ABSTRACT

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Leibniz algebras are generalizations of Lie algebras. Since the introduction of Leibniz algebras in 1993 by Jean-Louis Loday, many results for Lie algebras have been generalized to the Leibniz case, such as Lie's Theorem, Engel's Theorem, Cartan's Criterion and the Levi Decomposition. Since 2008, motivated by group theory, David Towers has defined c-ideals, c-sections, and completions of maximal subalgebras of Lie algebras. He has used these, as well as Cartan subalgebras and CAP-subalgebras, to characterize solvable and supersolvable Lie algebras.

We introduce definitions for c-ideals, CAP-subalgebras, c-sections and completions for maximal subalgebras of Leibniz algebras. We develop properties of these subalgebras of Leibniz algebras. We then give several characterizations of solvable and supersolvable Leibniz algebras based on the behavior of these subalgebras.

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Some Criteria for Solvable and Supersolvable Leibniz Algebras

by
Bethany Nicole Turner

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APPROVED BY:

Ernest Stitzinger
Co-chair of Advisory Committee

Kailash Misra
Co-chair of Advisory Committee

Nathan Reading

Mohan Putcha

Paul Huffman

DEDICATION

To the one under the bed.

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TABLE OF CONTENTS

CHAPTER

1

INTRODUCTION

Lie algebras, which arose though the study of Lie groups, have been well-studied since their introduction in the mid-nineteenth century. Since then they have found applications in both physics and applied mathematics.

Various generalizations of Lie algebras have been studied. For example, in 1955 Malcev algebras were introduced in [Mal55]. Leibniz algebras were defined in 1993 by Jean-Louis Loday [Lod93]. The major difference is that Leibniz algebras are not antisymmetric. Since that time, analogues of many major Lie algebra results have been proven for Leibniz algebras.

Among these are Lie's Theorem, Engel's Theorem, Levi decomposition, and Cartan's criterion [Dem13]. Much of the research into Leibniz algebras is focused on determining which properties of Lie algebras can be generalized, as in [Bar11] and [Bat13].

An important class of non-Lie Leibniz algebras are the cyclic Leibniz algebras. In recent years, methods have been developed for computing maximal subalgebras, Cartan subalgebras, minimal ideals, and Frattini subalgebras of cyclic algebras [McA14]. The cyclic Leibniz algebras have been classified over the complex numbers in [SS14]. These results are used throughout this work to construct non-Lie examples.

This work generalizes some results for Lie algebras that were originally motivated by group theory. Completions of maximal subgroups were defined by W.E. Deskins in [Des59], and further studied in [Des90], [BBE92] and [MB89]. Deskins defined the normal index of a maximal subgroup, and proved that a finite group is solvable if and only if all its maximal subgroups have prime power normal index. Ballester-Bolinches used completions and the normal index to characterize supersolvable groups.

The group theory concepts of c-supplemented subgroups, c-normal subgroups, and c-sections were defined and studied by Y. Wang and others in [BB00], [Wan] and [WS]. Wang proved in [Wan] that a finite group is solvable if and only if all its maximal subgroups are c-normal. In [WS], Wang and Shirong described the relationship between c-normal subgroups, c-sections and the normal index.

Since 2008, D. Towers has defined analogues of these group theory concepts for Lie algebras, and used them to characterize solvability and supersolvability. He defined c-ideals for Lie algebras in [Tow09], motivated by c-normal subgroups. He proved that a Lie algebra is solvable if and only if all its maximal subalgebras are c-ideals. In [Tow11] he defined the ideal index of a maximal subalgebra, and proved that a Lie algebra is solvable if and only if the ideal index of every maximal subalgebra equals its codimension. He defined c-sections of maximal subalgebras in [Tow15a].

CAP-subalgebras of Lie algebras have been studied in [HO70] and [Sti72]. In [Tow15b], Towers gave some properties of CAP-subalgebras in Lie algebras, and proved that a Lie algebra is solvable if and only if each of its maximal subalgebras is a CAP-subalgebra. He also proved that in a supersolvable Lie algebra, every subalgebra is a CAP-subalgebra.

The primary purpose of this work is to generalize, whenever possible, the results on solvable and supersolvable Lie algebras described above to the Leibniz case. Throughout, we also develop properties of the relevant subalgebras, and relationships between them. Whenever possible, we employ a technique of generalization of the Lie algebra proofs.

We give basic Leibniz algebra definitions in Chapter 2. We then introduce definitions of c-ideals, c-sections, completions, and CAP-subalgebras for the Leibniz case, illustrating with non-Lie Leibniz examples.

In Chapter 3 we develop properties of c-ideals, c-sections, completions, and CAP-subalgebras in Leibniz algebras. We characterize c-ideals and c-sections in simple Leibniz algebras, which differs from the Lie case. We also define the ideal index $\eta(A:M)$, and the c-section dimension $\eta^*(A:M)$ for maximal subalgebras, and give an equation relating these quantities. We give conditions under which every one-dimensional subalgebra is a c-ideal.

In Chapter 4 we give several characterizations of solvable Leibniz algebras. We prove that each of the following conditions is equivalent to solvability in a Leibniz algebra: (i) Every maximal subalgebra is a c-ideal, (ii) Every maximal subalgebra has ideal index equal to its codimension, (iii) Every maximal subalgebra has trivial c-section, (iv) Every maximal subalgebra has an abelian ideal completion, and (v) Every maximal subalgebra is a CAP-subalgebra. We prove that a Leibniz algebra is solvable if every Cartan subalgebra is a c-ideal, and we present a Lie counterexample to the converse.

In Chapter 5 we give characterizations of supersolvable Leibniz algebras. We show that a Leibniz algebra is supersolvable if and only if every maximal subalgebra satisfies $\eta(A:M)=1$. We present one result which we have not seen for Lie algebras, namely that Leibniz algebras are supersolvable if and only if all their subalgebras are CAP-subalgebras. We prove that one sufficient condition for supersolvability given by Towers is in fact a characterization of supersolvability in the non-Lie Leibniz case.

CHAPTER

2

PRELIMINARIES

We begin with basic definitions. Examples are included to highlight the fundamental differences between Lie algebras and Leibniz algebras. Unless otherwise noted, $\mathbb{F}$ is a field of arbitrary characteristic.

A **(left) Leibniz algebra** A is an $\mathbb{F}$-vector space with bilinear multiplication $[,]: A \times A \to A$ satisfying the (left) Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in A$.

For $a \in A$, the left multiplication operator $L_a : A \to A$ is defined by $L_a(b) = [a, b]$ for all $b \in A$. Similarly right multiplication $R_a : A \to A$ is defined by $R_a(b) = [b, a]$ for all $b \in A$. An operator $D : A \to A$ is a **derivation** if

$$D([b, c]) = [D(b), c] + [b, D(c)]$$

for all $b, c \in A$. So the left Leibniz identity arises from requiring that L_a be a derivation for all $a \in A$. The definition of derivation generalizes the product rule for derivatives, which is attributed to Gottfried Leibniz and thus inspired the name of Leibniz algebras.

If instead we require that each R_a be a derivation, we obtain a **(right) Leibniz algebra** which satisfies the (right) Leibniz identity

$$[[b, c], a] = [b, [c, a]] + [[b, a], c]$$

for all $a, b, c \in A$.

A Lie algebra satisfies both the left and right Leibniz identities. In fact, in a Lie algebra, each of these is equivalent to the Jacobi Identity. Leibniz algebras, however, need not be antisymmetric. Generally, left and right Leibniz algebras are different.

Example 2.0.1. Figure 2.1 illustrates ismorphism classes of all 2-dimensional Leibniz algebras. (1) and (3) are Lie (so both left and right Leibniz). Thus (2) and (4) are non-Lie Leibniz algebras. (2) is both a left and right Leibniz algebra, while (4) is left only.

Throughout this work, we will assume A is a finite-dimensional left Leibniz algebra. As with Lie algebras, we consider subalgebras, ideals, and normalizers. However, because

	x	y
x		y
y	-y	

(1)

	x	y
x	y	y
y		

(2)

	x	y
x		
y		

(3)

	x	y
x	y	
y		

(4)

Figure 2.1 2-dimensional Leibniz algebras

multiplication in a Leibniz algebra is not antisymmetric, we must modify some definitions to account for the difference between left and right multiplication.

For subsets $B, C \subseteq A$, we denote $[B, C] = \{[b, c] : b \in B, c \in C\}$. A subset $B \subseteq A$ is a **subalgebra** if $[B, B] \subseteq B$. A subalgebra $I \subseteq A$ is a **left ideal** of A if $[A, I] \subseteq I$, and a **right ideal** of A if $[I, A] \subseteq I$. An **ideal** of A is a subalgebra which is both a left and right ideal. A left ideal which is not a right ideal appears in Example 2.0.2. Basic properties of ideals in Lie algebras hold here as well; for example, the intersection of two ideals is an ideal. A similar modification is used to define the normalizer of a subalgebra.

For subsets $C, B \subseteq A$, the **left normalizer of** C **in** B is the set

$$N_B^l(C) = \{x \in B | [x, c] \in C \text{ for all } c \in C\},$$

while the **right normalizer of** C **in** B is the set

$$N_B^r(C) = \{x \in B | [c, x] \in C \text{ for all } c \in C\}.$$

The **normalizer of** C **in** B is the intersection $N_B^l(C) \cap N_B^r(C) = N_B(C)$. When $B = A$, we may omit the subscript and refer to $N^l(C), N^r(C)$, and $N(C)$. Example 2.0.2 illustrates that in a Leibniz algebra, the left and right normalizers of a subalgebra need not coincide.

Example 2.0.2. Let A be the 3-dimensional Leibniz algebra over $\mathbb{C}$ with nonzero multiplications given by $[x,z]=y, [y,z]=\alpha x+y, \alpha\in\mathbb{C}$. Let $C=\text{span}\{x\}$. Then the left normalizer of C, $N^l(C)=A$, while the right normalizer of C, $N^r(C)=\text{span}\{x,y\}$. Then the normalizer $N(C)=N^r(C)=\text{span}\{x,y\}$, while $N(C)\neq N^l(C)$. The subalgebra C is a left ideal since $[A,C]=0\subset C$, but not a right ideal since $z\in[C,A]$ but $z\notin C$.

There might exist elements $a\in A$ such that $[a,a]\neq 0$. Such elements can only exist in a non-Lie Leibniz algebra. We define a subspace

$$Leib(A)=\{[a,a]|a\in A\}.$$

$Leib(A)$ is an ideal of A, as shown in [Dem13]. In fact, $Leib(A)$ is the kernel of the natural homomorphism $A\to A'$, where $A'\cong A/Leib(A)$ is a Lie algebra and has been called the liezation of A [Gor13]. Hence, A is Lie if and only if $Leib(A)=0$. $Leib(A)$ is sometimes referred to as the **Leibniz ideal** of A.

Any non-Lie Leibniz algebra will contain a nontrivial, proper ideal, namely $Leib(A)$, so we modify our definition of simplicity from the Lie algebra case. A Leibniz algebra is **simple** if $[A,A]\neq Leib(A)$ and $Leib(A)$ is the only non-trivial, proper ideal of A. When A is Lie, $Leib(A)=0$, and ours coincides with the Lie definition.

Example 2.0.3. Consider A from Example 2.0.2. Then $a\in Leib(A)$ if and only if $a=[c_1x+c_2y+c_3z, c_1x+c_2y+c_3z]=c_2c_3\alpha x+(c_1+c_2c_3)y$ for some $c_1,c_2,c_3\in\mathbb{C}$. So $Leib(A)=\text{span}\{x,y\}=[A,A]$. So A is not simple, though $Leib(A)$ is the only proper ideal of A.

For any $n\geq 2$, we may construct an n-dimensional non-Lie Leibniz algebra. An n-dimensional **cyclic Leibniz algebra** is generated by the elements $\{a,a^2,\cdots,a^n\}$, with $[a,a^i]=$

a^{i+1} for $1 \le i \le n-1$. Then $[a, a^n] = \alpha_1 a + \cdots + \alpha_n a^n$, and in fact $\alpha_1 = 0$ [Dem13]. Furthermore $[a^i, x] = 0$ for all $x \in A$ and $i > 1$. Cyclic Leibniz algebras over $\mathbb{C}$ have been classified in [SS14]. Both (2) and (4) in Example 2.0.1 are cyclic with $a = x$. Three-dimensional Leibniz algebras over $\mathbb{C}$ (not necessarily cyclic) have been classified in [RR12]. All cyclic Leibniz algebras given as examples in this work are assumed to be defined over $\mathbb{C}$.

A subalgebra $M \le A$ is **maximal** if $M \le N \le A$ implies $M = N$ or $M = A$. If a Lie algebra L has exactly one maximal subalgebra, then $\dim(L) = 1$. However, the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = 0$ has exactly one maximal subalgebra, namely $[A, A]$. A method for computing all of the maximal subalgebras and minimal ideals of a cyclic Leibniz algebra is given in [McA14].

Our definitions of abelian, nilpotent, and solvable Leibniz algebras follow exactly from the Lie algebra definitions. A is **abelian** if $[A, A] = 0$. The **lower central series** $A = A^0 \supseteq A^1 \supseteq A^2 \supseteq \cdots$ is defined recursively by $A^{i+1} = [A, A^i]$. A is **nilpotent** if $A^n = 0$ for some n. A nilpotent algebra A is **nilpotent class** c if c is the smallest natural number such that $A^c = 0$. The **derived series** $A = A^{(0)} \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots$ is defined recursively by $A^{(i+1)} = [A^{(i)}, A^{(i)}]$. A is **solvable** if $A^{(n)} = 0$ for some n.

Similarly we may consider abelian, solvable and nilpotent subalgebras of A. $Leib(A)$ is an abelian ideal of A, since $L_{a^2} \equiv 0$ on A as a consequence of the Leibniz identity.

Because $A^{(i)} \subseteq A^{i+1}$, every nilpotent Leibniz algebra is solvable. The converse is generally false, as in Example 2.0.4. From Example 2.0.1, the 2-dimensional Lie algebra with $[x, y] = x$ is solvable since $L^{(3)} = 0$, but not nilpotent, since $L^n = \text{span}\{x\}$ for $n > 1$.

Example 2.0.4. Let A be the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = a^2$. Then $[A, A] = A^{(2)} = A^2$, $A^{(3)} = 0$, and $A^n = A^2 \neq 0$ for all $n \ge 2$. So A is solvable and not nilpotent.

$M_1 = \text{span}\{a^2 + a^3, a - a^3\}$ is a maximal subalgebra of A, which is solvable since $M_1^{(3)} \subseteq A^{(3)} = 0$, but not nilpotent since $M_1^n = \text{span}\{a^2 + a^3\} \neq 0$ for all $n \geq 2$. $M_2 = A^2$ is another maximal subalgebra, which is abelian, thus nilpotent.

Every cyclic Leibniz algebra is solvable, as $A^{(3)} = 0$. In fact, every non-Lie Leibniz algebra (not necessarily cyclic) with dimension ≤ 4 is solvable [Dem13]. Over $\mathbb{C}$, there is exactly one nilpotent cyclic Leibniz algebra, up to isomorphism, of each dimension $n \geq 1$. These are the algebras in which $[a, a^n] = 0$. No cyclic Leibniz algebra is abelian, as $0 \neq [a, a] \in A^{(2)}$.

The largest solvable ideal of A is called the **radical** of A and denoted $Rad(A)$. Because abelian ideals are solvable, $Leib(A)$ is a solvable ideal of A, so $Rad(A) \neq 0$ in a non-Lie Leibniz algebra. A Leibniz algebra is **semisimple** if $Rad(A) = Leib(A)$. A semisimple Leibniz algebra is Lie, by [Bat13], Corollary 2.7.

The largest nilpotent ideal of A is called the **nilradical** of A and denoted $Nil(A)$. Because $Leib(A) \subseteq Nil(A)$, $Nil(A) \neq 0$ in a non-Lie Leibniz algebra.

A subalgebra $B \leq A$ is **nil** if for all $b \in B$ and $a \in A$, there exists $n \in \mathbb{N}$ such that the left multiplication operator $L_b^n(a) = 0$. Every nil subalgebra of a finite-dimensional Leibniz algebra is nilpotent by Engel's Theorem. A Leibniz algebra with a nilpotent subalgebra which is not nil appears in Example 2.0.5.

Example 2.0.5. Consider the Lie algebra $S = sl(2, \mathbb{C})$ with relations $[e, f] = h, [e, h] = -2e$, and $[f, h] = 2f$, and its irreducible module $V = \text{span}\{v_0, v_1, \cdots, v_m\}$. Define $A = S \dotplus V$, where

$[V,S]=0$ and $[S,V]$ given by the module action of S on V:

$$\begin{aligned} e(v_i) &= (m-i+1)v_{i-1} \\ f(v_i) &= (i+1)v_{i+1} \\ h(v_i) &= (m-2i)v_i \end{aligned}$$

where $v_{-1}=v_{m+1}=0$. Then A is a non-Lie Leibniz algebra, with $Leib(A)=V$. The subalgebra $N=\text{span}\{h\}\dotplus V$ is nilpotent, but not nil since $L_h^n(e)\neq 0$ for any n.

Remark 2.0.6. If in Example 2.0.5 we define $[s,v]=-[v,s]$ for all $s\in S$, $v\in V$, we obtain a non-solvable Lie algebra with $Rad(A)=V$.

A subalgebra $B\leq A$ is a **Cartan subalgebra** if B is nilpotent and $N(B)=B$. If B is a Cartan subalgebra of A, then in fact $B=N(B)=N^r(B)$ [Gor13]. Example 2.0.7 gives a Cartan subalgebra not equal to its left normalizer, which can exist only in a non-Lie Leibniz algebra.

Cartan subalgebras are maximal nilpotent subalgebras. As for Lie algebras, solvable Leibniz algebras contain Cartan subalgebras. More results on the Cartan subalgebras of Leibniz algebras are given in [Bar11] and [Omi06].

Example 2.0.7. Let A be the 3-dimensional cyclic Leibniz algebra with $[a,a^3]=a^2$. Then $C=\text{span}\{a-a^3\}$ is a Cartan subalgebra of A. While $N^r(C)=N(C)=C$, $a\in N^l(C)$ so $N^l(C)\neq C$. Here, C is a left ideal, but not a right ideal, of A.

A is **supersolvable** if there is a chain $A=A_n\supseteq A_{n-1}\supseteq\cdots\supseteq A_2\supseteq A_1\supseteq A_0=0$ such that each A_i is an i-dimensional ideal of A, $1\leq i\leq n-1$. Every supersolvable Leibniz algebra is solvable. Over algebraically closed fields of characteristic zero, solvable Leibniz algebras are supersolvable as a consequence of Lie's Theorem. Therefore, every cyclic Leibniz algebra over $\mathbb{C}$ is supersolvable.

A solvable Leibniz algebra is supersolvable if and only if $\dim(A/M) = 1$ for all maximal subalgebras $M \leq A$, given by Barnes in [Bar11] as a generalization of his earlier result for Lie algebras in [Bar67]. Example 2.0.8 gives a Lie algebra which is solvable but not supersolvable.

Example 2.0.8. Let $V = \text{span}\{v_1, \cdots, v_p\}$ be a vector space over a field $\mathbb{F}$ of characteristic $p > 0$. Define operators $x, y, z : V \to V$ by:

$$\begin{aligned} x(v_j) &= v_{j+1} \\ y(v_j) &= (j+1)v_{j-1} \\ z(v_j) &= v_j \end{aligned}$$

where subscripts are taken mod p. Let $H = \text{span}\{x, y, z\}$ be the Lie algebra with the commutator bracket $[h_1, h_2] = h_1 h_2 - h_2 h_1$ for all $h_1, h_2 \in H$. Then $[y, x] = z$ is the only nontrivial product. H is a solvable Lie algebra.

Now let $L = H + V$ be a Lie algebra with $[h, v] = h(v)$ for all $h \in H, v \in V$. Then L is a solvable Lie algebra. However, L is not supersolvable since H is a maximal subalgebra with $\dim(L/H) = \dim(V) > 1$.

If C and D are ideals of A such that D is maximal among ideals properly contained in C, then the quotient C/D is a **chief factor** of A. If A is simple, the chief factors are $A/Leib(A)$ and $Leib(A)/0$. If A is supersolvable, $\dim(C/D) = 1$ for all chief factors C/D.

Example 2.0.9. The non-supersolvable algebra L from Example 2.0.8 has a chief factor C/D with dimension 1 given by $C = L$, $D = \text{span}\{x, z\} + V$. Another chief factor, $V/\{0\}$ has dimension $p > 1$.

The **Frattini subalgebra** of A, denoted $F(A)$, is the intersection of the maximal subalgebras of A. The **Frattini ideal** of A, denoted $\phi(A)$, is the largest ideal of A contained in $\phi(A)$. Thus if $F(A)$ is an ideal then $F(A) = \phi(A)$. The Frattini subalgebra of a cyclic Leibniz algebra may be computed explicitly as in [McA14]. The Frattini ideal of a Leibniz algebra has also been studied in [Bat13]. There, it is shown that $F(A) = \phi(A)$ when char($\mathbb{F}$) $= 0$. A counterexample in characteristic $p > 0$ is given in Example 2.0.12.

A is **elementary** if $\phi(B) = 0$ for every subalgebra B of A. A is an **E-algebra** if $\phi(B) \leq \phi(A)$ for all $B \leq A$. Elementary Leibniz algebras are E-algebras, but not conversely, as illustrated in Example 2.0.10.

Example 2.0.10. Let A be the 4-dimensional cyclic Leibniz algebra with $[a, a^4] = a^3$. Then the Frattini subalgebra $F(A) = \text{span}\{a^2 - a^4\}$. Here $\mathbb{F} = \mathbb{C}$ so we have $F(A) = \phi(A)$.Because A is solvable, $\mathbb{F}$ is perfect and $\phi(A) \neq 0$, A is not elementary by [Bat13], Corollary 3.9. Because A^2 is nilpotent, A is an E-algebra by [Bat13], Theorem 3.5.

Definition 2.0.11. For $B \leq A$, the **core of** B **in** A, denoted B_A, is the largest ideal of A contained in B. The **strict core of** B **in** A, denoted $k(B)$, is the sum of the ideals of A properly contained in B.

The core and strict core of B satisfy $0 \subseteq k(B) \subseteq B_A \subseteq B$. Any of these set inclusions may be strict. Thus $\phi(A) = F(A)_A$.

Example 2.0.12. Let L be the Lie algebra with char($\mathbb{F}$) $= 2$ and $[x, y] = z, [x, z] = -y, [y, z] = x$. Then $M = \text{span}\{x + y, z\}$ is a maximal subalgebra. The Frattini subalgebra is $F(A) = \text{span}\{x + y + z\}$ and the Frattini ideal is $\phi(A) = 0$. So $0 = k(M)$ and $0 = \phi(A) = M_A$.

Definition 2.0.13. A subalgebra B of A is **c-supplemented** if there is a subalgebra C of A with $A = B + C$ and $B \cap C \leq B_A$. If every subalgebra of A is c-supplemented, then A is a **c-supplemented Leibniz algebra**.

Definition 2.0.14. A is **completely factorizable** if for every subalgebra B, there is a subalgebra C such that $A = B + C$ and $B \cap C = 0$.

Example 2.0.15. The 3-dimensional cyclic Leibniz algebra with $[a, a^3] = a^3$ is c-supplemented but not completely factorizable, as there is no disjoint supplement to the subalgebra A^2.

C-ideals are generalizations of ideals which are related to c-supplements.

Definition 2.0.16. A subalgebra $B \leq A$ is a **c-ideal** if there is an ideal C of A such that

(i) $A = B + C$, and

(ii) $B \cap C \leq B_A$.

If B is a c-ideal, then B is c-supplemented. We will sometimes refer to the ideal C from Definition 2.0.16 as the **c-supplement** of B. Every ideal B of A is a c-ideal, because $A = B + A$ and $A \cap B = B \leq B_A$. However, c-ideals are not ideals in general, as demonstrated in Example 2.0.17.

Example 2.0.17. Let A be the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = a^3$. Then $M = \text{span}\{a - a^2, a - a^3\}$ is not an ideal because $[a - a^3, a] = a^2 \notin M$. However, M is a c-ideal because $A = M + A^2$ and $M \cap A^2 = \text{span}\{a^2 - a^3\} \leq M_A$.

Definition 2.0.18. If M is a maximal subalgebra of A, and C/D is a chief factor of A such that $A = M + C$ and $D \subseteq M$, then $(M \cap C)/D$ is a **c-section** of M.

In Chapter 3 we prove the existence, and uniqueness up to isomorphism, of c-sections for each maximal subalgebra.

Example 2.0.19. Let A and M be as in Example 2.0.17. Then $A^2/\text{span}\{a^2 - a^3\}$ is a chief factor satisfying $\text{span}\{a^2 - a^3\} \subseteq M$ and $A = M + A^2$. So $(M \cap A^2)/\text{span}\{a^2 - a^3\}$ is a c-section of M. Here $(M \cap A^2)/\text{span}\{a^2 - a^3\} \cong 0$, which we will show in Chapter 4 is the case for all c-sections in a solvable Leibniz algebra.

Definition 2.0.20. A subalgebra C of A is a **completion** for the maximal subalgebra M if C is not contained in M, but every proper subalgebra of C that is an ideal of A is contained in M. If C is also an ideal of A, then C is called an **ideal completion** for M. If C is an ideal completion for M such that $C/k(C)$ is abelian, C is an **abelian ideal completion**.

Using the above definition, C is an ideal completion for M if and only if $A = M + C$ and $k(C) \subseteq M$. We define the **index complex** of M as the set $I(M)$ of all completions of M. A **maximal completion** for M is a maximal element of $I(M)$, ordered by inclusion.

Example 2.0.21. Let A and M again be as in Example 2.0.17. Then $C = \text{span}\{a^3\}$ is an ideal of A not belonging to M, and $0 = k(C) \subseteq M$. So C is an ideal completion for M. C is in fact an abelian ideal completion since $C/k(C) \cong C$ is abelian.

Definition 2.0.22. Let C/D be a chief factor of A. A subalgebra $B \leq A$ **avoids** C/D if $B \cap C = B \cap D$, and **covers** C/D if $B + C = B + D$. Then B has the **covering and avoidance property** (equivalently B is a **CAP-subalgebra** of A) if B either covers or avoids every chief factor of A.

If M is a maximal subalgebra with ideal completion C, then $C/k(C)$ is a chief factor avoided by M.

Example 2.0.23. Let A be the 2-dimensional cyclic Leibniz algebra with $[a,a^2]=a^2$. Then the maximal subalgebras of A are $M_1 = \text{span}\{a^2\} = A^2$ and $M_2 = \text{span}\{a-a^2\}$. Because $\dim(A)=2$ these are in fact all of the nonzero, proper subalgebras of A. The chief factors of A are $F_1 = A/A^2$ and $F_2 = A^2/\{0\}$. Then:

- $M_1 \cap A = A^2 = M_1 \cap A^2$, so M_1 avoids F_1,
- $M_1 + A^2 = A^2 = M_1 + \{0\}$, so M_1 covers F_2,
- $M_2 + A^2 = A = M_2 + A$, so M_2 covers F_1, and
- $M_2 \cap A^2 = \{0\} = M_2 \cap \{0\}$, so M_2 avoids F_2.

Both M_1 and M_2 are CAP-subalgebras. M_1 is an ideal completion for M_2. Then $k(M_1)=0$, so $M_1/k(M_1) \cong F_2$ is a chief factor avoided by M_2. M_2 is a completion for M_1, but not an ideal completion. However since $k(A)=A^2=M_1$ and A/A^2 is abelian, A is an abelian ideal completion for M_1.

CHAPTER

3

SUBALGEBRA PROPERTIES

We now develop properties of c-ideals, completions for maximal subalgebras, c-sections and CAP-subalgebras of Leibniz algebras. We give a relationship between c-supplemented algebras and completely factorizable algebras. We give conditions under which every one-dimensional subalgebra is a c-ideal. We define the ideal index and c-section dimension of maximal subalgebras. We characterize c-sections and c-ideals in simple Leibniz algebras. We prove that the maximal subalgebras with trivial c-section are exactly those which are c-ideals.

The core B_A and strict core $k(B)$ of a subalgebra B are important to the definitions of c-ideal and completion. To further develop properties of these subalgebras, we need the following results about the behavior of B_A and $k(A)$.

Proposition 3.0.1. *Suppose $B \neq 0$ is a subalgebra of A.*

1. *B is an ideal if and only if $B_A = B$.*

2. *B is a minimal ideal if and only if $B_A = B$ and $k(B) = 0$.*

3. *B is the sum of two or more distinct, nontrivial ideals if and only if $B_A = k(B) = B$.*

4. *If B is a minimal subalgebra which is not an ideal, then $B_A = k(B) = 0$.*

Proof. Let $B \leq A$.

1. By definition $B_A \subseteq B$. $B \subseteq B_A$ if and only if B is an ideal of A.

2. If B is a minimal ideal then $B = B_A$ by (1) and $k(B) = 0$ by definition, as there are no ideals of A properly contained in B. Then $B = B_A$ implies B is an ideal by (1) and $k(B) = 0$ implies B is minimal.

3. Let $I_1, \cdots, I_n$ be ideals satisfying $B = I_1 + \cdots + I_n$, $n \geq 2$. Then B is an ideal so $B = B_A$ by (1), and $B = I_1 + I_2 \subseteq k(B) \subseteq B$ implies $B = k(B)$. Conversely, if $k(B) = B$ then B can be expressed as the sum of ideals of A properly contained in B by definition, so $B = I_1 + \cdots + I_n$ for some ideals I_i of A and $n \geq 2$.

4. If B is a minimal subalgebra then B properly contains no ideals of A, so $k(B) = B_A = 0$.

□

Example 3.0.2. Let A be the 3-dimensional cyclic Leibniz algebra with $[a,a^3]=a^2-2ia^3$. Consider the (maximal) subalgebra $B = \text{span}\{ia + a^2, a - a^3\}$. Then B is not an ideal. However $I = \text{span}\{ia^2 + a^3\}$ is an ideal of A properly contained in B, in fact the only one. So $k(B) = I = B_A$. In this case $0 \neq k(B) = B_A \neq B$.

Now consider the behavior of the strict core of a subalgebra with respect to quotients.

Lemma 3.0.3. *If U is a subalgebra of A and B is an ideal of A with $B \leq U$, then $k(U/B) \subseteq k(U)/B$.*

Proof. Suppose I is a proper ideal of U/B, so $I \subseteq k(U/B)$. Then $I = I'/B$ where I' is a proper ideal of U, so $I \subseteq k(U)/B$. □

Example 3.0.4 gives a counterexample to the converse of Lemma 3.0.3.

Example 3.0.4. Let A be the 4-dimensional cyclic Leibniz algebra with $[a,a^4] = a^3$. The minimal ideals of A are $I_1 = \text{span}\{a^2-a^4\}$, $I_2 = \text{span}\{a^3+a^4\}$ and $I_3 = \{a^3-a^4\}$. Let $U = I_1+I_2$. Then $k(U) = U$, so $k(U)/I_1 = U/I_1 \cong I_2$, while $k(U/I_1) \cong k(I_2) \cong 0$. So $k(U)/I_1 \nsubseteq k(U/I_1)$.

Lemma 3.0.5. *If B is a subalgebra of A and I an ideal of A with $I \leq B$, then $B_A/I = (B/I)_{A/I}$.*

3.1 C-supplemented Subalgebras

Recall Definition 2.0.13 of a c-supplemented subalgebra. This section is motivated by [Tow08]. Theorem 3.1.6 for Leibniz algebras is analagous to [Tow08], Proposition 2.4 for Lie algebras.

Lemma 3.1.1. *If A is completely factorizable then A is ϕ-free.*

Proof. We will prove the contrapositive. Suppose $\phi(A) \neq 0$. If A is completely factorizable, then there is some subalgebra $B \neq 0$ such that $A = \phi(A) + B$ and $\phi(A) \cap B = 0$. Because $\phi(A)$ does not belong to B, B is not maximal, so let M be a maximal subalgebra containing B. Then $\phi(A) \subseteq M$ and $B \subseteq M$ implies that $M = A$, a contradiction. So there is no such B, and therefore A is not completely factorizable. □

If A is completely factorizable then it is c-supplemented. The converse is generally false. The following example gives a non-Lie Leibniz algebra which is c-supplemented but not completely factorizable.

Example 3.1.2. Let A be the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = 0$. Then A is c-supplemted. But A is not completely factorizable because $A^2 = \text{span}\{a^2, a^3\}$ is not supplemented by a disjoint subalgebra.

Our proof of Lemma 3.1.3 refers to [Tow08].

Lemma 3.1.3. *The class of c-supplemented Leibniz algebras is closed under subalgebras and factor algebras.*

Proof. Let A be a Leibniz algebra.

1. Suppose B is c-supplemented in A and $B \leq K \leq A$. We show that B is c-supplemented in K.

   There is a subalgebra $C \leq A$ with $A = B + C$ and $B \cap C \leq B_A$. Then $(C \cap K) \leq K$, $K = A \cap K = (B + C) \cap K = B + (C \cap K)$, so $(C \cap K)$ is a supplement to B in K. Furthermore $(B \cap C \cap K) \leq (B \cap B) \leq B_A$. So B is c-supplemented in K, and K is a c-supplemented algebra. The class of c-supplemented Leibniz algebras is closed under subalgebras.

2. Suppose B is c-supplemented in A, so there exists a subalgebra C with $A = B + C$ and $(B \cap C) \leq B_A$. Then $A/I = B/I + (C+I)/I$ and $(B/I) \cap (C+I)/I = (B \cap (C+I))/I = (I + B \cap C)/I \leq B_A/I = (B/I)_{A/I}$, so B/I is c-supplemented in A/I and A/I is a c-supplemented algebra.

   Now suppose B/I is c-supplemented in A/I. So there is a subalgebra C/I of A/I such that $A/I = B/I + C/I$ and $(B/I) \cap (C/I) \leq (B/I)_{A/I} = B_A/I$. It follows that $A = B + C$ and $B \cap C \leq B_A$, so B is c-supplemented in A and A is a c-supplemented algebra. Thus the class of c-supplemented Leibniz algebras is closed under factor algebras.

□

Proposition 3.1.4. *Let B, D be subalgebras of A with $B \leq \phi(D)$. If B is c-supplemented in A, then B is an ideal of A and $B \leq \phi(A)$.*

Proof. Since B is c-supplemented, there is a subalgebra C satisfying $A = B + C$ and $B \cap C \leq B_A$. Then $D = D \cap A = D \cap (B + C) = B + D \cap C = D \cap C$ since $B \leq \phi(D)$. Then $B \leq D \leq C$, so $B = B \cap C \leq B_A$ and B is an ideal of A. Then $B \leq \phi(A)$ by [Tow73], Lemma 4.1. □

Corollary 3.1.5. *If A is c-supplemented then A is an E-algebra.*

Proof. Fix $D \leq A$. Then $\phi(D), D$ satisfy the conditions of 3.1.4. So $\phi(D) \leq \phi(A)$ and A is an E-algebra. □

We now present the main result of this section, which relates c-supplemented and completely factorizable Leibniz algebras.

Theorem 3.1.6. *Let A be a Leibniz algebra. The following are equivalent:*

1. *A is c-supplemented,*

2. *$A/\phi(A)$ is completely factorizable and every subalgebra of $\phi(A)$ is an ideal of A.*

Proof. ($1 \Rightarrow 2$) Suppose A is c-supplemented. Then every subalgebra of $\phi(A)$ is an ideal of A, as we see by letting $D = A$ in Proposition 3.1.4. It remains to show that $A/\phi(A)$ is completely factorizable.

Suppose first that $\phi(A) = 0$. Then we need to prove that A is completely factorizable. A is elementary, by Proposition 3.1.5. Let B be a subalgebra of A, and let D be a minimal subalgebra of A satisfying $A = B + D$. Then $B \cap D \leq \phi(D)$ [Bat13], but since A is elementary this becomes $B \cap D = 0$. So A is completely factorizable.

If A is not ϕ-free, then since A is c-supplemented, $A/\phi(A)$ is c-supplemented, by Lemma. Now $A/\phi(A)$ is both c-supplemented and ϕ-free, by the preceding argument, implies that $A/\phi(A)$ is completely factorizable.

($2 \Rightarrow 1$) Let B be a subalgebra of A. Then $(B + \phi(A))/\phi(A)$ is a subalgebra of $A/\phi(A)$, which is completely factorizable, so there exists a subalgebra $C/\phi(A) \leq A/\phi(A)$ such that $A/\phi(A) = (B+\phi(A))/\phi(A) + C/\phi(A)$ and $0 = (B+\phi(A))/\phi(A) \cap C/\phi(A) = (B+C+\phi(A))/\phi(A)$. Thus $A = B + C$ and $B \cap C \leq \phi(A)$, so $B \cap C$ is an ideal of A, so $B \cap C \leq B_A$. So A is c-supplemented. □

Example 3.1.7. The 3-dimensional cyclic Leibniz algebra with $[a, a^3] = 0$ is c-supplemented but not completely factorizable (Example 3.1.2). However $\phi(A) = A^{(2)}$, so $A/\phi(A)$ is one-dimensional and thus completely factorizable.

3.2 C-ideals

Recall from Definition 2.0.16 that the c-ideals of A are the c-supplemented subalgebras whose c-supplements are ideals. This section is motivated by [Tow09].

Lemma 3.2.1. *Let $B \leq A$.*

1. *If B is a c-ideal of A and $B \leq K \leq A$, then B is a c-ideal of K.*
2. *If I is an ideal of A and $I \leq B$, then B is a c-ideal of A if and only if B/I is a c-ideal of A/I.*

Proof. Let $B \leq A$.

1. Suppose that B is a c-ideal of A and $B \leq K \leq A$. Then there is a ideal C of A with $A = B + C$ and $B \cap C \leq B_A$. Now consider $C \cap K$, which is an ideal of K, and we will show is a c-supplement for B in K.

   First we show that $K = B + (C \cap K)$. We have $K = A \cap K = (B + C) \cap K$. Then $(B + C) \cap K = (B \cap K) + (C \cap K) = B + (C \cap K)$. It remains to show that $B \cap (C \cap K) \leq B_K$. That is $B \cap (C \cap K) \leq B_A \cap K \leq B_K$. So B is a C-ideal of K.

2. First suppose that B/I is a c-ideal of A/I. Then there is an ideal C/I of A/I such that $A/I = B/I + C/I$ and $(B/I) \cap (C/I) \leq (B/I)_{A/I} = B_A/I$. It follows that $A = B + C$ and $B \cap C \leq B_A$.

   Suppose conversely that I is an ideal of A with $I \leq B$ such that B is a c-ideal of A. Then there is an ideal C of A such that $A = B + C$ and $B \cap C \leq B_A$. Now $A/I = B/I + (C + I)/I$, where $(C+I)/I$ is an ideal of A/I and and $(B/I)\cap(C+I)/I=(B\cap(C+I))/I=(I+B\cap C)/I\le B_A/I=(B/I)_{A/I}$, so B/I is a c-ideal of A/I.

□

We often desire to relate the c-ideals of A to the c-ideals of $A/Leib(A)$. This is given in Corollary 3.2.3.

Lemma 3.2.2. *Let B be a subalgebra and I an ideal of A. If B is a c-ideal then $B+I$ is a c-ideal.*

Proof. First suppose B is a c-ideal. Then there is an ideal C such that $A=B+C$ and $B\cap C\subseteq B_A$. Then $A=(B+I)+C$. Then $(B+I)\cap C\subseteq(B\cap C)+(I\cap C)\subseteq B_A+I\subseteq(B+I)_A$. So B+I is a c-ideal. □

Corollary 3.2.3. *If B is a c-ideal of A, then $(B+Leib(A))/Leib(A)$ is a c-ideal of $A/Leib(A)$.*

Proposition 3.2.4 characterizes the c-ideals of a simple Leibniz algebra.

Proposition 3.2.4. *A is simple if and only if the only c-ideals of A are $Leib(A)$ and B such that $A=B\oplus Leib(A)$.*

Proof. Let A be a Leibniz algebra.

($\Rightarrow$) Suppose A is simple. Let B be a c-ideal of A. If $B_A=0$, then $Leib(A)$ must be the c-supplement to B implying that $A=Leib(A)\oplus B$. Otherwise $B_A=Leib(A)$ implies $Leib(A)\subseteq B$ so A is the c-supplement to B, implying that $A\cap B=B\subseteq Leib(A)$ so $B=Leib(A)$.

($\Leftarrow$) We prove the contrapositive. Suppose B is a c-ideal of A such that $A\neq B\oplus Leib(A)$ and $B\neq Leib(A)$. Then there is an ideal C such that $A=B+C$ and $B\cap C\subseteq B_A$. If $C=A$

then $B \neq Leib(A)$ is an ideal of A, so A is not simple. If $C = Leib(A)$ then $B \cap C \neq 0$ implies that $B_A \neq 0$ is an ideal of A not equal to $Leib(A)$, so A is not simple. □

Remark 3.2.5. If A is Lie, Proposition 3.2.4 becomes A is simple if and only if A has no proper c-ideals.

Example 3.2.6. Consider the simple Leibniz algebra $A = S \dotplus V$ from Example 2.0.5. Then the only ideal of A is $Leib(A) = V$. The only c-ideals of A then are S and V. There is no maximal subalgebra which is a c-ideal. For example, $M = \text{span}\{e, h\} \dotplus V$ is a maximal subalgebra of A, but $M_A = V$ and the only ideal supplement to M in A is A itself. Then because $M \cap A = M \nsubseteq M_A = V$, M is not a c-ideal.

3.2.1 One-dimensional subalgebras

In a Lie algebra L, every element $x \in L$ generates the one-dimensional subalgebra span$\{x\}$, because $[x, x] = 0 \in \text{span}\{x\}$ for all $x \in L$. In a Leibniz algebra, this need not be the case. In fact, in a cyclic Leibniz algebra, span$\{a\}$ is never a subalgebra since $a^2 \neq 0$. We can describe the one-dimensional subalgebras of a Leibniz algebra according to whether they are c-ideals. The results in this section are motivated by [Tow09].

Theorem 3.2.7. *Let span$\{x\}$ be a one-dimensional subalgebra of A. Then span$\{x\}$ is a c-ideal if and only if:*

1. *span$\{x\}$ is an ideal of A, or*

2. $x \notin A^2$.

Proof. First note that span$\{x\}$ is a subalgebra if and only if $[x,x]=0$. Suppose span$\{x\}$ is a c-ideal of A. Then there is an ideal K such that $A = K + \text{span}\{x\}$ and $K \cap \text{span}\{x\} \leq (\text{span}\{x\})_A$. But span$\{x\}$ has dimension one so $K \cap \text{span}\{x\} = \text{span}\{x\}$ or $K \cap \text{span}\{x\} = 0$. In the first case, $\text{span}\{x\} = (\text{span}\{x\})_A$ is an ideal of A. In the second case, $x \notin K$ and $K \leq A^2$ implies that $x \notin A^2$.

For the converse, because every ideal is a c-ideal, we need only show that $x \notin A^2$ implies that span$\{x\}$ is a c-ideal. So suppose $x \notin A^2$. Then there is a subspace $K \leq A$ of codimension 1 in A such that $A^2 \leq K$ and $x \notin K$. Then $A = \text{span}\{x\} \oplus K$ and K is an ideal of A, so span$\{x\}$ is a c-ideal of L. □

Theorem 3.2.7 is true for Lie algebras ([Tow09], Proposition 5.1), and our proof is the same, despite the fact that a non-Lie Leibniz algebra may contain many elements which do not belong to a one-dimensional subalgebra.

Example 3.2.8. Let A be the two-dimensional cyclic Leibniz algebra with $[a, a^2] = a^2$. Then span$\{a - a^2\}$ is a one-dimensional subalgebra which is a c-ideal, but not an ideal, of A. Note this agrees with Theorem 3.2.7 because $a - a^2 \notin A^2 = \text{span}\{a^2\}$.

We now characterize Leibniz algebras in which all one-dimensional subalgebras are c-ideals. A subalgebra C of A is **almost abelian** if $C = \text{span}\{x\} \oplus D$ where D is abelian and $[x,d] \subseteq D$ for all $d \in D$.

Theorem 3.2.9. *Let A be a Leibniz algebra over any field F. Then all one-dimensional subalgebras of A are c-ideals of A if and only if:*

1. $A^3 = 0$*; or*

2. *$A = B \oplus C$, where B is an abelian ideal of A and C is an almost abelian ideal of A.*

Proof. Suppose $A^3 = 0$. Let Fx be a subalgebra of A so $[x, x] = 0$. If $x \notin A^2$ then Fx is a c-ideal of A by Theorem 3.2.7. Otherwise $x \in A^2$ so Fx is an ideal, thus a c-ideal.

Suppose now that A is as in (2). Let $A = Z \oplus B \oplus Fx$ and choose $z + b + \alpha x \in A$. We will assume that $S = \text{span}\{z + b + \alpha x\}$ is a one-dimensional subalgebra, and will verify that S is a c-ideal of A. If $z \neq 0$, then let $Z = Z_1 \oplus Fz$. Then $M = Z_1 \oplus B \oplus Fx$ is an ideal of A with $M \cap S = \text{span}\{a + \alpha x\}$ contained in the core of M and $A = S + M$. So suppose $z = 0$. If $\alpha = 0$, then Fb is an ideal of A so $F(b + \alpha x) = Fb$ is a c-ideal. If $\alpha \neq 0$, then choosing $M = Z \oplus B$ shows that $F(b + \alpha x)$ is a c-ideal of A.

Now suppose that every one-dimensional subalgebra of A is a c-ideal and that $A^3 \neq 0$. Write $B = A/Leib(A)$. We know that $A \cong Leib(A) \oplus B$, where B is Lie. If $B^3 = 0$ then $A^3 = 0$, a contradiction. So by [Tow15a], since every one-dimensional subalgebra of B is a c-ideal, $B = C \oplus D$ where C is abelian and D is almost abelian, so $A = (Leib(A) \oplus C) \oplus D$ and the proof is complete. □

3.3 C-sections

Recall Definition 2.0.18 of a c-section of a maximal subalgebra. For every maximal subalgebra M, a c-section exists, and all c-sections of M are isomorphic. A c-section of M is isomorphic to a c-section of any quotient M/B. The results in this section are motivated by [Tow15a].

Theorem 3.3.1. *For every maximal subalgebra M of A there is a unique c-section up to isomorphism.*

Proof. Let M be a maximal subalgebra. First we verify that a c-section exists. Let C be minimal among ideals that do not belong to M. So $L = M + C$ and if C/D is a chief factor then $D \subset M$ by minimality of C. So $D \subset M \cap C$ and $(M \cap C)/D$ is a c-section.

Now let $(M \cap C)/D$ be a c-section. We will show that this is isomorphic to a c-section of M in which $D = M_A$. Clearly $D \subseteq M_A \cap C \subseteq C$, so since C/D is a chief factor and $M_A \cap C$ is an ideal of A we either have $M_A \cap C = D$ or $M_A \cap C = C$. If the latter holds then $C \subseteq M_A$, giving $L = A$, a contradiction. So $M_A \cap C = D$. Define $E = C + M_A$. Then $E/M_A \cong C/D$ is a chief factor and $(M \cap E)/M_A$ is a c-section. Furthermore,

$$\frac{M \cap E}{M_A} = \frac{M_A + (M \cap C)}{M_A} \cong \frac{M \cap C)}{M_A \cap C} = \frac{M \cap C}{D}.$$

So any c-section of M is isomorphic to some $(M \cap E)/M_A$. Now let $(M \cap E_1)/M_A$ and $(M \cap E_2)/M_A$ be two c-sections of M. If $E_1 \cong E_2$ then these c-sections are isomorphic. Otherwise consider that E_1/M_A and E_2/M_A are chief factors, so E_1/M_A and E_2/M_A are minimal ideals of the primitive Leibniz algebra A/M_A. Therefore $E_1/M_A \cong E_2/M_A$. □

Definition 3.3.2. Define $Sec(M)$ to be an algebra isomorphic to any c-section of M, and define $\eta^*(A : M) = \dim(Sec(M))$. Theorem 3.3.1 guarantees that $\eta^*(A : M)$ is well-defined.

Lemma 3.3.3. *Let $B \subseteq M \subseteq A$, where M is maximal in A and B is an ideal of A. Then $Sec(M) \cong Sec(M/B)$.*

Proof. M/B is a maximal subalgebra of A/B. Let $(C/B)/(D/B)$ be a chief factor of A/B such that $D/B \subseteq M/B$ and $C/B \nsubseteq M/B$. Then C/D is a chief factor of A such that $D \subseteq M$ and $A = C + M$. Hence $Sec(M) \cong C \cap M/D \cong Sec(M/B)$. □

Example 3.3.4. Let $L = H \dot{+} V$ and C/D be as in Example 2.0.9. Then $M = S \dot{+} V$ where $S = \text{span}\{x, z\}$ is a maximal subalgebra with $Sec(M) = (M \cap C)/D \cong S$. In the quotient algebra L/V, we have $Sec(M/V) \cong S \cong Sec(M)$.

Proposition 3.3.5. *If A is simple, then $Sec(M) \cong M/Leib(A)$ for all maximal subalgebras M with $Leib(A) \subseteq M$, and $Sec(M) \cong M \cap Leib(A)$ for all maximal subalgebras M with $Leib(A) \nsubseteq M$.*

Proof. Suppose A is simple. So $0 \subseteq Leib(A) \subseteq A$ is the only chief series for A. Let M be maximal. If $Leib(A) \subseteq M$ then $Sec(M) \cong M/Leib(A)$. If $Leib(A) \nsubseteq M$ then $Sec(M) \cong M \cap Leib(A)$. □

Remark 3.3.6. If A is Lie, Proposition 3.3.5 becomes A is simple if and only if $Sec(M) \cong M$ for all maximal subalgebras M.

Example 3.3.7. Consider the simple Leibniz algebra $A = S \dot{+} V$ from Examples 2.0.5 and 3.2.6. Then the chief factors are A/V and $V/\{0\}$. Consider the maximal subalgebra $M = \text{span}\{e, h\} \dot{+} V$. Then $Sec(M) \cong (M \cap A)/V \cong M/V \cong M/Leib(A)$.

3.4 Completions

Recall Definition 2.0.20 of a completion for a maximal subalgebra. Every maximal subalgebra M has an ideal completion. Any two distint ideal completions C and D for M will satisfy $C/k(C) \cong D/k(D)$, so the dimension may be identified with M. We prove that in a non-solvable Leibniz algebra, $Rad(A)$ is the intersection of the maximal subalgebras of A which have no abelian ideal completion. The results in this section are motivated by [Tow11].

Proposition 3.4.1. *If M is a maximal subalgebra of A then $I(M)$ is non-empty; in fact, $I(M)$ contains an ideal of A.*

Proof. Let C be a minimal element in the set of ideals not belonging to M, ordered by inclusion. Since M is maximal, $A = M + C$. If $k(C) = C$, then $C = I_1 + \cdots + I_n$ where each I_i is an ideal of A properly contained in C. Then some $I_i \nsubseteq M$, contradicting the minimality of C. So $k(C) \subsetneq C$. Then the minimality of C implies that $k(C) \subseteq M$, so $C \in I(M)$. □

Proposition 3.4.1 implies that any ideal supplement of a maximal subalgebra M must contain a completion for M.

Corollary 3.4.2. *If M is a maximal subalgebra, and C is an ideal of A that does not belong to M, then C contains a completion for M.*

Lemma 3.4.3. *Any ideal completion C for a maximal subalgebra M is maximal in $I(M)$.*

Proof. Suppose not. Then $C \subseteq D$ for some completion D for M with $D \neq C$. Then $C \subseteq k(D) \subseteq M$, a contradiction. □

Example 3.4.4 gives a counterexample to the converse of Lemma 3.4.3.

Example 3.4.4. Let L be the two-dimensional non-abelian Lie algebra, as in Example 2.0.1 (2). Then $M = \text{span}\{x + y\}$ is a maximal subalgebra, with maximal completion $C = \text{span}\{x\}$ which is not an ideal. (Note that $D = \text{span}\{y\}$ is an abelian ideal completion for M.)

Though a maximal subalgebra may have multiple, distinct ideal completions, Theorem 3.4.5 describes a simple relationship between all ideal completions for a given M.

Theorem 3.4.5. *Let M be a maximal subalgebra of A and let C, D be two ideal completions of M. Then $C/k(C) \cong D/k(D)$.*

Proof. Let A be a counterexample of minimal dimension. Let M be a maximal subalgebra of A with ideal completions C and D such that $C/k(C)$ and $D/k(D)$ are not isomorphic.

Suppose for a contradiction that $k(C)\cap k(D)\neq 0$. Then $M/(k(C)\cap k(D))$ is a maximal subalgebra of $A/(k(C)\cap k(D))$, with ideal completions $C^{'} = C/(k(C)\cap k(D))$ and $D^{'} = D/(k(C)\cap k(D))$. Then by minimality of A, $D^{'}/k(D^{'})\cong C^{'}/k(C^{'})$, a contradiction. So $k(C)\cap k(D)=0$.

Now $C\cap k(D)$ is an ideal of A, and $C\cap k(D)\subseteq C\cap M$, therefore $C\cap k(D)\subseteq k(C)$. Therefore $C\cap k(D)\subseteq k(C)\cap k(D)=0$. Similarly, $D\cap k(C)=0$.

Let $K = k(C)+k(D)$. Consider the maximal subalgebra M/K of A/K. We will show that $(C+k(D))/K$ is an ideal completion for M/K. First, $A=M+C$ implies that $A/K = M/K+(C+k(D))/K$. Now let X/K be an ideal of A/K properly contained in $(C+k(D))/K$; that is, $X/K\subseteq k((C+k(D))/K)$. Then $X\cap C$ is an ideal of A, and $X\cap C$ is properly contained in C since $X\cap C = C$ implies $C\subseteq X$ and thus $X = C+k(D)$, a contradiction. So $X = X\cap(C+k(D))\subseteq k(C)+k(D)\subseteq M$. So $X/K\subseteq M/K$. So $k((C+k(D))/K)\subseteq M/K$.

Similarly, $(D+k(C))/K$ is an ideal completion for M/K. But then if $K\neq 0$, the minimality of A implies that $(C+k(D))/K\cong(D+k(C))/K$. But $(C+k(D))/K\cong C/k(C)$ and $(D+k(C))/K\cong D/k(D)$, so for these two to be isomorphic is a contradiction. Therefore $K = k(C)+k(D)=0$.

Because $k(C)+k(D)=0$, $k(C)=k(D)=0$, and so C and D are minimal ideals of A. We will now show that $M\cap C = M\cap D = 0$. $M\cap C$ is a left ideal of A, because $[A,M\cap C] = [M+D,M\cap C]=[M,M\cap C]+[D,M\cap C]=\subset M\cap C$. Similarly $M\cap C$ is a right ideal of A. So $M\cap C$ is a proper ideal of A contained in C; therefore $M\cap C=0$. Similarly $M\cap D=0$.

Finally we show that $C \cong D$, establishing a contradiction. We have

$$C \cong \frac{C+D}{D} = \frac{(C+D)\cap(M+D)}{D} = \frac{(C+D)\cap M+D}{D} \cong (C+D)\cap M.$$

Similarly $D \cong (C+D)\cap M$, so $C \cong D$. This is a contradiction. We conclude that $C/k(C) \cong D/k(D)$. □

Theorem 3.4.5 and its proof appear in [Tow11] for Lie algebras. The result allows for the following definition.

Definition 3.4.6. Given a maximal subalgebra M of A, define the **ideal index** of M by $\eta(A:M) = \dim(C/k(C)$, where C is any ideal completion for M. Theorem 3.4.5 guarantees that $\eta(A:M)$ is well-defined.

Example 3.4.7. Consider the Leibniz algebra $A = sl(2,\mathbb{C}) \dotplus V$ from Example 2.0.5. Then $M = \text{span}\{e,h\} \dotplus V$ is a maximal subalgebra. A is an ideal completion for M, but $k(A) = Leib(A) = V$ since A is simple, so $A/k(A) \cong sl(2,\mathbb{C})$ is not abelien. So A is not an abelian ideal completion, and in fact there is no abelian ideal completion for M. Furthermore $\eta(L:M) = \dim(A) = 3$.

Define $\phi^*(A)$ to be the intersection of all maximal subalgebras of A which have no abelian ideal completion. If there are no such subalgebras then define $\phi^*(A) = A$. Towers proved in [Tow11] that $Rad(L) = \phi^*(L)$ for Lie algebras L. This result is motivated by Deskins, who gave a similar description of the maximal normal solvable subgroup of a group in [Des90], Theorem B. We obtain a characterization of $Rad(A)$ for Leibniz algebras.

Theorem 3.4.8. $Rad(A) = \phi^*(A)$

Proof. First we show that $Rad(A) \subseteq \phi^*(A)$. If $Rad(A) \subseteq M$ for all maximal subalgebras M, then the result is clear. So, suppose there is a maximal subalgebra M with $Rad(A) \nsubseteq M$. We will now find an abelian ideal completion of M. Let C be minimal in the set of ideals that are inside $Rad(A)$ but not M. Then C is an ideal completion of M, and $C/k(C)$ is a minimal solvable ideal, and so abelian. Thus, if $Rad(A) \nsubseteq M$, then M has an ableian ideal completion. So $Rad(A) \subseteq \phi^*(A)$.

Now let $A = Leib(A) \oplus B$ where $B \cong A/Leib(A)$. Then $\phi^*(B) = Rad(B)$ by [Tow11], Theorem 2.1. Then $\phi^*(A) = \phi^*(Leib(A) \oplus B) \subseteq \phi^*(Leib(A)) \oplus \phi^*(B) = Rad(Leib(A)) \oplus Rad(B) \subseteq Rad(A)$. □

Example 3.4.9. Let A be the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = a^3$. Then $M = A^2 = \text{span}\{a^2, a^3\}$ is a maximal subalgebra with $\dim(A/M) = 1$. Note that M is a c-ideal because it is an ideal, and $Sec(M) = 0$ because $(M \cap A)/M$ is a c-section for M. Thus $\eta^*(A : M) = 0$. Finally A is an ideal completion for M because $k(A) = M$, so $\eta(A : M) = 1$.

Finally, abelian ideal completions are preserved by quotients, as in Lemma 3.4.10.

Lemma 3.4.10. *Let M be a maximal subalgebra, and B, C ideals with $B \leq M$, $B \leq C$. Then C is an abelian ideal completion for M if and only if C/B is an abelian ideal completion for M/B.*

Proof. First suppose C is an abelian ideal completion for M. Then $C \nsubseteq M$ implies $C/B \nsubseteq M/B$. Let I/B be an ideal of A/B properly contained in C/B. Then $I \subseteq k(C) \subseteq M$ implies $I/B \subseteq M/B$. So $k(C/B) \subseteq M/B$. So C/B is an abelian ideal completion for M/B.

Now let C/B be an abelian ideal completion for M/B. Then $C/B \nsubseteq M/B$ implies $C \nsubseteq M$. We have $k(C/B) \subseteq M/B$. Let I be an ideal of A properly contained in C. It remains to show that $I \subseteq M$. But $I/B \subseteq k(C/B) \subseteq M/B$ implies $I \subseteq M$. So $k(C) \subseteq M$ and C is an abelian ideal completion for M. □

3.5 CAP-subalgebras

Recall Definition 2.0.22 of a CAP-subalgebra. This section is motivated by [Tow15b].

Lemma 3.5.1. *Let B be a subalgebra of A, and C/D a chief factor of A. Then:*

1. *B covers C/D if and only if $B \cap C + D = C$; and*

2. *B avoids C/D if and only if $(D + B) \cap C = D$.*

3. *If $B \cap C + D$ is an ideal of A, then B covers or avoids C/D. In particular, ideals are CAP-subalgebras.*

4. *A is simple if and only if $Leib(A)$ is the only non-trivial proper CAP-subalgebra of A.*

Proof. Let C/D be a chief factor of A and $B \leq A$ a subalgebra.

1. $(\Rightarrow)$ Suppose B covers C/D so $B + C = B + D$. We already have $B \cap C + D \subseteq C$ because $D \subseteq C$. Now fix $c \in C$. If $c \in B$ then $c \in B \cap C + D$. Otherwise $c \in C \subset B + C = B + D$ so $c = b + d$ for some $b \in B, d \in D$.

   $(\Leftarrow)$ Suppose $B \cap C + D = C$. We already have $D \subseteq C$ implies $B + D \subseteq B + C$. Now fix $b \in B, c \in C$. Then $c = b' + d$ where $b' \in B \cap C, d \in D$. So $b + c = b' + b + d \in B + D$. So $B + C = B + D$.

2. $(\Rightarrow)$ Suppose B avoids C/D so $B \cap C = B \cap D$. We already have $D \subseteq C$ implies $D \subseteq (D + B) \cap C$.

($\Leftarrow$) Suppose $(D+B)\cap C = D$. We already have $D \subseteq C$ implies $B \cap D \subseteq B \cap C$. Now fix $b \in B \cap C$.

3. Suppse $I = B \cap C + D$ is an ideal of A. Then $D \subseteq I \subseteq C$ so either $I = C$ or $I = D$. If $I = C$ then B covers C/D by (1). Otherwise $B \cap C \subseteq D$ so $B \cap C = B \cap D$ and B avoids C/D. So I is a CAP-subalgebra. Since $B \cap C + D$ is an ideal for any ideal B of A, the ideals of A are CAP-subalgebras.

4. ($\Rightarrow$) Suppose A is simple. Then $A/Leib(A)$ and $Leib(A)/\{0\}$ are the chief factors. Let B be a non-trivial, proper CAP-subalgebra. We will show $B = Leib(A)$.

   Suppose B avoids $Leib(A)/\{0\}$. Then $Leib(A) \cap B = 0$. If B avoids $A/Leib(A)$ then $A \cap B = B = Leib(A) \cap B = 0$, a contradiction. So B covers $A/Leib(A)$, thus $A = Leib(A) + B$. Therefore $A = Leib(A) \oplus B$, so $A^2 = Leib(A)$, contradicting that A is simple.

   So B covers $Leib(A)/\{0\}$, and $B + Leib(A) = B$. So $Leib(A) \subseteq B$. Then if B covers $A/Leib(A)$ we have $A = B$, a contradiction, so B avoids $A/Leib(A)$. Then $A \cap B = Leib(A) \cap B$. So $B = Leib(A) \cap B$ and $Leib(A) \subseteq B$ implies $B = Leib(A)$.

   ($\Leftarrow$) Suppose $Leib(A)$ is the only non-trivial, proper CAP-subalgebra of A. Now suppose A has an ideal $I \neq Leib(A)$. Then I is a CAP-subalgebra by (3), a contradiction. So there is no such I and A is simple.

$\square$

Remark 3.5.2. Lemma 3.5.1 appears in [Tow08] for Lie algebras as Lemma 2.1. Part (4) is modified to account for the definition of simplicity in Leibniz algebras.

Lemma 3.5.3. *Let B be a CAP-subalgebra of the Leibniz algebra A, and let I be an ideal of A. Then $B+I$ is a CAP-subalgebra of A.*

Proof. Let C/D be a chief factor of A. We will show that $B+I$ either covers or avoids C/D. Because B is a CAP-subalgebra, B either covers or avoids C/D. If B covers C/D, then $B+C=B+D$, in which case $B+I+C=B+I+D$, so $B+I$ covers C/D as well. Similarly, if I covers C/D then so does $B+I$.

Since every ideal is a CAP-subalgebra, the remaining case is that both B and I avoid C/D; that is, $B\cap C=B\cap D$ and $I\cap C=I\cap D$. We will show that $B+I$ avoids C/D as well. We have $(B+I)\cap D\subseteq(B+I)\cap C$ because $D\subseteq C$. For the other direction, fix $x\in(B+I)\cap C$. Then $x=b+i$ for some $b\in B$, $i\in I$, and $x\in C$. Then $b\in(I+C)\cap B=(I+D)\cap B=D+(I\cap C)=D$. Then $x\in D$ implies that $(B+I)\cap C\subseteq(B+I)\cap D$. So $B+I$ avoids C/D, and therefore $B+I$ is a CAP-subalgebra. □

3.6 Relationships

In this section we relate the subalgebras previously discussed. We show that $\eta(A:M)=\eta^*(A:M)+\dim(A/M)$ for all maximal subalgebras M. The maximal subalgebras which are c-ideals are precisely those with trivial c-sections. Such M satisfy $\eta(A:M)=\dim(A:M)$. Ideal index is preserved under quotient algebras.

Theorem 3.6.1. *For a maximal subalgebra M, $\eta^*(A:M)=\eta(A:M)-dim(A/M)$.*

Proof. Let M be a maximal subalgebra of A and let C be an ideal completion for M. Note that $C/k(C)$ is a chief factor with $k(C)\subseteq M$ and $C\nsubseteq M$. So $M\cap C/k(C)$ is a c-section of M and $\dim(M\cap C/k(C))=\eta^*(A:M)$. Note also that $A/M=(M+C)/M=C/(C\cap M)$. Thus,

$$\begin{aligned}\eta(A:M) &= \dim(C/k(C)) \\ &= \dim(C)-\dim(k(C)) \\ &= \dim(C)-\dim(C\cap M)+\dim(C\cap M)-\dim(k(C)) \\ &= \dim(C/(C\cap M))+\dim(C\cap M/D) \\ &= \dim(A/M)+\eta^{*}(A:M).\end{aligned}$$

□

Theorem 3.6.2. *Let M be a maximal subalgebra of A. Then M is a c-ideal of A if and only if Sec(M) = 0.*

Proof. Firt suppose $Sec(M) = 0$. Let C be minimal among ideals of A not contained in M. So $A = M + C$. Then if C/D is a chief factor, $D \subseteq M$ by minimality of C so $M \cap C \cong D$. So $M \cap C$ is an ideal of A because D is; therefore, $M \cap C \subseteq M_A$ and thus M is a c-ideal.

Conversely, suppose that M is a c-ideal. Then there exists an ideal C of A such that $A = M + C$ and $M \cap C \subseteq M_A$. We will show that $C/(M \cap C)$ is a chief factor, allowing us to conclude that $Sec(M) = (C \cap M)/(C \cap M) = 0$.

We know that $M \cap C$ is an ideal of A, as is C. Let K be an ideal of A with $M \cap C \subset K \subseteq C$. Then $K \nsubseteq M$, so $A = M + K$ and $M \cap C = M \cap K$. This yields that $\dim(A) = \dim(M) + \dim(K) - \dim(M \cap K) = \dim(M) + \dim(C) - \dim(M \cap C)$, so $K = C$ and the proof is complete.

□

Corollary 3.6.3. *Let M be a maximal subalgebra of A. Then M is a c-ideal if and only if $\eta(A:M) = dim(A/M)$.*

Example 3.6.4. Let A be the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = a^3$ and consider the maximal subalgebra and c-ideal $M = \text{span}\{a-a^2, a-a^3\}$. Then $A^2 = \text{span}\{a^2, a^3\} \nsubseteq M$ but $k(A^2) = \text{span}\{a^2 - a^3\} \subseteq M$ so A^2 is an ideal completion for M. Then $\eta(A : M) = \dim(A^2/k(A)) = 1 = \dim(A/M)$.

Proposition 3.6.5. *Let M be a maximal subalgebra of A and let B be an ideal of A with $B \subseteq M$. Let C/B be an ideal completion of M/B in A/B, let $k(C/B) = K/B$, and let D be an ideal completion of M in A. Then dim$(C/K) \cong$ dim$(D/k(D))$.*

Proof. Using the same approach as in Theorem 3.6.1, we have

$$\begin{aligned}
\dim(C/K) &= \dim((C/B)/(K/B)) \\
&= \dim(C/B) - \dim(K/B) \\
&= \dim(C/B) - \dim(C/B \cap M/B) + \dim(C/B \cap M/B) - \dim(K/B) \\
&= \dim(C/B) - \dim((C \cap M)/B) + \dim((C \cap M)/B) - \dim(K/B) \\
&= \dim(C/(C \cap M)) + \dim((C \cap M)/K) \\
&= \dim(A/M) + \eta^*(A : M) \\
&= \eta(A : M).
\end{aligned}$$

We use the fact that $(C \cap M)/K$ is a c-section of M in A to determine that $\dim((C \cap M)/K) = \eta^*(A : M)$. This is because $C/K = (C/B)/(K/B) = (C/B)/k(C/B)$ is a chief factor, $A = M + C$, and $K \subseteq M$. Then, because D is an ideal completion for M, we have $\dim(D/k(D)) = \eta(A : M) = \dim(C/K)$. □

Remark 3.6.6. For a Lie algebra L, $C/K \cong D/k(D)$ in Proposition 3.6.5 [Tow11]. Our result is sufficient to prove Corollary 3.6.7.

Corollary 3.6.7. *Let M be a maximal subalgebra of A and let B be an ideal of A with $B \subseteq M$. Then $\eta(A/B : M/B) = \eta(A : M)$.*

Proof. Let C/B be an ideal completion of M/B in A/B. By Proposition 3.6.5 we have $\eta(A/B : M/B) = \dim((C/B)/(K/B)) = \eta(A : M)$. □

Example 3.6.8. Let A be the 3-dimensional cyclic Leibniz algebra with $[a, a^3] = a^2 + 2ia^3$. Then $A^2 = \text{span}\{a^2, a^3\}$ is a maximal subalgebra of A, and $B = \text{span}\{ia^2 - a^3\}$ is an ideal contained in M. Then $\eta(A : M) = 1 = \eta(A/B : M/B)$.

CHAPTER

4

SOLVABLILITY CRITERIA

We now develop characterizations of and criteria for solvability in Leibniz algebras. The results in this chapter are motivated by Towers' work in [Tow09],[Tow11],[Tow15a], and [Tow16]. Whenever possible, we use a modification of the Lie algebra proof for our generalizations. In particular, we examine the Lie algebra $A/Leib(A)$, because of the following fact.

Lemma 4.0.1. *Let A be a Leibniz algebra, and I a solvable ideal of A. If A/I is solvable, then A is solvable.*

Proof. Since I and A/I are solvable, $I^{(n)} = (A/I)^{(m)} = 0$ for some n and m. Consider the canonical homomorphism $\pi : A \longrightarrow A/I$. Then $\pi(A^{(m)}) = \pi(A)^{(m)} = (A/I)^{(m)} = 0$, and therefore $A^{(m)} \subseteq \ker\pi = I$. So $A^{(n+m)} = (A^{(m)})^{(n)} \subseteq I^{(n)} = 0$. So A is solvable. □

This leads to the following useful corollary.

Corollary 4.0.2. *Let A be a Leibniz algebra. If the Lie algebra $A/Leib(A)$ is solvable, then A is solvable.*

Proof. $Leib(A)$ is a solvable ideal of A. Apply Lemma 4.0.1. □

4.1 C-ideals

We can obtain information about a Leibniz algebra by studying the behavior of its maximal subalgebras. For example, a Leibniz algebra is nilpotent if and only if every maximal subalgebra is an ideal [Dem13]. In [Tow09], Towers proved that a Lie algebra is solvable if and only if every maximal subalgebra is a c-ideal. Recall from Definition 2.0.16 that a subalgebra $B \leq A$ is a c-ideal if there is an ideal C satisfying $A = B + C$ and $B \cap C \leq B_A$.

Theorem 4.1.1. *All maximal subalgebras of A are c-ideals of A if and only if A is solvable.*

Proof. First suppose that all maximal subalgebras of A are c-ideals, and let $M/Leib(A)$ be a maximal subalgebra of $A/Leib(A)$. Then M is a maximal subalgebra of A and thus a c-ideal. So $M/Leib(A)$ is a c-ideal of $A/Leib(A)$ by Lemma 3.2.1. So every maximal subalgebra of the Lie algebra $A/Leib(A)$ is a c-ideal, and therefore $A/Leib(A)$ is solvable by [Tow09], Theorem 3.1. Thus A is solvable.

Now suppose that A is solvable. Let M be a maximal subalgebra of A. We will show that M is a c-ideal. Because A is solvable, $A^{(k)} = 0$ for large enough k, so there exists some $k \geq 2$ such that $A^{(k)} \leq M$ but $A^{(k-1)} \nleq M$. Then since $A^{(k-1)}$ is an ideal, $M \cap A^{(k-1)} \leq M_A$. We also have $A = M + A^{(k-1)}$, so M is a c-ideal of A. □

We may restate Theorem 4.1.1 in terms of the ideal index $\eta(A : M)$ of each maximal subalgebra M.

Corollary 4.1.2. *A is solvable if and only if $\eta(A : M) = dim(A/M)$ for all maximal subalgebras M of A.*

Proof. If A is solvable if and only if M is a c-ideal if and only if $\eta(A : M) = \dim(A/M)$, by Theorem 4.1.1 and Corollary 3.6.3, respectively. □

Here is another restatement of Theorem 4.1.1 in terms of c-sections.

Corollary 4.1.3. *A is solvable if and only if $Sec(M) = 0$ for all maximal subalgebras M.*

Proof. A is solvable if and only if M is a c-ideal if and only if $Sec(M) = 0$, by Theorems 4.1.1 and 3.6.2, respectively. □

If we restrict our attention to Leibniz algebras over fields of characteristic zero, we may relax the sufficient condition for solvability from Theorem 4.1.1. Again we use an analagous result for Lie algebras, from [Tow09]. The following proof depends on the fact that $Rad(A) \neq 0$ in a non-Lie Leibniz algebra.

Theorem 4.1.4. *Let A be a Leibniz algebra over a field of characteristic 0. Then A has a solvable maximal subalgebra that is a c-ideal of A if and only if A is solvable.*

Proof. If A is a Lie algebra, the result appears in [Tow09]. So, we will assume A is non-Lie. Suppose A has a solvable maximal subalgebra M which is a c-ideal of A. We will show that A is solvable. For a contradiction, let A be a counterexample of minimum dimension. Because M is a c-ideal of A and M_A is an ideal of A contained in M, we have by Lemma 3.2.1 that M/M_A is a c-ideal of A/M_A. Furthermore, M/M_A is a maximal solvable subalgebra of A/M_A because M is a maximal solvable subalgebra of A. Also note that M_A is solvable because it is contained in the solvable subalgebra M.

Suppose for a contradiction that $M_A \neq 0$. Then $\dim(A/M_A) < \dim(A)$, and since A is a counterexample of minimum dimension, the existence of the solvable maximal ideal M/M_A of A/M_A implies that A/M_A is solvable. But A/M_A and M_A both solvable implies A is solvable, a contradiction. So we must have $M_A = 0$. Now let $R = \text{rad}(A)$ and observe that $R = M_A = 0$. This implies that A is Lie, a contradiction. So A must be solvable.

For the converse, suppose A is solvable. By Theorem 4.1.1, every solvable maximal subalgebra of A is a c-ideal, which completes the proof. □

Example 4.1.5 illustrates that the c-ideal in Theorem 4.1.4 needs to be a solvable maximal subalgebra.

Example 4.1.5. Consider the 5-dimensional Leibniz algebra $A = \text{span}\{e, f, h, x, y\}$ with nonzero multiplications given by $[h,e] = 2e, [h,f] = -2f, [h,x] = x, [h,y] = -y, [e,h] = -2e, [e,f] = h, [e,y] = -x, [f,h] = 2f, [f,e] = -h, [f,x] = y$. Then $M = \text{span}\{e, f, h\} \equiv sl_2$ is a maximal subalgebra of A, and it is a c-ideal with c-supplement $I = \text{span}\{x, y\}$. However, A is not solvable. In fact, as indicated by Theorem 4.1.4, A has no solvable maximal subalgebras. $Rad(A) = I$ is not maximal.

Again, we may restate Theorem 4.1.4 in terms of both the ideal index and c-sections. We collect these in a corollary.

Corollary 4.1.6. *Let A be a Leibniz algebra over a field of characteristic zero. Then:*

1. *A is solvable if and only if A has a solvable maximal subalgebra M with $Sec(M) = 0$;*

2. *A is solvable if and only if A has a solvable maximal subalgebra M with $\dim(A/M) = \eta(A : M)$.*

Proof. Let A be a Leibniz algebra over a field of characteristic zero.

1. Theorems 4.1.4 and 3.6.2.

2. Theorem 4.1.4 and Corollary 3.6.3.

□

We now turn our attention to Cartan subalgebras. These are maximal nilpotent subalgebras. In [Tow09], Towers proved that if every maximal nilpotent subalgebra of a Lie algebra L is a c-ideal, then L is solvable. As in Theorem 4.1.4, if we restrict our attention to Leibniz algebras over fields of characteristic zero, we can relax the sufficient condition for solvability. If every Cartan subalgebra of such a Leibniz algebra is a c-ideal, then the algebra is solvable. To prove this we need the following lemma, which appears for Lie algebras in [Dix56].

Lemma 4.1.7. *Let L be a Leibniz algebra, over a field of characteristic 0, with Levi factor S. Let $R = rad(L)$. Let A be a Cartan subalgebra of $C_R(S)$, and B be a Cartan subalgebra of S. Then $C = A + B$ is a Cartan subalgebra of L.*

Proof. Note that $L = R \oplus S$, so $A \leq R$ and $B \leq S$ implies that $C = A \oplus B$. Because both A and B are Cartan, they are nilpotent, thus their sum C is nilpotent. It remains to show that C is self-normalizing. But $C \leq L$ so $C \subseteq N(C)$, therefore we must prove that $N(C) \subseteq C$.

Fix $x \in N(C)$. Then $x \in L$, so $x = y + z$ for some $y \in R, z \in S$. Now fix $u \in B \subseteq C$. Because $x \in N(C)$, we have $[u, x] \in C$. So there is a unique expression $[u, x] = a + b$ where $a \in A$, $b \in B$. Observe that $[u, x] = [u, y] + [u, z]$. Now $z \in S, u \in B \leq S$ implies $[u, z] \in S$, but $x \in C = A \oplus B$ implies $[u, z] \in B \subseteq C$. This means $[u, z] \in A \subseteq C$. Therefore $[u, z] = [u, x] - [u, y] \in C$, from which we may conclude $y \in N^l(C)$. Also, $[z, B] \subseteq B$ implies $z \in B \subseteq C$.

For all $t \in B$, the above argument shows that $[t, y] \in A \subseteq C_R(S)$. So $[t, [t, y]] = 0$, therefore $(\text{ad}\, t)^2(y) = 0$. Now $(\text{ad}(t)(y) = 0$, thus $y \in C_R(B) \subseteq C_R(S) = A$. So $y \in A \subseteq C$. Now $y, z \in C$ implies $y + z = x \in C$. Thus $N(C) \subseteq C$, which completes the proof. □

We can now prove our solvability criterion.

Theorem 4.1.8. *Let A be a Leibniz algebra over a field of characteristic 0. Suppose every Cartan subalgebra of A is a c-ideal of A. Then A is solvable.*

Proof. Suppose that every Cartan subalgebra is a c-ideal of A, and let A have a nonzero Levi factor S. Let H be a Cartan subalgebra of S, and let B be a Cartan subalgebra of its centralizer in the solvable radical of A. Then $C = H + B$ is a Cartan subalgebra of A by Lemma 4.1.7. Therefore C is a c-ideal of A. So there is an ideal K of A such that $A = C + K$ and $C \cap K \in C_A$.

Because K is an ideal, there is some $r \geq 2$ such that $A^{(r)} \leq K$. But $S \leq A^{(r)} \leq K$, so $C \cap S \leq C \cap K \leq C_A$, and therefore $C \cap S \leq C_A \cap S = 0$, a contradiction. So $S = 0$, thus A is solvable. □

Example 4.1.9, whose construction appears in [Jac79], p. 52, gives a counterexample to the converse of Theorem 4.1.8. To obtain a converse, we must replace solvability with the stronger condition that $\cap_{i=1}^{\infty} A^i = A^{\infty}$ is abelian.

Example 4.1.9. Let $V = \text{span}\{e_1, \cdots, e_p\}$ be a vector space over a field $\mathbb{F}$ of characteristic $p > 0$. Define operators $x, y : V \to V$ by:

$$\begin{aligned} x(e_i) &= e_{i+1} \\ y(e_i) &= (i-1)e_i \end{aligned}$$

where subscripts are taken mod p. Let $L = \text{span}\{x, y\}$ be the Lie algebra with commutator bracket $[x, y] = xy - yx$, so that $[x, y] = x$ and L is nonabelian. Now let $L' = L \dotplus V$. L' is a Lie algebra.

Consider the subalgebra $C = \mathbb{F}y + \mathbb{F}e_1$. C is abelian, thus nilpotent. Furthermore $x \notin N(C)$ since $[x, y] \notin C$, and $e_j \in N(C)$ if and only if $j = 1$, so C is a Cartan subalgebra of L'. But $C_A = 0$. Therefore, because $\mathbb{F}x \dotplus \text{span}\{e_2,, \cdots, e_p\}$ is not an ideal of L', C is not a c-ideal.

In Example 4.1.9, L' is solvable, because V is an abelian ideal of L' and $L'/V \cong L$, which is solvable. However, $(L')^{\infty} = \mathbb{F}x \oplus V$ is not abelian. For Leibniz algebras A in which A^{∞} is abelian, every Cartan subalgebra is a c-ideal. This resut follows from the following theorem.

Theorem 4.1.10. *Let L be a Lie algebra such that the ideal $L^{\infty} = \cap_{i=1}^{\infty} L^i$ is abelian. Then L has a Cartan subalgebra, and H is a Cartan subalgebra of L if and only if $L = H \oplus L^{\infty}$.*

Proof. This is [Win72], Theorem 4.4.1.1. □

The following lemma appears in [Tow09], and motivates our generalization to Leibniz algebras.

Lemma 4.1.11. *Let L be a Lie algebra with L^∞ abelian. Then every Cartan subalgebra of L is a c-ideal.*

Proof. Let H be a Cartan subalgebra of L. Then $L = H \oplus L^\infty$, by Theorem 4.1.10. Then since L^∞ is an ideal, H is a c-ideal. □

We now extend this result to Leibniz algebras.

Theorem 4.1.12. *Let A be a Leibniz algebra with $\cap_{i=1}^{\infty} A^i = A^\infty$ abelian. Then every Cartan subalgebra of A is a c-ideal.*

Proof. Denote $Leib(A) = L$. Then $(A/L)^\infty$ is abelian. Let C be a Cartan subalgebra of A. Then $(C+L)/L$ is a Cartan subalgebra of A/L by [Bar11], Corollary 6.3. Then $(C+L)/L$ is a c-ideal of A/L. So C is a c-ideal of A by Lemma 3.2.2. □

Example 4.1.13. Let A be the 4-dimensional cyclic Leibniz algebra with $[a, a^4] = a^3$. Then the unique Cartan subalgebra of A is $C = \text{span}\{a^4 - a^2\}$, which is an ideal and thus a c-ideal.

The preceeding example illustrates the fact that every cyclic Leibniz algebra satisfies the conditions of Theorem 4.1.12.

Corollary 4.1.14. *Let A be a cyclic Leibniz algebra. Then the unique Cartan subalgebra C of A is a c-ideal.*

Proof. $A^\infty \subseteq A^2$, which is abelian. Apply Theorem 4.1.12. □

Compare Corollary 4.1.14 with the observation that the Cartan subalgebra of a cyclic Leibniz algebra is generally not an ideal. The 3-dimensional cyclic Leibniz algebra has Cartan subalgebra $C = \text{span}\{a - a^3\}$, which is a left ideal, but not a right ideal as $[a - a^3, a] = a^2 \notin C$. However, C is a c-ideal with c-supplement A^2.

4.2 C-Sections

Recall Definition 2.0.18 of a c-section of a maximal subalgebra. We have seen that $Sec(M) = 0$ for all maximal subalgebras M if and only if A is solvable. We can strengthen this result as follows. These results are based on Lie algebra results in [Tow15a].

A subalgebra B of A is **nil** if L_b acts nilpotently on A for all $b \in B$; that is, for all $b \in B$ and $a \in A$, there exists $n \in \mathbb{N}$ such that $L_b^n(a) = 0$. A nil subalgebra is nilpotent, but not conversely. The Cartan subalgebra C from Example 4.1.9 is nilpotent, but not nil since $L_y^n(x) = x \neq 0$ for all n.

Theorem 4.2.1. *Sec(M) is nil for every maximal subalgebra M of A if and only if A is solvable.*

Proof. If A is solvable, then every maximal subalgebra has a trivial (hence, nil) c-section. Suppose that $Sec(M)$ is nil for every maximal subalgebra M. Consider a maximal subalgebra $M/Leib(A)$ of $A/Leib(A)$. Then M is maximal in A, so $Sec(M)$ is nil. Then $Sec(M) \cong Sec(M/Leib(A))$ by Lemma 3.3.3, so $Sec(M/Leib(A))$ is nil in $A/Leib(A)$. So $A/Leib(A)$ is solvable, by [Tow15a], Theorem 2.6. So A is solvable. □

If we require an algebraically closed field of nonzero characteristic, we may replace the condition "nil" with the stronger condition "nilpotent."

Theorem 4.2.2. *Let A be a Leibniz algebra over an algebraically closed field of characteristic* $p > 0$*. Then Sec(M) is nilpotent for every maximal subalgebra* M *of A if and only if A is solvable.*

Proof. Let A be a Leibniz algebra over an algebraically closed field of characteristic $p > 0$.

($\Rightarrow$) Suppose $Sec(M)$ is nilpotent for all maximal subalgebras M of A. Let $M/Leib(A)$ be maximal in $A/Leib(A)$, so M is maximal in A and $Sec(M)$ is nilpotent. Then $Sec(M/Leib(A)) \cong Sec(M)$ is nilpotent, so the Lie algebra $A/Leib(A)$ is solvable by [Tow15a], Theorem 2.5. So A is solvable.

($\Leftarrow$) If A is solvable, then $Sec(M) = 0$ by Corollary 4.1.3, so $Sec(M)$ is nilpotent. $\square$

Example 4.2.3. Let L be the 3-dimensional, non-solvable Lie algebra over a field of characteristic 2 given in Example 2.0.12. Then $M = \text{span}\{x + z, y\}$ is a maximal subalgebra, and then because L is simple, $Sec(M) \cong M$ by Proposition 3.3.5. Additionally, M is not nilpotent, as $L_y^n(x+z) = x + z \neq 0$ for all $n > 0$. Corollary 4.2.2 guarantees that L has at least one maximal subalgebra M with $Sec(M)$ not nilpotent.

4.3 Completions

We now characterize solvable Leibniz algebras by the existence of abelian ideal completions for each maximal subalgebra. Recall Definition 2.0.20 of a completion for a maximal subalgebra.

Theorem 4.3.1. *A is solvable if and only if every maximal subalgebra has an abelian ideal completion.*

Proof. Let A be solvable and let $M \leq A$ be a maximal subalgebra. Then there is some $k \geq 1$ such that $A^{(k)} \subset M$ but $A^{(k-1)} \not\subset M$. Then by Corollary 3.4.2, $A^{(k-1)}$ contains an ideal completion C for M, with $C/k(C) \subseteq A^{(k-1)}/k(C)$ abelian. So M has an abelian ideal completion.

Now suppose that every maximal subalgebra of A has an abelian ideal completion. Let $M/Leib(A)$ be maximal in $A/Leib(A)$. Then let C be an abelian ideal completion of M, so that by Lemma 3.4.10 $M/Leib(A)$ has an abelian ideal completion as well. Then since $A/Leib(A)$ is Lie, it is solvable as shown in [Tow11]. So A is solvable. □

Example 4.3.2. Consider the non-solvable Leibniz algebra $A = S \dot{+} V$ from Example 2.0.5 and its maximal subalgebra $M = \text{span}\{e, h\} \dot{+} V$. M has only one ideal completion, which is A itself. Then $A/k(A) \equiv S$ is nonabelian, so M has no abelian ideal completion.

4.4 CAP-subalgebras

Our main result in this section is that a Leibniz algebra is solvable if and only if all its maximal subalgebras are CAP-subalgebras. We also show that in a nilpotent Leibniz algebra, every Cartan subalgebra is a CAP-subalgebra. While this result follows immediately from Theorem 5.3.1, our proof in this section takes a different approach. Recall Definition 2.0.22 of a CAP-subalgebra.

Theorem 4.4.1. *A is solvable if and only if every maximal subalgebra is a CAP-subalgebra.*

Proof. Suppose every maximal subalgebra of A is a CAP-subalgebra. Let M be maximal. We know a c-section for M exists, so let $(M \cap C)/D$ be a c-section. So $A = M + C$ and $D \subseteq M$. Then since M is a CAP-subalgebra we have that either $M + C = M + D$ or $M \cap C = M \cap D$.

If $M+C=M+D$, then $A=M+D$ and $M\cap D=D\subseteq M_A$ so M is a c-ideal. Otherwise $M\cap C=M\cap D=D$ so $Sec(M)=0$ and M is a c-ideal. Because M is a c-ideal, A is solvable.

Now suppose A is solvable and let M be maximal. Let C/D be a chief factor. Then if C and D are nonzero we have $M+C=A=M+D$ so M covers C/D. Otherwise D is 0 and C is minimal so $M\cap C=0=M\cap D$, so M avoids C/D. □

Remark 4.4.2. Theorem 4.4.1 is given in [Tow15b] for Lie algebras, but our proof is different.

Example 4.4.3. Let $L=sl(2,\mathbb{C})$. Then $M=\text{span}\{e,h\}$ is a maximal subalgebra which is not a CAP-subalgebra. Because L is simple, the only chief factor is $L/\{0\}$. Then $L\cap M=M\neq\{0\}\cap M$ and $L+M=L\neq\{0\}+M$.

The following theorem is given for Lie algebras in [Tow15b].

Lemma 4.4.4. *Let A be any Leibniz algebra, let U be a supplement to an ideal B in A, and suppose that $B^k\subseteq U$ for some $k\in\mathbb{N}$. Then U is a CAP-subalgebra of A.*

Proof. We have $A=B+U$. Let C/D be a chief factor of A. Now consider $D+[B,C]$, which is an ideal of A. Because C is an ideal we have $[B,C]\subseteq C$ and thus $D\subseteq D+[B,C]\subseteq C$. Since C/D is a chief factor, either $D+[B,C]=C$ or $D=[B,C]=D$. Suppose first that $D+[B,C]=C$. Then $H+U=K+U$ so U covers H/K. Now suppose that $D+[B,C]=D$. This means $[B,C]\subseteq D$. Then $D+(U\cap C)$ is an ideal of A. So the result follows from Lemma 3.5.1 (3). □

This allows for the following theorem.

Theorem 4.4.5. *Let A be a nilpotent Leibniz algebra. Then every c-ideal is a CAP-subalgebra.*

Proof. Let A be nilpotent and U be a c-ideal. Then there is some ideal B in A such that $A = U + B$. Because A is nilpotent, we have that $B^k = 0 \subseteq U$ for some $k \in \mathbb{N}$. So U is a CAP-subalgebra by Lemma 4.4.4. □

Corollary 4.4.6. *If A is the cyclic Leibniz algebra with $[a, a^n] = 0$, then the unique Cartan subalgebra of A is a CAP-subalgebra.*

Proof. A is nilpotent, so this is Corollary 4.1.14 and Theorem 4.4.5. □

Example 4.4.7 gives a counterexample to the converse of Theorem 4.4.5.

Example 4.4.7. Example 2.0.23 gives a supersolvable Leibniz algebra A in which every c-ideal (in fact, every subalgebra) is CAP-subalgebra; however, A is not nilpotent.

Remark 4.4.8. Theorem 4.4.5 may be viewed as a corollary to Theorem 5.3.1.

CHAPTER

5

SUPERSOLVABILITY CRITERIA

We now characterize supersolvable Leibniz algebras in terms of their c-ideals, CAP-subalgebras, and the ideal index of their maximal subalgebras.

5.1 C-ideals

We obtain a sufficient condition for supersolvability in terms of c-ideals. Recall from Definition 2.0.16 that a subalgebra $B \leq A$ is a c-ideal if there is an ideal C satisfying $A = B + C$ and

$B \cap C \leq B_A$. This section is motivated by [Tow09]. Our main result is Theorem 5.1.1.

Theorem 5.1.1. *Let A be a solvable Leibniz algebra in which every maximal subalgebra of each maximal nilpotent subalgebra of A is a c-ideal of A. Then A is supersolvable.*

Lemmas 5.1.2, 5.1.3, and 5.1.4 will be used in the proof of Theorem 5.1.1. They appear for Lie algebras in [Tow09].

Lemma 5.1.2. *Let A be a Leibniz algebra, let I be an ideal of A, and let U/I be a maximal nilpotent subalgebra of A/I. Then $U = C + I$, where C is a maximal nilpotent subalgebra of A.*

Proof. If $I \leq \phi(U)$, then $U/\phi(U)$ is nilpotent, and therefore U is nilpotent by [Bat13]. Because U/I is a maximal sub algebra of A/I, U is a maximal subalgebra of A, so $U = U + I$, giving the result.

So suppose that $I \nleq \phi(U)$. So there is some maximal subalgebra M of U with $I \nsubseteq M$. Then $U = I + M$ since I is an ideal. If B is minimal with respect to $U = I + B$, then $I \cap B \leq \phi(B)$ [Bat13], Lemma 3.6. Also $U/I \cong B/(I \cap B)$ is nilpotent, which means that B is nilpotent. If we now choose C to be the biggest nilpotent subalgebra of U such that $U = I + C$, then C is a maximal nilpotent subalgebra. □

Lemma 5.1.3. *Let A be a Leibniz algebra in which every maximal subalgebra of each maximal nilpotent subalgebra of A is a c-ideal of A, and let I be a minimal abelian ideal of A. Then every maximal subalgebra of each maximal nilpotent subalgebra of A/I is a c-ideal of A/I.*

Proof. Suppose that U/I is a maximal nilpotent subalgebra of A/I. Then $U = C + I$, where

C is a maximal nilpotent subalgebra of A, by Lemma 5.1.2. Let B/I be a maximal subalgebra of U/I. We will show that B/I is a c-ideal of A/I.

Because B is a subalgebra of $U = C + I$, we have $B = B \cap (C + I) = B \cap C + I$ because $B \cap I = I$. Then $B \cap C + I = D + I$ for some maximal subalgebra D of C with $B \cap C \leq D$. Now by our assumption D is a c-ideal of A, so there is an ideal K of A with $A = D + K$ and $D \cap K \leq D_A$.

If $I \leq K$, then K/I is an ideal of A/I. Furthermore $A/I = (D + K)/I = ((D + I)/I) + (K/I)$. But from above we have $B = D + I$, so this is $A/I = B/I + K/I$. Also $(B/I) \cap (K/I) = (B \cap K)/I$. But $B = D + I$ so $(B \cap K)/I = ((D + I) \cap K)/I = (D \cap K + I)/I$. But $D \cap K$ is an ideal of A contained in D so $(D \cap K + I)/I \leq (D_A + I)/I$. Finally $D + I = B$ so $D_A + I \leq B$ and is an ideal of B, therefore $(D_A + I)/I \leq (B/I)_{A/I}$.

If $I \nleq K$, then $I \cap K = 0$ since I is minimal and $I \cap K$ is an ideal of A contained in I. Then $(I + K)/K$ is a minimal ideal of of A/K since I is a minimal ideal of A. So $(I + K)/K$ is a minimal ideal of A/K, which is nilpotent, thus $\dim(I) = 1$ and $AI \leq I \cap K = 0$. It follows that $I < C$ and $B = D$. We have $A = B + K$ and $B \cap K \leq B_A$, so $A/I = (B/I) + ((K + I)/I)$ and $(B/I) \cap ((K + I)/I) = (B \cap (K + I))/I = (B \cap K + I)/I \leq (B_A + I)/I \leq (B/I)_{A/I}$. So B/I is a c-ideal of A. $\square$

Lemma 5.1.4. *Let A be a Leibniz algebra in which every maximal subalgebra of each maximal nilpotent subalgebra of A is a c-ideal of A, and suppose that I is a minimal abelian ideal of A and M is a core-free maximal subalgebra of A. Then I is one-dimensional.*

Proof. We have that $A = I \oplus M$ and I is the unique minimal ideal of A (this comes from I minimal, M maximal and M core-free). Because I is abelian it is nilpotent, so we may choose C to be a maximal nilpotent subalgebra of A containing I. If $C = I$, then I is iteslf a maximal nilpotent subalgebra. Choose B to be a maximal subalgebra of I, so that $I = B + Fi$ because I is abelian, and $B_A = 0$ since I is a minimal ideal and B is contained in I. Then B is a c-ideal of A by our assumption, so there is an ideal K of A with $A = B + K$ and $B \cap K \leq B_A = 0$. But now $A = I + K = K$, giving $B = 0$ and $\dim I = 1$.

So suppose that $C \neq I$. Then $I \cap C = I$ and $A = I + M$ so $C = (I + M) \cap C = I + M \cap C$. Let B be a maximal subalgebra of C containing $M \cap C$. Then B is a c-ideal of A, so there is an ideal K of A with $A = B + K$ and $B \cap K \leq B_A$. If $I \leq B_A \leq B$, we have $C = I + M \cap C \leq B$, a contradiction since B is a maximal subalgebra of C. Hence $I \nleq B$, so $B \leq M \cap C$ thus $B_A = 0 = B \cap K$ and $A = B \oplus K$. Now $C = B + C \cap K$ and $B \cap C \cap K = B \cap K = 0$, so $C = B \oplus (C \cap K)$. As C is nilpotent this means that $\dim(C \cap K) = 1$. But $I \leq C \cap K$, so $\dim I = 1$, as required. □

We now prove Theorem 5.1.1.

Proof. Let A be a minimal counter-example, and let I be a minimal abelian ideal of A. Then A/I satisfies the same hypothesis, by Lemma 5.1.3. So A/I is supersolvable. We will prove that $\dim(I) = 1$ to conclude that A is supersolvable.

If there is another minimal ideal B of A, then $I \cap B = 0$ and $I \cong (I + B)/B \leq A/B$. But A/B is supersolvable by our assumption, so its minimal ideals are one-dimensional, and thus $\dim(I) = 1$. So we assume that I is the unique minimal ideal of A. Also, if $I \leq \phi(A)$, we have that $A/\phi(A)$ is supersolvable, so A is supersolvable by [Bur15]. So we assume that $I \nleq \phi(A)$. It follows that $A = I \oplus M$, where M is a core-free maximal subalgebra of L. So $\dim(I) = 1$ by Lemma 5.1.4.

Because A/I supersolvable, there is a chain of ideals

$$A/I = A_n/I \supseteq A_{n-1}/I \supseteq \cdots \supseteq A_1/I \supseteq A_0/I = 0$$

where each A_i/I is an i-dimensional ideal of A/I. Then

$$A = A_n \supseteq A_{n-1} \supseteq \cdots \supseteq A_1 \supseteq I = A_0 \supseteq 0$$

is a chain of ideals of A, with each A_i having dimension $(i+1)$ in the $(n+1)$-dimensional algebra A. So A is supersolvable. □

Example 5.1.5 gives a Lie counterexample to the converse of Theorem 5.1.1.

Example 5.1.5. Let $A = H \dotplus V$ be as in Example solvable-not-supersolvable. So A is solvable. The maximal nilpotent subalgebras of A have the form $M = M^{'} + V$, where $M^{'}$ is a maximal nilpotent subalgebra of H. Note that every maxial nilpotent subalgebra of H has dimension 2. Let M be a maximal nilpotent subalgebra of A. Let B be a maximal subalgebra of M. Then $B = M^{''} + V$ for some maximal subalgebra $M^{''}$ of $M^{'}$. So $\dim(M^{''}) = 1$. So $(M^{''})_A = M^{''}$ or 0. In the first case $M^{''}$ is a c-ideal. Otherwise $M^{''}$ is a one-dimensional subalgebra of H. Since $M^{''}$ is not an ideal we have $M^{''} = \text{span}\{h\}$ for some $h \notin H^2$.

5.2 Completions

A solvable Leibniz algebra is supersolvable if and only if every maximal subalgebra M satisfies $\dim(A/M) = 1$. In [Tow11], Towers proved that a Lie algebra L is supersolvable if and only if the ideal index $\eta(L : M) = 1$ for all maximal subalgebras M of L. We generalize this result to Leibniz algebras.

Recall from Definition 2.0.20 that the subalgebra C is a completion for M if $A = M + C$ and $k(C) \subseteq M$. The ideal index $\eta(A : M)$ from Definition 3.4.6 is the dimension of $C/k(C)$

for an ideal completion C of M.

Theorem 5.2.1. *A is supersolvable if and only if $\eta(L:M)=1$ for all maximal subalgebras M of A.*

Proof. Let A be a Leibniz algebra.

($\Rightarrow$) Suppose A is supersolvable. Let M be a maximal subalgebra of A. Then $\dim(A/M)=\eta(A:M)$ because A is solvable, and $\dim(A/M)=1$ because A is supersolvable. So $\eta(A:M)=1$.

($\Leftarrow$) Now suppose $\eta(A:M)=1$ for all maximal subalgebras M of A. Let $M'/Leib(A)$ be maximal in $A/Leib(A)$. Then M' is maximal in A so $\eta(A/Leib(A):M'/Leib(A))=\eta(A:M')=1$. So the Lie algebra $A/Leib(A)$ is supersolvable by [Tow11], Corollary 2.8. So A is solvable, thus $\dim(A/M)=\eta(A:M)=1$ for all maximal subalgebras M of A. So A is supersolvable. □

Example 5.2.2. The 2-dimensional cyclic Leibniz algebra with $[a,a^2]=a^2$ has two maximal subalgebras, span$\{a-a^2\}$ and span$\{a^2\}$, each of which satisfies $\eta(A:M)=1$. This is because A is the only ideal completion of each one-dimensional, maximal subalgebra, and $k(A)=Leib(A)=\text{span}\{a^2\}$. Then $\eta(A:M)=\dim(A/k(A))=1$.

Theorem 5.2.3. *If A has a supersolvable maximal subalgebra M with $\eta(A:M)=1$ and $Nil(A)\nsubseteq M$, then A is supersolvable. If A is non-Lie, the converse is true.*

Proof. ($\Rightarrow$) If A is ϕ-free, then $Nil(A)=Asoc(A)$, by [Bat13], Theorem 2.4. So there is a minimal abelian ideal of I of A with $I\nsubseteq M$. Then $A=M\oplus I$. Then I is an ideal completion for M with $k(I)=0$, so $\eta(A:M)=1=\dim(I/k(I))=\dim(I)$. Therefore $\dim(A/M)=1=\eta(A:M)$, so A is solvable by Corollary 4.1.2, and supersolvable by Theorem 5.2.1.

If A is not ϕ-free, then $\eta(A/\phi(A) : M/\phi(A)) = \eta(A : M) = 1$ by Corollary 3.6.7, and $Nil(A/\phi(A)) = Asoc(A/\phi(A)) = Nil(A)/\phi(A)$ by [Bat13], Theorem 2.4. So $A/\phi(A)$ satisfies the hypothesis of the theorem, and is ϕ-free. So by the preceeding paragraph $A/\phi(A)$ is supersolvable, whence A is supersolvable, by [Bur15].

($\Leftarrow$) We prove the contrapositive. Suppose that for all supersolvable maximal subalgebras M of A, either $\eta(A : M) \neq 1$ or $Nil(A) \subseteq M$. We show that A is not supersolvable.

Let A be a counterexample of minimum dimension. So A and $A/Nil(A)$ are supersolvable. We will show that $\dim(A/Nil(A)) < \dim(A)$ and that $A/Nil(A)$ satisfies the hypothesis of the theorem, then by minimality of A this will contradict that $A/Nil(A)$ is supersolvable.

If A has no supersolvable maximal subalgebras, then A is not supersolvable. So let M be a supersolvable maximal subalgebra of A. Since A is supersolvable, M is a c-ideal and $1 = \dim(A/M) = \eta(A : M)$. So we must have $Nil(A) \subseteq M$. So A is not nilpotent.

If A is non-Lie then $Nil(A) \neq 0$ so $\dim(A/Nil(A)) < \dim(A)$. If $M = Nil(A)$ then $A/Nil(A) \cong A/M$ is 1-dimensional and supersolvable, contradicting the minimality of A. So $Nil(A)$ is a proper subset of M. Then if $M/Nil(A)$ is a supersolvable maximal subalgebra of $A/Nil(A)$, $Nil(A/Nil(A)) = 0 \subset M/Nil(A)$, so $A/Nil(A)$ satisfies the hypothesis of the theorem and is not supersolvable, a contradiction. So A is not supersolvable. □

Remark 5.2.4. The first direction ($\Rightarrow$) of Theorem 5.2.3 is proven for Lie algebras in [Tow11]. The converse is proven here for Leibniz algebras A with $Nil(A) \neq 0$.

5.3 CAP-subalgebras

In Chapter 4, we proved that A is solvable if and only if every maximal subalgebra is a CAP-subalgebra; now we make a similar characterization of supersolvable Leibniz algebras. In [Tow15b], Proposition 2.9,Towers proved that if a Lie algebra L is supersolvable then every subalgebra is a CAP-subalgebra. In fact, the converse is true as well, which is Theorem 5.3.1.

Theorem 5.3.1. *A is supersolvable if and only if every subalgebra is a CAP-subalgebra.*

Proof. Let A be a Leibniz algebra.

($\Rightarrow$) Suppose A is supersolvable. Let B be a subalgebra of A, and C/D a chief factor. Then $\dim(C/D) = 1$. We will show that B either covers or avoids C/D. Suppose first that $B \cap C \subseteq D$. Then $B \cap C \subseteq B \cap D \subseteq B \cap C$, so that $B \cap C = B \cap D$, and B avoids C/D. Now suppose that $B \cap C \nsubseteq D$. Because $\dim(C/D) = 1$, we must have $C = D + (B \cap C)$, so that $C + B = D + B$, and B avoids C/D. So B is a CAP-subalgebra.

($\Leftarrow$) We prove the contrapositive. Suppose A is not supersolvable. We will construct a subalgebra B that is not a CAP-subalgebra. Because A is not supersolvable, we may choose a chief factor C/D with $\dim(C/D) > 1$. Let $c \in C$ such that $c \notin D$. If $c^2 \in D$, then let $B = D \cup \text{span}\{c\}$, which is a subalgebra because D is an ideal. If $c^2 \notin D$, then let $B = D \cup \text{span}\{c^2\}$, which is a subalgebra since $[c^2, c^2] = 0 \in D$. Then $B + D = B \neq C = B + C$ so B does not cover C/D. Also, $B \cap D = D \neq B = B \cap C$, so B does not avoid C/D. So if every subalgebra is a CAP-subalgebra, then A is supersolvable. $\square$

Example 5.3.2 illustrates why supersolvability, rather than solvability, is necessary in

Theorem 5.3.1. Because solvable and supersolvable Leibniz algebras coincide over fields of characteristic zero, our example is over a field of prime characteristic.

Example 5.3.2. Let $A = H + V$ as in Example 2.0.8. Then A is a solvable, non-Lie Leibniz algebra over a field of characteristic $p > 0$. A is not supersolvable. $V/\{0\}$ is a chief factor and any proper subset S of V is a (non-maximal) subalgebra which neither covers nor avoids $V/\{0\}$. So S is a subalgebra which is not a CAP-subalgebra. However, by Theorem 4.4.1 all maximal subalgebras of L are CAP-subalgebras.

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