possible to obtain such a description.

Theorem. *Let A be a finite-dimensional Lie admissible algebra with multiplication $*$ over an algebraically closed field of characteristic 0, such that $A^{(-)}$ is a simple Lie algebra. Then if A satisfies the identity $(x,x,x)=0$, then there exists a linear form τ on A and a scalar $\beta \in F$, such that*

$$x * y = \frac{1}{2}[x,y] + \tau(x)y + \tau(y)x + \beta x\#y, \tag{8}$$

where $x\#y$ is either equal to zero, for all $x,y \in A$, or $A^{(-)} \cong sl(n+1,F)$ and

$$x\#y = xy + yx - \frac{2}{n+1}tr(xy)E,$$

where E is the identity matrix. Furthermore, A is power-associative, if and only if $\beta = 0$ in (8).

Corollary. *If, the algebra A from the hypotheses of the theorem is flexible, then*

$$x * y = \frac{1}{2}[x,y] + \beta x\#y;$$

if, in addition, A is power-associative, then A is a Lie algebra.

§5. Malcev Algebras and Binary Lie Algebras

5.1. Structure and Representation of Finite-Dimensional Malcev Algebras. We have defined Malcev algebras and binary Lie algebras in §1, which arose in (Malcev, 1955) as two natural generalizations of Lie algebras. After expanding the Jacobian in identity (1.9), it may be rewritten (in view of anticommutativity) in the following form:

$$xyzx + yzx^2 + zx^2y = xy \cdot xz, \tag{1}$$

where parentheses were omitted in left normalized products $xyzx = (xy \cdot z)x$, $yzx^2 = (yz \cdot x)x$, etc, for the convenience of notation. If the characteristic of the base field F is different from 2, then the following identity follows from (1):

$$xyzt + yztx + ztxy + txyz = ty \cdot xz, \tag{2}$$

which, together with the anticommutativity identity $x^2 = 0$, is appropriate to take as a definition of Malcev algebras in the case char $F = 2$ too, since first of all, for $t = x$ it again turns into identity (1), and secondly, it has a number of advantages: it is multilinear and transforms into itself after cyclic

permutations of the variables x, y, z, t, hence all the variables participate equally in (2). Every Lie algebra satisfies identity (2); on the other hand, an anticommutative algebra satisfying identity (1) or (2) is a binary Lie algebra. Thus, the class of Malcev algebras is placed between Lie algebras and binary Lie algebras.

Up to now, the theory of finite-dimensional Malcev algebras has almost as a completed form as the theory of Lie algebras. We will give here only basic facts on the structure theory of Malcev algebras.

In studying various classes of algebras one of the most essential questions is the question of describing simple algebras of that class. In case of Malcev algebras it is natural to ask the question about classification of simple Malcev algebras that are not Lie algebras, and there is almost complete answer to this question: non-Lie simple Malcev algebras over an arbitrary field F of characteristic different from 2 have been described, even without the assumption of finite-dimensionality (Kuz'min, 1971; Filippov, 1976a).

Let $\mathbb{O} = \mathbb{O}(\alpha,\beta,\gamma)$ be the Cayley-Dickson algebra over F (cf. 2.1). Then $\mathbb{O} = F \dotplus M$, where $M = \{x \in \mathbb{O} \mid t(x) = 0\}$ and multiplication in $\mathbb{O}$, for elements $a, b \in M$ is defined by the following formula:

$$a \cdot b = -(a,b) + a \times b, \tag{3}$$

where $(,)$ is a symmetric non-degenerate bilinear form on M and $(\times)$ is anticommutative multiplication on M. We denote the constructed 7-dimensional anticommutative algebra $(M, \times)$ by $M(\alpha,\beta,\gamma)$; it is defined in the case when char $F = 2$ too, and it is a central simple Malcev algebra over F; if char $F \neq 3$, then the algebra $M(\alpha,\beta,\gamma)$ is not a Lie algebra.

Theorem. *Every central simple Malcev algebra over a field F of characteristic $\neq 2$ is either a Lie algebra or an algebra of type $M(\alpha,\beta,\gamma)$. In particular, there are no non-Lie simple Malcev algebras of characteristic 3.*

Using the right alternativity of the Cayley-Dickson algebra $\mathbb{O}(\alpha,\beta,\gamma) = F \dotplus M$, we obtain the following $(a, b \in M)$:

$$(a \cdot b) \cdot b = -(a,b)b - (a \times b, b) + (a \times b) \times b = a \cdot b^2 = -(b,b)a,$$

hence

$$(a \times b) \times b = -(b,b)a + (a,b)b, \quad (a \times b, b) = 0. \tag{4}$$

In particular, (4) implies that the bilinear form $(,)$ is uniquely determined by the multiplication operation $(\times)$ on M, and then, by formula (3), multiplication on $\mathbb{O}$ is defined too. Thus, two Malcev algebras of types $M(\alpha,\beta,\gamma)$ and $M(\alpha',\beta',\gamma')$ over F are isomorphic if and only if the corresponding Cayley-Dickson algebras $\mathbb{O}(\alpha,\beta,\gamma)$ and $\mathbb{O}(\alpha',\beta',\gamma')$ are isomorphic. Another useful criterion for isomorphism of the algebras of type $M(\alpha,\beta,\gamma)$ consists in equivalence of bilinear forms $(,)$ defined on them.

For every $n \geqslant 3$ there exist central simple anticommutative algebras of dimension $2^n - 1$, over the field F of characteristic 2, satisfying identity (1), but only for $n = 3$ do they satisfy identity (2) (Kuz'min, 1967a).

In case char $F = 0$ the structure of finite-dimensional Malcev algebras has been studied in more depth. We define a Killing form K on the space of Malcev algebras A, by setting $K(x, y) = \mathrm{tr}\,(R_x R_y)$, where R_x is the operator of right multiplication by x in A. The form K is symmetric and associative: $K(x, y) = K(y, x), K(xy, z) = K(x, yz)$. The algebra A is semisimple (i.e. its solvable radical $S(A)$ equals 0), if and only if its Killing form is non-degenerate. In case of Lie algebras, this claim becomes the well-known Cartan criterion for semisimplicity of Lie algebras. A semisimple algebra A decomposes into the direct sum of simple algebras, which are, by what has been said, either simple Lie algebras or 7-dimensional algebras of type $M(\alpha, \beta, \gamma)$, over its centroid $\Gamma \supseteq F$ $(\alpha, \beta, \gamma \in \Gamma)$.

The radical $S(A)$ coincides with the orthogonal complement of A^2, with respect to K. In particular, the algebra A is solvable if and only if $K(A, A^2) = 0$. In addition, the inclusions $S \cdot A \subseteq N, SD \subseteq N$ hold, where D is an arbitrary derivation of the algebra A, and $N = N(A)$ is the greatest nilpotent ideal (nilradical) in A. Both the radical and nilradical of a Malcev algebra have the property of ideal heredity: if $B \triangleleft A$, then $S(B) = B \cap S(A), N(B) = B \cap N(A)$.

An automorphism $\phi \in \mathrm{Aut}\, A$ is called special, if it is a product of automorphisms of the form $\exp D$, where D is a nilpotent inner derivation of the form $R_{xy} + [R_x, R_y]$. Just as in the case of alternative and Jordan algebras, a theorem on splitting off of the radical and conjugacy of semisimple quotients with respect to special automorphisms holds for Malcev algebras of characteristic 0. This result generalizes the classical Levi-Malcev-Harish-Chandra theorem for Lie algebras.

A Cartan subalgebra of a Malcev algebra A over an arbitrary field F is defined in the same fashion as in the case of Lie algebras: it is a nilpotent subalgebra H that coincides with its normalizer $\mathfrak{N}(H) = \{x \in A \mid H \cdot x \subseteq H\}$. Such subalgebras necessarily exist if $|F| \geqslant \dim A$. If char $F = 0$ and F is algebraically closed, then the Cartan subalgebras are mutually conjugated via special automorphisms.

An effective way to study finite-dimensional Malcev algebras of arbitrary characteristic is through the representations theory or Malcev modules. In agreement with identity (2), a linear transformation $\rho : A \longrightarrow \mathrm{End}\, V$ is called a (right) representation of a Malcev algebra A if, for all $a, b, c \in A$, the following relation holds:

$$\rho(ab \cdot c) = \rho(a)\rho(b)\rho(c) - \rho(c)\rho(a)\rho(b) + \rho(b)\rho(ca) - \rho(bc)\rho(a); \qquad (5)$$

in this case, V is called a *Malcev A-module*. Since the algebra A is anticommutative, the notion of a Malcev A-module is equivalent to the notion of a bimodule: it suffices to set $am = -ma$ $(a \in A, m \in V)$. A special case of a representation is a regular representation $x \mapsto R_x$.

The representation theory of nilpotent Malcev algebras is fully analogous to the corresponding theory for Lie algebras. An important role here is played by a theorem on nilpotency of the associative algebra A_ρ^*, generated by the operators $\rho(x)$, with the condition that $\rho(x)$ are nilpotent (an analogue of Engel's theorem). If, in addition, ρ is an almost exact representation (i.e. the kernel of ρ does not contain non-zero ideal of the algebra A), then the algebra A is also nilpotent.

The representation ρ is called split if, for every $x \in A$, the eigenvalues of the matrix $\rho(x)$ are in F. For split representations of solvable Malcev algebras of characteristic 0, an analogue of a Lie theorem on triangularization holds, i.e. on the existence of an A-invariant flag of subspaces (submodules) of the module V. The following claim is an analogue of the classical Weil theorem: every representation of a semisimple Malcev algebra of characteristic 0 is completely reducible. If V is an exact irreducible A-module (char $F = 0$), then the algebra A is simple, and one of the following cases holds: 1) $A \cong M(\alpha, \beta, \gamma)$ and V is a regular A-module, 2) A is a Lie algebra and V is a Lie module, 3) $A \dotplus V \cong A_1 \dotplus V_1$, where $A_1 = \mathrm{sl}(2, F), \dim M_1 = 2, \rho(a) = a^*$, where a^* is the matrix adjoint to the matrix $a \in A_1$.

A well known first Whitehead lemma on derivation of a semisimple Lie algebra to a bimodule generalizes to representations of semisimple Malcev algebras; this implies, in particular, that every derivation of a semisimple Malcev algebra is inner.

5.2. Finite-Dimensional Binary Lie Algebras (*BL*-Algebras). Engel's theorem in its classical formulation remains valid for a BL-algebra A of arbitrary characteristic: if every operator R_x is nilpotent, then the algebra A is nilpotent (Kuz'min, 1967b). As in the case of Malcev algebras, a BL-algebra A contains the greatest nilpotent ideal $N(A)$ – the nilradical of the algebra A. If A is a nilpotent algebra and ρ is a finite-dimensional binary Lie representation of A by nilpotent operators, then the enveloping associative algebra A_ρ^* of the representation ρ is nilpotent. However, the majority of the results on finite-dimensional BL-algebras relates to the case of characteristic 0. In the sequel, A will denote a finite-dimensional BL-algebra of characteristic 0. If A is solvable and V is a binary Lie A-module, then the algebra A^2 is nilpotent and acts nilpotently on V. The solvable algebra $A \neq 0$ contains an abelian ideal $I \neq 0$, and if the base field is algebraically closed, then A contains an one-dimensional ideal. Semisimple BL-algebras and their representations are described in the following

Theorem (Grishkov, 1980). *If A is semisimple, V is a binary Lie A-module and if V_0 is the annihilator of the algebra A in module V, then A is a Malcev algebra and V/V_0 is a Malcev A-module.*

In general, A does not necessarily decomposes into a semidirect sum of a semisimple subalgebra and the solvable radical S, but, among all the subalgebras $B \leqslant A$, for which $A = B + S$, there exists a subalgebra B_0, which is a semidirect sum of a semisimple Lie subalgebra L and an ideal C, whose radical R is in the center (the annihilator) of A, $C^2 = C, \overline{C} = C/R$ is the direct sum of 7-dimensional simple Malcev algebras (the base field is algebraically closed); if V is a binary Lie A-module, then $VR = 0$. In particular, the Levi decomposition for A exists, if A/S is a Lie algebra.

Example. Let M be a simple 7-dimensional Malcev algebra over F, let V be a finite-dimensional space with $1 \leqslant \dim V \leqslant 14$ and let $\sigma : M \times M \longrightarrow V$ be an arbitrary skew-symmetric function. Then the algebra $(M, V, \sigma) = M \dotplus V$ with the multiplication $(a_1 + v_1) \cdot (a_2 + v_2) = a_1a_2 + \sigma(a_1, a_2)$ $(a_i \in M, v_i \in V, i = 1, 2)$ is a binary Lie algebra. Since the radical V of the algebra $A = (M, V, \sigma)$ coincides with its center, then it obviously does not split off if $A^2 = A$, which is easily achievable for an appropriate choice of σ.

5.3. Infinite-Dimensional Malcev Algebras. Studies of Malcev algebras, without the assumption of finite-dimensionality are mainly based on the analysis of identities of the free Malcev algebra with finite or countable number of generators. Since the structure of the identities in Malcev algebras essentially depends on the characteristic, we will consistently assume here that $\operatorname{char} F \neq 2$.

A number of properties of infinite-dimensional algebras, such as local nilpotency, local finiteness, being algebraic (every subsequent notion is weaker than the preceeding one) brings them closer to finite-dimensional algebras. An algebra A is called locally finite, if every finite set of of its elements generates a finite-dimensional subalgebra. An algebra A is called an *algebraic algebra* if, for all $x, y \in A$, there exists a natural number n dependent on x, y, such that xy^n belongs to a subalgebra with the generators $x, xy, \ldots, xy^{n-1}$. A special case of the algebraic property is the weak Engel condition $E : xy^{n(x,y)} = 0$. Every Malcev algebra A contains the greatest locally finite ideal $L(A)$ as well as the greatest locally nilpotent ideal $LN(A)$ (Kuz'min, 1968a). In the class of algebraic Malcev algebras, the extension of locally finite algebra by a locally finite is again a locally finite algebra, thus $L(A/L(A)) = 0$. If A is weakly Engel, then $L(A) = LN(A)$ (Kuz'min, 1968a). The following theorem reduces the question of local nilpotency for Malcev algebras to the corresponding question for Lie algebras. For a Malcev algebra A to be locally nilpotent, it is necessary and sufficient that condition E holds in A and that every Lie homomorphic image of A is locally nilpotent (Filippov, 1976b). In particular, a Malcev algebra of characteristic $p > 2$, satisfying the condition E_{p+1}, i.e. the identity $xy^{p+1} = 0$ is locally nilpotent, since this holds for Lie algebras. As in the case of Lie algebras, this implies an affirmative solution of the weak Burnside problem for Moufang loops of prime period (Grishkov, 1985; Kostrikin, 1986, cf. also §6).

If $\operatorname{char} F \neq 2, 3$, then, along with the Jacobians $J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y$, we consider the function $g(x, y, z, t, u)$ with a number of remarkable properties:

$$g(x, y, z, t, u) = J([x, y, xz], t, u) + J([x, t, yz], x, u),$$

where $[x, y, z] = 3xy \cdot z - J(x, y, z)$. The function $g(x, y, z, t, u)$ is skew-symmetric in y, z, t, u and equals 0, if $y, z \in A^3$ or $y, z, t \in A^2$; in particular, the algebra A^2 satisfies the identity $g = 0$. Since the function g is non-zero in the free Malcev algebra M_k, with $k \geqslant 5$ generators, for $A = M_k$ $(k \geqslant 5)$, the identities for A and A^2 differ. The algebra M_4 satisfies the identity $g = 0$ and, for $k \geqslant 5$, M_k satisfies the identity $g(x_1, \ldots, x_5)x_6 \ldots x_{k+2} = 0$, but does not satisfy the identity $g(x_1, \ldots x_5)x_6 \ldots x_{k+1} = 0$, thus, for every $k \geqslant 4$ the identities for M_k and M_{k+1} are different (Filippov, 1984). Moreover, the algebras M_k, for $k \geqslant 5$ have a non-trivial center and thus, trivially, they are not prime. A free Malcev algebra with countable number of generators also has a non-trivial center: it contains for instance the elements of the form $g(x_1, \ldots x_5)y^2$, which are in general different from 0. The semiprime Malcev algebras satisfy the identity $g = 0$.

Prime and semiprime Malcev algebras of characteristic 3 are Lie algebras, thus, in particular, non-Lie simple Malcev algebras of characteristic 3 do not exist. Let A be a prime non-Lie Malcev algebra of characteristic $\neq 2, 3$. Then the center Z of the algebra of right multiplications $R(A)$ is different from 0 and is naturally contained in the centroid Γ. Since Γ is a commutative integral domain, there exists the quotient field Q of Z (coinciding with the quotient field of Γ) and the canonical embedding $A \longrightarrow A_Q = Q \otimes_Z A$. The algebra A_Q turns out to be a 7-dimensional central simple algebra over the field Q.

The commutator algebra $A^{(-)}$ of an arbitrary alternative algebra A is a Malcev algebra. In this regard, there arises a question: is it true that every Malcev algebra of characteristic $\neq 2, 3$ is embeddable into a commutator algebra of a suitable alternative algebra? The answer to this question is affirmative for semiprime Malcev algebras, but the problem remains open in the general case.

§6. Quasigroups and Loops

6.1. Basic Notions. A non-empty set Q with a binary operation $(\cdot)$ is called a *quasigroup*, if the equations $a \cdot x = b$, $y \cdot a = b$ are uniquely solvable, for all $a, b \in Q$. The solutions of these equations are denoted as $x = a \setminus b, y = b/a$, and the binary operations $\setminus, /$ are respectively called the left and the right division. They are related to the multiplication $(\cdot)$, via the following identities:

$$x \cdot (x \setminus y) = x \setminus (x \cdot y) = y, \quad (x/y) \cdot y = (x \cdot y)/y = x. \tag{1}$$

It is clear that the notion of a quasigroup generalizes the notion of a group: group is a quasigroup for which the operation of multiplication is associative. A quasigroup with unity is called a *loop*.

A quasigroup may be equivalently defined as a set with three basic operations $(\cdot), \backslash, /$, related via identities (1). Then a loop is a quasigroup with the additional identity $x \backslash x = y/y$.

Finite quasigroups may be defined with the aid of Cayley tables.

Example.

G_1	e	a	b	c	d
e	e	a	b	c	d
a	a	e	d	b	c
b	b	c	e	d	a
c	c	d	a	e	b
d	d	b	c	a	e

G_2	e	a	b	c	d
e	e	a	b	c	d
a	a	e	d	b	c
b	b	c	a	d	e
c	c	d	e	a	b
d	d	b	c	e	a

G_3	e	a	b	c	d
e	e	a	b	c	d
a	a	b	d	e	c
b	b	c	a	d	e
c	c	d	e	a	b
d	d	e	c	b	a

G_4	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	c	d	a	e
c	c	d	e	b	a
d	d	b	a	e	c

These are apparently mutually non-isomorphic non-associative loops. Incidentally, the smallest order of a non-associative loop is 5.

If we omit the borders in the Cayley table of a finite quasigroup, the resulting square has the property that the elements in every row and every column (their number equals the order of the quasigroup) do not repeat. Squares with this property are called *latin squares.* They have been apparently studied by Euler. Thus, quasigroups and latin squares are closely linked. Shuffling around rows and columns of a latin square or renaming its entries (i.e. applying a permutation of the set Q, where the entries of the square come from) does not change the latin property of the square. Algebraically this leads to an important property, namely that of isotopy of quasigroups. Quasigroups Q, Q' are said to be isotopic, if there is a triple of bijective transformations $\alpha, \beta, \gamma : Q \longrightarrow Q'$, such that $(xy)\gamma = x\alpha \cdot y\beta$, for all $x, y \in Q$. The triple (α, β, γ) is called an *isotopy* of Q to Q'. Composition of isotopies is defined in a natural way: $(\alpha, \beta, \gamma) \cdot (\alpha_1, \beta_1, \gamma_1) = (\alpha\alpha_1, \beta\beta_1, \gamma\gamma_1)$. If $Q = Q'$, then the isotopy is called an *autotopy*; the autotopies form a group with respect to the given operation. The transformations $R_a : x \mapsto xa$, $L_a : x \mapsto ax$ are called the right and the left translation respectively. Let us fix an arbitrary pair of elements $a, b \in Q$ and let us define a new operation $(\circ)$ on Q, by setting $xy = xb \circ ay$. Then $Q(\circ)$ is a quasigroup and $(R_a, L_a, 1)$ is an isotopy of Q to $Q(\circ)$. It is clear that $e = ab$ is the unity of the quasigroup $Q(\circ)$, i.e. $Q(\circ)$ is a loop. Thus, every quasigroup is isotopic to a loop. For $b = a$, the loop $Q(\circ)$ is called an LP-isotope of the quasigroup Q.

The notion of isotopy does not play a special role for groups, because of Albert's theorem: if two groups are isotopic, they are isomorphic. This statement is a consequence of a more general statement: if a loop is isotopic to a group, then they are isomorphic. Loop properties preserved under isotopies are called *universal*; the aforementioned implies that associativity is an example of a universal property. Universal identities in quasigroups may be described in the language of the so-called $*$-automata (Gvaramiya, 1985). The category of quasigroups is embeddable into the category of invertible $*$-automata, and a quasigroup formula, in particular an identity, is universal, if and only if it is expressible in terms of the automata category containing it.

By using isotopies, a number of quasigroups may be obtained from a given quasigroup. Another way consists in passing to the anti-isomorphic quasigroup or in substituting multiplication operation on Q by the operation of the left or the right division. The resulting quasigroups are called parastrophs of the quasigroup Q.

6.2. Analytic Loops and Their Tangent Algebras. Let us first recall some basic facts from the theory of Lie groups and Lie algebras. First of all, there are different variants of definitions of Lie groups, depending on requirements of topological nature. The topological space of a Lie group G is an n-dimensional manifold – topological, differentiable, or analytic, and the group operations (multiplication and inversion) are assumed to be respectively continuous, differentiable appropriately many times, or analytic. The question of equivalence of the weakest of these definitions with the strongest was the essence of the fifth Hilbert problem, which was affirmatively solved in 1952 (Gleason, Montgomery, Zippin, (cf. Kaplansky, 1971)).

Every neighbourhood U of the identity element $e \in G$, homeomorphic to the Euclidean space $\mathbb{R}^n$, is a local Lie group: there exists a neighbourhood $U_1 \subseteq U$ of e, such that, for $x, y \in U_1$, the operations of multiplication and inversion are defined (with the values in U); the functions $f^i(x, y)$, where $f^i(x, y)$ is the i-th coordinate of the vector $f(x, y) = x \cdot y$ are either continuous, or differentiable, or analytic, depending on the initial requirements.

We will hold on to an intermediate variant and assume that the functions $f^i(x, y)$ are twice continuously differentiable in U_1. The point e is taken to be the coordinate origin; thus $f(x, 0) = f(0, x) = x$. The set of all the tangent vectors at e, to all the differential paths $g(t)$ starting at e, forms the *tangent space* $T_e = \mathbb{R}^n$. Let $a, b \in T_e$ be the tangent vectors for the paths $g(t), h(t)$. Then the tangent vector to the path $k(t)$, where $k(t^2) = g(t)h(t)[h(t)g(t)]^{-1}$ is a bilinear function of the vectors a, b; in this way, T_e becomes an n-dimensional algebra over the field $\mathbb{R}$, called the *tangent algebra* of the (local) Lie group G. The algebra $T_e = L(G)$ is obviously anticommutative (satisfies the identity $x^2 = 0$); associativity of multiplication in G implies that $L(G)$ satisfies the Jacobi identity $xy \cdot z + yz \cdot x + zx \cdot y = 0$, i.e. $L(G)$ is a Lie algebra. Another way of constructing the Lie algebra $L(G)$ is possible too. To this end, expand the functions $f^i(x, y)$ into the Taylor series in a neighbourhood of the coordinate origin $|x|, |y| < \epsilon$

$$f^i(x, y) = x^i + y^i + a^i_{jk} x^j y^k + o(\epsilon^2)$$

(the summation is over the repeating indices) and set $c^i_{jk} = a^i_{jk} - a^i_{kj}$. Then $L(G)$ is defined as the algebra with a basis $e_1, \dots, e_n$ and the multiplication table $e_i e_j = c^k_{ij} e_k$.

A. I. Malcev has pointed out that neither of the two ways of defining the tangent algebra really needs associativity of multiplication in G, as well as that they can be applied to differentiable and, in particular, to *analytic loops*. The curve $k(t)$, considered above may be defined in case of loops by the equality $k(t^2) = g(t)h(t)/[h(t)g(t)]$.

There is a close relation between Lie groups and their tangent algebras. For instance, two connected Lie groups are locally isomorphic if and only if, their Lie algebras are isomorphic; connected simply connected Lie group is uniquely determined by its Lie algebra. Thus it is clear why Lie algebras are one of the main instruments in studying Lie groups. However, in a more general case of loops, without associativity, the notion of a tangent algebra turns out to be not very meaningful, if at least power-associativity is not assumed.

A continuous curve $g(t)$, defined for sufficiently small values of t is called a *locally one-parameter subgroup*, if the following identity $g(t+s) = g(t)\cdot g(s)$ holds in its domain. Such a curve is always differentiable and if a is its tangent vector in e, then the coordinates $g^i(t)$ satisfy the following system of ordinary differential equations

$$\frac{dg^i(t)}{dt} = v^i_j(g)a^j, \quad i = 1,\ldots,n, \tag{2}$$

where $v^i_j(x) = \frac{\partial}{\partial y^j} f^i(x,0)$. A crucial moment for the theory of local Lie groups is the theorem about the existence of the local one-parameter subgroups $g(t) = g(a;t)$ with a given arbitrary tangent vector $a \in T_e$. Because of $g(\alpha a;t) = g(a;\alpha t), \alpha > 0$, the vector $g(a;1)$ is defined, for sufficiently small a. The transformation $\exp : a \mapsto g(a;1)$ defines a diffeomorphism of a neighbourhood V of the coordinate origin in T_e to some neighbourhood U of the element e in G. The inverse transformation $\log : U \mapsto V$ introduces the so-called canonical coordinates of the 1st kind. The formula $a \circ b = \log(\exp a \cdot \exp b)$ provides V with a structure of a local Lie group, isomorphic to the local group U; the one-parameter subgroups in V are defined by the equations $\tilde{g}(a;t) = at$.

A natural boundary of generality of the assumptions, under which a similar situation occurs is the condition of power-associativity. And, in fact, this condition turns out to be sufficient too. Let $g(t)$ be a solution of the system (2) with the initial condition $g(0) = (0,\ldots,0) = e$, defined for $|t| < \alpha$. By approximating the curve $g(t)$ by the cyclic subgroups with generators $g(u), u \to 0$, we can show that $g(t)$ is a one-parameter subgroup in its whole domain. Thus it is in the class of power-associative loops where the exponential transformation makes sense and where the canonical coordinates of 1st kind are defined (Kuz'min, 1971).

The fact that a passage from one system of canonical coordinates of 1st kind to another system of canonical coordinates of 1st kind is given by analytic functions (in fact even linear) implies, for instance, invariance of the differentiable structure in a local differentiable power-associative loop.

If G is a Lie group, then, in the canonical coordinates of the 1st kind the multiplication operation on G (more exactly, in a neighbourhood of the unity of G) is expressible through the operations of addition and multiplication in the Lie algebra, with the aid of the so-called *Campbell-Hausdorff series*

$$x \circ y = x + y + \frac{1}{2}xy + \frac{1}{12}(xy^2 + yx^2) + \frac{1}{24}yx^2y + \ldots, \tag{3}$$

where parentheses have been left out in the left normalized products of elements of $L(G)$. Hausdorff gave a constructive method of finding the summands of the series (3) (cf. Chebotarev, 1940). Consider x, y to be generators of the free Lie algebra (or the free anti-commutative algebra) and denote the right-hand-side of (3) by $u(x,y)$. Then $u(x,y)$ satisfies a symbolic differential equation

$$x\frac{\partial u}{\partial x} - xV^{-1}(y)\frac{\partial u}{\partial y} = 0,$$

where the operator $s\frac{\partial}{\partial t}$ stands for the differential replacement of occurrences of t by occurrences of s, $xV^{-1}(y) = \sum_{n=0}^{\infty}(-1)^n b_n xy^n$ and b_n are rational numbers with the derived function $\sum b_n t^n = t/(e^t - 1)$ ($b_k = B_k/k!$, where B_k are the so-called Bernoulli numbers). Arranging $u(x,y)$ in powers of x: $u = u_0 + u_1 + \ldots$, we get a system of recurrent relations for determining u_i: $u_0 = y$, $ku_k = xV^{-1}(y)\frac{\partial u_{k-1}}{\partial y}, k \geqslant 1$. In particular,

$$u_1 = xV^{-1}(y) = x + \frac{1}{2}xy + \frac{1}{12}xy^2 - \frac{1}{6!}xy^4 + \frac{1}{6\cdot 7!}xy^6 + \ldots$$

Alternative (dissociative) loops where every two elements generate a subgroup, occupy an intermediate place between power-associative loops and groups. If G is an alternative differentiable loop and if $g(t), h(t)$ are its local one-parametric subgroups, then the products of the form $g(t_1)h(s_1)\ldots g(t_n)\,h(s_n)$, for small t_i, s_i do not depend on the distribution of the parentheses (initially it is checked for rational t_i, s_i). Thus, $g(t_i), h(s_i)$ lie in a local Lie subgroup. After switching to the canonical coordinates of the 1st kind, we find that the multiplication in G, in the neighbourhood of the coordinate origin is expressible by the ordinary Campbell-Hausdorff formula (3), and that the tangent algebra $L(G)$ is a binary Lie algebra: every two of its elements generate a Lie subalgebra. Formula (3) shows that G is determined uniquely, up to a local isomorphism, by its tangent algebra, and since the right-hand-side of (3) is an analytic function of its coordinates x, y, G has a structure of a local analytic loop, compatible with the initial differential structure.

For every finite-dimensional binary Lie algebra L over $\mathbb{R}$, with the aid of the Campbell-Hausdorff series, a local analytic alternative loop is constructed, whose tangent algebra is isomorphic to L.

For a long time the attention of algebraists has been attracted by the so-called *Moufang loops*, defined by any of the following mutually equivalent identities:

$$(xy \cdot z)y = x(y \cdot zy), \tag{4}$$
$$(yz \cdot y)x = y(z \cdot yx), \tag{5}$$
$$xy \cdot zx = (x \cdot yz)x. \tag{6}$$

They have been first considered by Moufang, in whose honor they have been named, in connection to studies on non-Desarguesian projective planes. The following fundamental theorem is due to her: if G is a Moufang loop, then every three elements $a, b, c \in G$, connected with the relation $ab \cdot c = a \cdot bc$, generate a subgroup. In particular, for $c = e$, this implies the statement about the alternativeness of the Moufang loops. Let us give two examples of analytic Moufang loops.

Alternative rings and algebras satisfy the identities (4)–(6) (cf. 2.3). If A is a finite-dimensional alternative algebra with unity, over the field $\mathbb{R}$ and if a is its invertible element, then a^{-1} is a polynomial in a (over $\mathbb{R}$). Thus, the set $W(A)$ of invertible elements of the algebra A is closed with respect to multiplication and forms an analytic Moufang loop globally. Its tangent algebra is isomorphic to the commutator algebra $A^{(-)}$, whose space coincides with A and the multiplication $[,]$ is related to the multiplication in A by the formula $[a, b] = ab - ba$.

Multiplicativity of the norm in the Cayley-Dickson algebra $\mathbb{O} = \mathbb{O}(\alpha, \beta, \gamma)$ over $\mathbb{R}$ implies that the elements in $\mathbb{O}$, with the norm equal to 1 also form an analytic Moufang loop H, globally. If $\mathbb{O}$ is a division algebra, then the space of this loop is a 7-dimensional sphere S^7, and if $\mathbb{O}$ is a split Cayley-Dickson algebra, then the space H is analytically isomorphic to the direct product $S^3 \times \mathbb{R}^4$. The tangent algebra of this loop is isomorphic to a 7-dimensional simple Malcev algebra $M(\alpha, \beta, \gamma)$ (cf. 5.1).

By writing down operation of multiplication in an arbitrary analytic Moufang loop G with the aid of the Campbell-Hausdorff series (in canonical coordinates of the 1st kind), A. I. Malcev discovered that the tangent algebra $L(G)$ satisfies identity (4.1), i.e. turns out to be a Malcev algebra (in modern terminology). Thus, a Malcev algebra is associated to every analytic (or differentiable) Moufang loop, in the same way as a Lie algebra is associated to a Lie group. The question arose, however, about the existence of the inverse correspondence: Is there any analytic Moufang loop, even local, corresponding to any finite-dimensional real Malcev algebra? For an arbitrary Lie algebra L, the multiplication defined in the neighbourhood of the coordinate origin, via the Campbell-Hausdorff series, produces a local Lie group $g(L)$. An analogous result turned out to be true for Malcev algebras too (Kuz'min, 1971).

Let L be a finite-dimensional Malcev algebra over $\mathbb{R}$ and let G be a local analytic loop, constructed on L, by the Campbell-Hausdorff formula. G is alternative, because L is a binary Lie algebra. Substituting y by $x^{-1}y$ in (6) and multiplying both sides of that equality by x^{-1} on the right, we conclude that, in the class of alternative loops, the Moufang identities are equivalent to the following identity

$$(y \cdot zx)x^{-1} = x(x^{-1}y \cdot z). \tag{6'}$$

Assume that x, y, z are generators of the free Malcev algebra and define a formal alternative loop by series (3), and then denote the left-hand-side of (6′) by $\theta_1(x, y, z)$ and the right-hand-side by $\theta_2(x, y, z)$. Then, we can show that each function θ_i satisfies the following symbolic differential equation

$$y\frac{\partial\theta}{\partial y} - \left[yV^{-1}(z) - \frac{1}{6}J(x, y, z)V(x)\right]\frac{\partial\theta}{\partial z} = 0; \tag{7}$$

this equation induces a system of recurrent relations for the participating functions θ, homogeneous in y. Since the components of the zero power (computable as the values of the functions θ_i, for $y = 0$) also coincide for both functions ($\theta_i(x, 0, z) = (zx)x^{-1} = z$, $\theta_2(x, 0, z) = x(x^{-1}z) = z$), all the other components of these functions are equal, and $\theta_1 = \theta_2$. Thus, a formal Moufang loop is assigned via the Campbell-Hausdorff series to a free Malcev algebra, while a local analytic Moufang loop G corresponds to a finite-dimensional Malcev algebra L; this gives the affirmative answer to the question posed above. Note that, for $x = 0$, equation (7) turns into the defining equation for the functions $\theta(0, y, z) = y \circ z$.

The fundamental results about relations between local Lie groups and global Lie groups carry over to analytic Moufang loops. Namely, every local analytic Moufang loop is locally isomorphic to an analytic global Moufang loop. If G and G' are connected analytic Moufang loops where G is simply connected and if ϕ is a local homomorphism of G to G', then ϕ is uniquely extendable to a homomorphism $\tilde{\phi}$ globally, and if G' is also simply connected and ϕ is a local isomorphism, then $\tilde{\phi}$ is an isomorphism of G to G'. Thus, there exists an up to isomorphism unique simply connected analytic global Moufang loop G with a given tangent Malcev algebra, and every connected analytic Moufang loop G' with the same tangent algebra can be obtained from G by factoring out mod a discrete central normal subgroup. The space of a simply connected analytic Moufang loop, with a solvable tangent Malcev algebra, is homeomorphic to the Euclidean space $\mathbb{R}^n$ (Kerdman, 1979).

The analogous statements in the more general case of binary Lie algebras and analytic alternative loops are incorrect: a finite-dimensional binary Lie algebra over $\mathbb{R}$ may be not a tangent algebra of any global analytic alternative loop.

Example. The unique non-Lie Malcev algebra A of dimension 4 has the following multiplication table: $e_1e_2 = e_3, e_1e_4 = e_1, e_2e_4 = e_2, e_3e_4 = -e_3, e_1e_3 = e_2e_3 = 0$. The algebra A is solvable and is the tangent algebra of an analytic Moufang loop whose space coincides with $\mathbb{R}^4$, while the multiplication in the coordinate form is defined by the following formulas:

$$z_1 = x_1e^{y_4} + y_1, z_2 = x_2e^{y_4} + y_2, z_3 = x_3 + y_3e^{x_4} + x_1y_2 - x_2y_1, z_4 = x_4 + y_4.$$

An interesting generalization of the theory of local analytic Moufang loops is connected with the notion of a Bol loop. A loop G is called a *left Bol loop*, if it satisfies the following identity:

$$(y \cdot zy)x = y(z \cdot yx) \tag{8}$$

(compare with (5)). The loop anti-isomorphic to it satisfies the following identity:

$$(xy \cdot z)y = x(yz \cdot y) \tag{8'}$$

and is called the right Bol loop. Moufang loops are both left and right Bol loops; conversely, the set of identities (8), (8′) implies the identity $xy{\cdot}x = x{\cdot}yx$ (flexibility), therefore left and right Bol loop is a Moufang loop. In the sequel, we mean a left Bol loop, when we speak of a Bol loop.

Example. The set of positive definite Hermitian matrices of order n with the operation of "multiplication" $a \circ b = \sqrt{ab^2a}$ is an analytic global Bol loop. The inversion operation in this loop is an automorphism: $(x \circ y)^{-1} = x^{-1} \circ y^{-1}$. More generally, let G be a group with an involutive automorphism ϕ and assume that the set $\{x \cdot (x\phi)^{-1} \mid x \in G\}$ allows taking unique square roots. Then A is a Bol loop with respect to the operation $a \circ b = \sqrt{ab^2a}$. We remark also that in this case, every element $x \in G$ is uniquely representable in the form $x = ah$, where $a \in A$, $h \in H = \{y \in G \mid y\phi = y\}$, thus A can be identified in a natural way with the space of the left conjugacy classes of G mod H.

The Bol loops are power-associative, thus, they have canonical coordinates of the 1st kind. If G is a local differentiable Bol loop of class $C^k, k \geqslant 5$, then the multiplication operation in G is analytic with respect to the canonical coordinates of the 1st kind. The tangent space T_e of a locally analytic Bol loop is a binary-ternary algebra with anticommutative multiplication and a trilinear operation $[,,]$ satisfying the following identities:

$$[x, x, y] = 0, \tag{9}$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0, \tag{10}$$

$$[a, b, [x, y, z]] = [[a, b, x], y, z] + [x, [a, b, y], z] + [x, y, [a, b, z]], \tag{11}$$

$$xy \cdot zt = [x, y, zt] - [z, t, xy] + [x, y, t]z - [x, y, z]t. \tag{12}$$

Identities (9)–(12) define the class of *Bol algebras.* In canonical coordinates of 1st kind multiplication in G is expressible through the mentioned operations in $T_e = B(G)$ by the following formula:

$$a \circ b = a + b + \frac{1}{2}ab - \frac{1}{4}(ab^2 - ba^2) + \frac{1}{3}[a, b, b] - \frac{1}{6}[b, a, a] + \dots \tag{13}$$

Thus, to every locally analytic Bol loop G, the tangent Bol algebra $B(G)$ is assigned uniquely. Conversely, every finite-dimensional Bol algebra over

the field $\mathbb{R}$ is a tangent algebra of some local analytic Bol loop and two such loops are locally isomorphic if and only if their tangent algebras are isomorphic (Sabinin, Miheev, 1982). In case of Moufang loops, the ternary operation $[,,]$ is expressible through the binary, as follows:

$$[x, y, z] = \frac{1}{3}(2xy \cdot z - yz \cdot x - zx \cdot y), \tag{14}$$

and the series on the right-hand-side of (13) turns into the Campbell-Hausdorff series. Identities (9)–(11) define the so-called class of *Lie triple systems* (L. t. s.). Any Lie algebra is a Lie triple system with respect to the operation of double multiplication or, any of its subspaces closed under this operation; every L. t. s. has a standard embedding into a Lie algebra. Every Jordan algebra is a L. t. s. with respect to the operation $[x, y, z] = (x, z, y)$. A Malcev algebra A is a L. t. s. T_A with respect to operation (14), thus L. t. s. is one of the means of studying Malcev algebras.

Bol loops have an intrinsic application in differential geometry. For example, every locally symmetric affinely connected space X (cf. Helgason, 1962, p. 163; Sabinin, Mikheev, 1985), can be provided, in a natural way, with the structure of a local Bol loop X_e, in a neighbourhood of an arbitrary point $e \in X$; furthermore, e is the unity of the loop X_e and the geodesics, through e, are local one-parameter subgroups. A special role in this case is played by analytic Moufang loops and Malcev algebras: a solution of the problem of describing n-dimensional torsion-free affinely connected spaces, with n independent infinitely small translations is related to them (Sabinin, Mikheev, 1985).

6.3. Some Classes of Loops and Quasigroups. The class of loops closest to groups and most researched is that of Moufang loops. The loop property to be a Moufang loop is universal in the sense of 6.1. The Moufang loops also have the following invariant under isotopy. We introduce a derived operation $x + y = xy^{-1}x$ in a Moufang loop $Q(\cdot)$ and we call $Q(+)$ the core of Q. If two Moufang loops are isotopic, then their cores are isomorphic. The loop property of being a left Bol loop is also universal.

Commutative Moufang loops (CML) have been studied specially thoroughly. They arise in studying rational points on cubic hyperplanes (Manin, 1972) and are characterized by one identity $x^2 \cdot yz = xy \cdot xz$.

Let k be an infinite field and let V be a cubic hypersurface defined over k. By definition, V is defined by a homogeneous equation of the third degree $F(T_0, \dots, T_n) = 0$, where $(T_0, \dots, T_n)$ is the system of homogeneous coordinates in the projective space P^n over k. We will assume that the form F is irreducible over the algebraic closure $\overline{k}$ of the field k. Let V_r be the set of non-singular k-points of V (the tangent hyperplane to V is defined in such points). A ternary relation of collineation is introduced on the set V_r: the triple of points (x, y, z) is collinear, if x, y, z are on one line l defined over k which is either contained in V or intersects V only in the points x, y, z (each of the points x, y, z appears as many times as the order of osculation of V with line l at that point). It is obvious that the relation of collineation is symmetric, i.e. is preserved under any permutations of x, y, z; moreover, for every $x, y \in V_r$ there exists a point $z \in V_r$, such that x, y, z are collinear. If z is uniquely determined by given x and y, then by setting $x \circ y = z$ we can define a structure of a totally-symmetric quasigroup on V_r (the equality $a \circ b = c$ is preserved under all the permutations of the symbols a, b, c). For $n > 1$ however, the set V_r does not in general have this property, hence one performs an additional factorization mod a "permissible" equivalence relation S, with a totally-symmetric quasigroup $Q = V_r/S$ as a result. Fixing an arbitrary element E in Q and introducing a new multiplication $(\cdot)$ in Q by the formula $X \cdot Y = E \circ (X \circ Y)$, we get a CML $Q(\cdot)$ with unity E and the identity $x^6 = 1$. The loop $Q(\cdot)$ is locally finite.

Let G be an arbitrary CML and let $(x, y, z) = (xy \cdot z)(x \cdot yz)^{-1}$ be the *associator* of the elements $x, y, z \in G$. We define the descending central series $G = G_0 \geqslant G_1 \geqslant \dots$ of the loop G, by taking $G_{i+1} (i \geqslant 0)$ to be the subloop generated by the associators of the form $(x, y, z), x \in G_i, y, z \in G$. The loop G is said to be centrally nilpotent of class k, if $G_k = (e)$ and if k is the smallest number with this property.

By the Bruck-Slaby theorem, every n-generated CML is centrally nilpotent of class $\leqslant n-1$ (cf. Bruck, 1958). The intriguing question about the exactness of this estimate had been open for a long time; the affirmative answer was given in 1978 (Malbos, 1978; Smith, 1978; cf. also Bénéteau, 1980).

If two CML are isotopic, then they are isomorphic.

We can arrive at the notion of a Moufang loop in the following way. Let ${}^{-1}x, x^{-1}$ be defined by the equalities ${}^{-1}x \cdot x = x \cdot x^{-1} = e$. A loop G is called an IP-loop (a loop with the inversion property), if it satisfies the identity ${}^{-1}x \cdot xy = yx \cdot x^{-1} = y$ (in this case ${}^{-1}x = x^{-1}$). Every Moufang loop is an IP-loop, because of the alternativity property. If all the LP-isotopes of G (cf. 6.1) are IP-loops, then G is a Moufang loop.

Medial quasigroups, defined by the identity $xu \cdot vy = xv \cdot uy$, appear in various applications. The main theorem for them is the Bruck-Toyoda theorem: Let $Q(\cdot)$ be a medial quasigroup; then there exists an abelian group $Q(+)$ such that $x \cdot y = x\phi + y\psi + c$, where ϕ, ψ are commuting automorphisms of the group $Q(+)$ and c is a fixed element.

Another type of quasigroups has apparently first attracted attention of the algebraists; these are the *distributive quasigroups* satisfying the following identities of the left and the right distributivity:

$$x \cdot yz = xy \cdot xz, \quad yz \cdot x = yz \cdot zx.$$

There is an interesting relation between the distributive quasigroups and the CML: every LP-isotop $Q(\circ)$ of a distributive quasigroup $Q(\cdot)$ is a CML and isomorphic LP-isotopes $Q(\circ)$ correspond to different elements $a \in Q$.

The defining identities of a distributive quasigroup Q mean that the left and the right translations are automorphisms of Q, and they generate respectively the left and the right associated groups. The left associated group of a finite distributive quasigroup is solvable and, moreover, its commutator is nilpotent; furthermore, a finite distributive quasigroup decomposes into the direct product of its maximal p-subquasigroups. The latter result carries over to locally finite distributive quasigroups too.

The following analogue of Moufang's theorem holds for distributive quasigroups: if four elements a, b, c, d are related by the medial law $ab \cdot cd = ac \cdot bd$, then they generate a medial subsemigroup. A distributive quasigroup may be not medial globally. The left distributive identity does not imply the right distributivity. Let $Q(\cdot)$ be a Moufang loop, where the mapping $x \to x^2$ is a permutation, and $Q(+)$ is its core (i.e. $x + y = xy^{-1}x$). Then $Q(+)$ is a left distributive quasigroup, isotopic to a left Bol loop. $Q(+)$ is a distributive quasigroup if and only if $Q(\cdot)$ satisfies the identity $xy^2x = yx^2y$.

Example. Let G be the free group with the identity $x^3 = 1$ and with $r \geqslant 3$ generators. Then G is a finite group of order $3^{m(r)}$, where $m(r) = r + \binom{r}{2} + \binom{r}{3}$; G is a non-associative CML with respect to the operation $x \circ y = x^{-1}yx^{-1}$, and it is a distributive quasigroup, with respect to the operation $x + y = xy^{-1}x$; it is non-medial for $r \geqslant 4$.

A special place in the theory of quasigroups is occupied by the direction related to finding fairly general conditions on the identity of the quasigroup $Q(\cdot)$, so that the operation of multiplication on Q is representable in the form $x \cdot y = x\alpha + y\beta + c$, where $Q(+)$ is a group, α, β are its automorphisms and c is a fixed element. A typical example of such an identity is the medial identity to which the aforementioned Bruck-Toyoda theorem applies. The identity $w_1 = w_2$ is called balanced, if w_1, w_2 are non-associative words of the same composition and every variable occurs in w_1, w_2 only once. A balanced identity $w_1 = w_2$ is called completely cancellable, if w_i contains a subword $u_i, i = 1, 2$, where u_1, u_2 are of the same composition and of length > 1. If a quasigroup satisfies a balanced identity, which is not completely cancellable, then it is isotopic to a group.

6.4. Combinatorial Questions of the Theory of Quasigroups. The class of totally-symmetric quasigroups (TS-quasigroups) that we encountered in 6.3 is interesting from the point of view of combinatorics. It is obvious that this class is defined by the identities $xy = yx$, $x \cdot xy = y$. An idempotent ($x^2 = x$) TS-quasigroup is called a *Steiner quasigroup*. Steiner quasigroups arise in connection to the so-called *Steiner triple systems* studied in combinatorial analysis. A system S of unordered triples (a, b, c) of elements of the set Q is called a Steiner triple system, if the elements of every triple are mutually different and if every pair of elements $a, b \in Q$ occurs in a unique triple $(a, b, c) \in S$. Setting $a \cdot b = c$, if $(a, b, c) \in S$ and $a^2 = a$, we get a Steiner quasigroup; conversely, every Steiner quasigroup $Q(\cdot)$ determines a Steiner triple system S on Q. A finite Steiner quasigroup of order n exists if and only if n is either of the form $6k + 1$ or $6k + 3$. Every distributive quasigroup is an extension of a normal Steiner subquasigroup, by a medial quasigroup. In the above example of a distributive quasigroup, related to the group G of period 3, $G(+)$ is a Steiner quasigroup.

Steiner quadruples are defined analogously to Steiner triplets: every pair of elements $a, b \in Q$ occurs in a unique quadruple $(a, b, c, d) \in S$. The *Stein quasigroups*, satisfying the identities $x \cdot xy = yx, xy \cdot yx = x$ are related to them. Every two different elements a, b of a Stein quasigroup Q generate a subquasigroup of order 4 with the following Cayley table

	a	b	c	d
a	a	c	d	b
b	d	b	a	c
c	b	d	c	a
d	c	a	b	d

Thus, 2-generated subquasigroups in the Stein quasigroup Q form a quadruple Steiner system Q. Conversely, the Stein quasigroup $Q(\cdot)$ is easily constructible on every quadruple Steiner system, with the aid of the Cayley table above. A finite Stein quasigroup of order n exists if and only if n is either of the form $12k+1$ or $12k+4$ (Stein, 1964).

If a Stein quasigroup is isomorphic to a group, then the latter must be of nilpotence degree 2.

A number of applications of the theory of quasigroups to the solution of combinatorial questions is based on the close relation between finite quasigroups and the latin squares, mentioned in 6.1. One of the questions of this kind concerns investigations of orthogonal latin squares. Two latin squares of order n are called *orthogonal*, if in their superposition, there are exactly n^2 different ordered pairs of elements. A pair of orthogonal quasigroups corresponds to a pair of orthogonal latin squares. The quasigroups $Q(A)$ and $Q(B)$ (A, B are the multiplication operations on Q) are called orthogonal, if the system of equations $A(x, y) = a$, $B(x, y) = b$ is uniquely solvable, for all $a, b \in Q$. By joint efforts of Bose, Shrikhande and Parker (1960), it was shown that there are orthogonal latin squares of every order $n \neq 2, 6$. Latin squares form an orthogonal system, if they are mutually orthogonal. An orthogonal system of quasigroups (OSQ) corresponds to an orthogonal system of latin squares. The following theorem holds: If $A_1, \ldots A_t$ is an orthogonal set of latin squares of order $n \geqslant 3$, then $t \leqslant n-1$. For $t = n-1$, this orthogonal system and the corresponding system of quasigroups are called full. The full OSQ may be constructed with the aid of Galois fields, setting for instance, $A_i(x, y) = x + \lambda_i y$, where $\lambda_i \neq 0$. Thus, for every $n = p^\alpha \geqslant 3$, where p is a prime number, there exists a full OSQ of order n. Full OSQ are related to projective planes (cf. 2.2), since every projective plane may be coordinatized by a full OSQ. Furthermore the following holds:

Theorem. *A projective plane of order $n \geqslant 3$ may be constructed if and only if there exists a full OSQ of order n.*

The OSQ with $k \leqslant n-1$ quasigroups of order n correspond to algebraic k-nets (cf. 6.5), which are generalizations of projective planes.

The multiplication operation in the Stein quasigroup $Q(\cdot)$ is orthogonal to the operation $x \circ y = y \cdot x$. There are six more classes of quasigroups that are definable by identities of the Stein identity type, for which some conditions of orthogonality are satisfied.

The notion of orthogonality is closely related to the notion of a transversal in a latin square. A transversal of a latin square of order n is a sequence of n different elements in different rows and columns of the latin square. A quasigroup is called admissible, if the corresponding latin square contains at least one transversal. In this case the quasigroup has the so-called complete permutation. A permutation θ of elements of $Q(\cdot)$ is called complete, if the mapping $\theta' : x \mapsto x \cdot x\theta$ is also a permutation. For instance, the identity permutation in any group of odd order or, more generally, in a power-associative loop of odd order, is a complete permutation. In the Klein group $K_4 = \{e, a, b, c\}$ the permutation $\theta = \begin{pmatrix} e & a & b & c \\ e & b & c & a \end{pmatrix}$ is a complete permutation, since $\theta' = \begin{pmatrix} e & a & b & c \\ e & c & a & b \end{pmatrix}$. On the other hand, the cyclic groups of even order have no complete permutations, i.e. are not admissible. It is known that the number of different transversals in a group is a multiple of the order of the group.

A quasigroup with an orthogonal quasigroup has a complete permutation. Two permutations ϕ_1, ϕ_2 on the set Q are non-intersecting, if $x\phi_1 \neq x\phi_2$, for every $x \in Q$. The following holds:

Theorem. *A quasigroup Q of order n has an orthogonal quasigroup, if and only if, Q contains n mutually non-intersecting complete permutations.*

A group, on the other hand, has an orthogonal quasigroup if and only if it is admissible.

The existence of a transversal in a quasigroup Q of order n gives a possibility to go from Q to a quasigroup of order $n+1$, if we project the elements of this transversal onto an additional row and column indexed by k, where $k \notin Q$, and place the new element k in their place and in the cell (k, k). This passage from a semigroup of order n to a quasigroup of order $n+1$ is called extension and allows for different generalizations.

The notion of orthogonality of latin squares of order n may be generalized requiring that in their superposition there are exactly r different ordered pairs, $n \leqslant r \leqslant n^2$. In this case the latin squares and the corresponding quasigroups are called r-orthogonal (for $r = n^2$ we get the ordinary orthogonality). In relation to the notion of r-orthogonality, there is a question of describing the spectrum R_n of partial orthogonality of the class of quasigroups of order n, i.e., for every n, finding all the values of r such that there are r-orthogonal quasigroups of order n. This question remains open in general. There are descriptions of the spectrum R_n, for small n: $R_3 = \{3,9\}, R_4 = \{4,6,8,9,12,16\}, R_5 = \{5,7,10-19,21,25\}$.

Example. The following two squares are 12-orthogonal:

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

3	4	2	1
1	3	4	2
4	2	1	3
2	1	3	4

A notion of r-orthogonal systems of quasigroups is introduced in analogy with the OSQ. Up to now, only the $(n+2)$-orthogonal systems of quasigroups of order n have been investigated. It has been proved that the number of quasigroups in systems of this kind does not exceed $2^{[\frac{n}{2}]-1}$ and possibilites of constructing full systems of this kind (i.e. systems with $2^{[\frac{n}{2}]-1}$ quasigroups) over groups have been investigated.

The theory of quasigroups has practical applications, through its combinatorial aspect: in the theory of information coding and in planning experiments.

6.5. Quasigroups and Nets. The notion of a 3-web has a significant role in differential geometry; the algebraic analogue of this notion is that of a 3-net. A family N consisting of objects of two forms – lines l and points p with the incidence relation $p \in l$ is called an *(algebraic) 3-net*, if the set of all the lines $\mathcal{L}$ is divided into three mutually non-intersecting classes $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ and if the following conditions hold: 1) every point p is incident to one and only one line in every class; 2) every two lines in different classes are incident to some point p. Instead of "p is incident to l" we will say "p belongs to l", "l contains p". If $P(l) = \{p \mid p \in l\}$, then all the classes $\mathcal{L}_i$ are of the same cardinality as $P(l)$ and are of the same cardinality as some "index" set Q. Let $\phi_i : Q \longrightarrow \mathcal{L}_i$ be some bijective transformations, $i = 1,2,3$. Let us introduce the coordinates of lines and points in N. If $\phi_i(a) = l \in \mathcal{L}_i$, then assign the pair $[a,i]$ to the line l. If the lines $[a,1],[b,2]$ contain the point p, then assign the pair (a,b) to the point p. Let us define a multiplication operation $(\cdot)$ on Q, setting $a \cdot b = c$, if the line $[c,3]$ contains the point (a,b). Q is a quasigroup with respect to this operation; mutually isotopic quasigroups correspond to different triples of "index transformations" ϕ_i. Thus a class of isotopic quasigroups uniquely corresponds to every 3-net. Conversely, one can relate with any given quasigroup $Q(\cdot)$ (and its isotopes) the algebraic 3-net N whose lines are the pairs $[a,i]$, $a \in Q$, $i = 1,2,3$, and whose points are the pairs (a,b), $a,b \in Q$ with the incidence relation $(a,b) \in [a,1],[b,2],[a \cdot b,3]$.

The closure conditions, that also arose in differential geometry, play a great role in the theory of 3-nets. The points $(a,b),(c,d)$ are called collinear if $a \cdot b = c \cdot d$. It is said that a 3-net N satisfies some closure condition, if collinearity of some pairs of points implies the collinearity of another pair of

points. Thus, the closure conditions are equivalent to some quasi-identities of a special form, in the coordinate quasigroup Q. Since the closure conditions do not depend on notation of the lines, they are preserved under isotopy of quasigroups (i.e. have the universal property). In some cases, the closure conditions (quasi identities) are equivalent to the identities in $Q(\cdot)$, which are also universal, i.e. are invariant with respect to the isotopies of loops. An algorithm for constructing the closure figures, corresponding to the universal identities is known (Belousov, Ryzhkov, 1966).

Example 1. The Thomsen condition is of the form $x_1y_2 = x_2y_1$, $x_1y_3 = x_3y_1 \Rightarrow x_2y_3 = x_3y_2$. It is satisfied if and only if $Q(\cdot)$ is isotopic to an abelian group.

Example 2. The Reidemeister condition is of the form $x_1y_2 = x_2y_1$, $x_1y_4 = x_2y_3$, $x_3y_2 = x_4y_1 \Rightarrow x_4y_3 = x_3y_4$. It is satisfied if and only if $Q(\cdot)$ is isotopic to a group.

The notion of a 3-net generalizes to that of k-nets ($k \geqslant 3$), which differ from 3-nets in that there are exactly k lines passing through every point and those lines belong to k different classes $\mathcal{L}_1, \dots, \mathcal{L}_k$. The number k is called the genus of the net. The points and lines of the k-net N may be denoted again as the pairs $(a,b),[a,i]$, where a,b run through the base set $Q, i = 1,\dots,k, [a,i] \in \mathcal{L}_i$. If $[c,i]$ is a line in $\mathcal{L}_i$ passing through the point (a,b), we set $A_i(a,b) = c$; the outcome of this is that we assign the binary operation A_i on Q to every family $\mathcal{L}_i$. Since, for $i \neq j$ the system of equations $A_i(x,y) = a$, $A_j(x,y) = b$ is uniquely solvable for every $a,b \in Q$, the operations $A_1,\dots,A_k$ form an orthogonal system (OSO), which, in particular, may be OSQ. Change of the coordinates of points and lines of k-nets corresponds to an isostrophy of the corresponding OSO (or OSQ); isostrophy is a generalization of isotopy. The order of a k-net is the number of points on every line, i.e. the order of the set Q. The genus k and the order n are connected by the relation $k \leqslant n+1$. If $k = n+1$, then such a net is nothing else but an affine plane.

Let $\sum = \{A_i \mid i = 1,\dots,k\}$ be the coordinate OSO. We have a bijective correspondence $\theta_{ij} : (a,b) \longrightarrow (x,y)$, if we assign, to every point (a,b) the intersection point (x,y) of the lines $[a,i]$, $[b,j]$ $(i \neq j)$. Let us denote $A_m\theta_{ij}(a,b) = V_{ijm}(a,b)$; then V_{ijm} is a quasigroup operation on Q. The system $\overline{\sum}$ of all the quasigroups V_{ijm} is called a covering for $\sum$. A family of points and lines of a k-net N is called a configuration, if there are three lines (edges) of this configuration passing through every point (vertex) of this configuration and if every edge contains at least two vertices. It turns out that some functional equation on the quasigroups in $\overline{\sum}$ corresponds to every configuration in N and conversely. Thus, the configuration with 4 vertices corresponds to the equation of general associativity, the configuration with 5 vertices corresponds to the equation of general distributivity etc.

Even a more general notion of a spacial net is considered in the theory of nets; an orthogonal system of n-ary operations (n-quasigroups) corresponds to them. Such nets have not been studied much up to now and it is partly related to the fact that there are different variants of defining orthogonality for n-quasigroups.

Formation of the quasigroup theory goes back to the beginning of the thirties. It has recently intensively developed in a number of countries (USSR, USA, France, Hungary, etc.) and has become one of the independent parts of modern algebra.

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