Chapter 3

What is Mathematics? More Basics

Numbers are a part of mathematics; but unfortunately, when asked, too many people think that numbers are all there is to mathematics. When pressed, some say that geometry and algebra are part of mathematics; some even say that mathematics is a language – the language of science. I claim that mathematics includes all of these things but much more. Mathematics can arise in various forms in any human endeavor, some of which have nothing to do with numbers! I will partially demonstrate this claim in this book.

3.1 The Definition of Mathematics Used in this Book

Definitions. In formal mathematics one of the most important concepts is that of *definition*. Definitions, i.e., the meanings of words/symbols, stated as precisely as is humanly possible, are indispensable stones in the foundation upon which mathematics is built. Even though this book is not completely formal, I owe you the best definition of mathematics that I know. The following is the beginning not the end of a discussion; here is the definition of *mathematics* that I use in this book.

Definition. *Mathematics* is the search for and study of patterns.

For example, patterns in counting give rise to numbers, arithmetic and eventually number theory and algebra. Patterns in space give rise to geometry. Patterns in human thought give rise to mathematical logic. Patterns in motion, in part, give rise to calculus, cf., [136]. Patterns of patterns can be very interesting also. For me there are no limits, any pattern is worth a look – especially if it captures your curiosity.

Exercise 3.1 What is Mathematics?

(i) Consider the following quote: *"Thus (pure) mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."*(Bertrand Russell) What do you think this means? Comment.

(ii) Again, consider: *"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.'* (Albert Einstein, 1921). Is there any relationship of this quote of Einstein to the previous one due to Russell?

(iii) An amazing thing is that mathematics is so useful. I attribute this miracle to the fact that ultimately mathematics rests on patterns observed in Nature. A poetic statement of this miracle is: *"The mathematician may be compared to a designer of garments, who is utterly oblivious of the creatures whom his garments may fit. To be sure, his art originated in the necessity for clothing such creatures, but this was long ago; to this day a shape will occasionally appear which will fit into the garment as if the garment has been made for it. Then there is no end of surprise and delight."* See [126]. Comment.

(iv) A pattern is an example of a "simplification," an extraction of information common to a number of different situations, events, or processes. One hallmark of genius is the ability to sift through available data and extract what is "truly important," to simplify appropriately. Compare the following quote from [358], *"A good simplification should minimize the loss of information relevant to the problem of concern."* Does this indicate that starting with a problem can lead to patterns?

(v) Do any (or all) of the above quotes have anything to do with the search for and study of patterns?

When using pure mathematics or studying applications, always be as clear as possible in your own mind about the definitions of everything!

Axioms and Assumptions: Hidden or Not. If definitions are one type of foundation stone for mathematics, *axioms*, sometimes called *assumptions* are another. Axioms are statements that are assumed to be true; they are often statements of patterns. In pure mathematics the axioms are usually clearly stated, cf., II. When applying mathematics to a "real life" problem, there are often *hidden assumptions*. If hidden assumptions are not rooted out and made explicit, severe and unnecessary difficulties may interfere with the process of finding solutions. One of my favorite examples of hidden assumptions giving rise to difficulties is the following *actual* log of a radio transmissions between between a U.S. naval ship and the Canadian authorities off the coast of Newfoundland in October, 1995 (as released by the Chief of Naval Operations 10, 1995 and quoted in *The New York Times*, Sunday, July 5, 1998):

Canadians: Please divert your course 15 degrees to the south to avoid a collision.

Americans: Recommend you divert your course 15 degrees to the north to avoid a collision.

Canadians: Negative. You will have to divert your course 15 degrees to the south to avoid a collision.

Americans: This is the Captain of a U.S. Navy ship. I say again, divert YOUR course.

Canadians: No. I say again, you divert YOUR course.

Americans: THIS IS THE AIRCRAFT CARRIER U.S.S. LINCOLN, THE SECOND-LARGEST SHIP IN THE UNITED STATES' ATLANTIC FLEET. WE ARE ACCOMPANIED BY THREE DESTROYERS, THREE CRUISERS AND NUMEROUS SUPPORT VESSELS. I DEMAND THAT YOU CHANGE YOUR COURSE 15 DEGREES NORTH. I SAY AGAIN, THAT'S ONE FIVE DEGREES NORTH, OR COUNTER-MEASURES WILL BE UNDERTAKEN TO ENSURE THE SAFETY OF THIS SHIP.

Canadians: This is a lighthouse. Your call.

Another quite common hidden assumption that can shut down an urgently needed discussion/debate is the assumption that Aristotelian (yes-no, true-false, black-white) logic applies to a situation when actually fuzzy (or measured) logic which allows any numerical truth value between 0 (false) and 1 (true) is far more applicable, cf., Sections 1.4 and 8.2.

Exercise 3.2 All or Nothing Arguments are Usually Needless

(i) Person A insists that Mr. X, our local congressional representative, is an environmentalist because he has introduced many pro-environmental bills in the legislature, and because he loves the outdoors. Person B vehemently disagrees, saying no one who drives a gas guzzling vehicle as Mr. X does could be an environmentalist.

Use fuzzy/measured logic to more deeply analyze the above discussion and hopefully make it more productive.

(ii) Discuss: "You are either with me or against me." (Said by Roman Tribune Messala, representing the Roman Emperor, to Judah Ben-Hur, Charlton Heston, in the 1959 movie, *Ben-Hur*. Also uttered by some contemporary politicians.)

(iii) Suppose on a 0 to 1 scale of "objectivity values," if a statement is a "pure fact" or is "purely objective" it is assigned a 1. If it is "pure opinion" or "purely subjective" it is assigned a 0. Can you find a reasonably complicated sentence that is pure fact, i.e., purely objective? Who agrees/disagrees with you?

(iv) Suppose in a conversation/debate you are being critical of American (or fill in the blank with some other country) foreign and environmental policies. The other party turns hostile and says: "Can't you say anything good about America?" Using careful definitions and fuzzy logic how might you productively respond?

3.2 The Logic of Nature and the Logic of Civilization

A case can be made, and we do so in part in this book, that humans face some daunting problems: global warming with accompanying disruptions to climate and agriculture, collapse of most of the earth's wild fish populations, vanishing native forests, fresh water shortages, presence of persistent pollutants nearly everywhere, species extinctions, human populations beyond the carrying capacity of ecosystems that support them, and so on.

There are various possible futures that are mathematically possible. Many are grim, and were I writing a polemic I would now recount in detail the most ferocious possibilities to encourage action on the reader's part. However, with limited space and time I will instead focus on solutions. Human extinction, being one of the near term possible futures, I will make the assumption that most humans would like to avoid such.

Technology Will Rescue Us? There is a view with many adherents, some with best-selling books, that most if not all of our problems (will) have technological solutions. But consider that chlorofluorocarbons, CFCs, DDT, leaded gasoline were all widely heralded and appreciated technological solutions to certain pressing problems, cf., Section 4.5. Each of these solutions created problems, which in some cases were far greater than the initial problem that was "solved." If you would like to read an entire book of similar examples, see [681]. Thus we are dealing with a *pattern*! I enshrine this pattern as an axiom, which can also be considered a corollary[1] of the Connection Axiom:

[1]A corollary of a mathematical statement, like a theorem which has been proven or an axiom that has been assumed, is a proposition which logically follows from said statement, often with little proof required.

The Principle of Unintended Consequences: *In a complex system, it is not possible to do just one thing.*

Now from many perspectives it would not be appropriate to ban technology; besides, many of us find technology a lot of fun. In fact, there are examples of technology that have had immense positive consequences, like the polio vaccine. Thus an approach to this situation employed in many societies is the *Precautionary Principle*, which I invite you to investigate. This principle states, briefly, that the burden of proof of safety for the (commercial or widespread as opposed to experimental) introduction of a technology falls on those who would introduce said technology. Since every solution to a problem will have additional consequences, some unimaginable at the time, it is wise to take the time to investigate what those unintended consequences might be. A major difficulty is that very often fortunes are to be made or the benefits are so "obvious" with a new technology, that there is a rush to commercialization without adequate caution. Now, reliance on the precautionary principle may not avoid all technological disasters; no system is perfect; most species (probably ours included) will eventually go extinct. But using the Precautionary Principle greatly increases the chances that we will avoid technological disaster and might even postpone human extinction.

A Completely Different Approach – The Logic of Civilization Must be Compatible with the Logic of Nature. While pondering technological solutions, I suggest that there are problems whose very nature require a completely different approach. For example, suppose you are driving a bus down a long road greatly enjoying the scenery, the sense of motion, the wind blowing in the open windows. However, there is a brick wall at the end of this road that for some reason you cannot see, or refuse to see, even though a few (but only a few) of the many folks on board are shouting: "Slow down, turn off this road, we are going to hit that brick wall." Any improvements to your bus's efficiency, innovations that allow you to power the bus with wind and sun, and so on, do nothing to alter the fact that your trip will not end well – unless you stop and re-evaluate why it is that you are traveling down that particular road, in that particular direction.

I propose that we take the time to examine the fundamental *logic* of our civilization. What are our collective hidden assumptions? Do we need to change the way we think? Will this change in our thinking lead to changes in our behavior that Nature requires for our long term survival? I contend that from the point of view of humans, *Nature has its own logic.* There are laws of Nature which humans may or may not preceive – but nevertheless operate. The law of gravity is an example of a law of Nature that can be perceived at many different levels. The Spanish architect Gaudi built a multi-story apartment complex with a narrow, railingless, external staircase which Gaudi said brought the residents in touch with their own mortality as they climbed. This is a visceral level at which we all understand gravity. Newton created a mathematical level of the understanding of gravity with his inverse square law, and there is yet a more detailed level of understanding with relativistic refinements due to Einstein. There are laws of physics, chemistry, biology, economics, sociology, "human nature," mathematics and so on, which I assert are human efforts to understand the *logic of Nature*. I will discuss the role of the laws of thermodynamics in economics in VII, for example. Clearly an entire library of books could be devoted to this topic; however, I will restrict attention here to just one axiom that I see as essential to the long-term survival of humans. If adopted as a fundamental logical principle this axiom would completely change the way most humans now behave on earth. The consequences of this rather innocent axiom are truly revoluitonary.

The Bio-Copernican Axiom. *Humans are not the "center" of the biosphere,*[2] *i.e., earth's system of life. Humans are a part of Nature, not apart from Nature. Humans are subject to the laws of Nature.*

Notions equivalent to this axiom probably were (are?) part of the fabric of some indigenous peoples' cultures before contact and dissolution by "modern man." Before briefly discussing this axiom and some of its implications, I would like to revisit the history of the original Copernican Axiom, a history that provides contemporary insights that go deeper than a simple analogy.

Aristarchus of Samos (310 B.C.– ca. 230 B.C.) was a Greek astronomer and mathematician and the first person of which I am aware to present an explicit heliocentric, i.e., sun-centered, model of our solar system. His model was not adopted; instead, the geocentric, i.e., earth-centered, model of Aristotle and Ptolemy was accepted as the obviously correct model for millennia. Nicolaus Copernicus (1473–1543) redeveloped the theory of the heliocentric solar system and wrote a book in Latin on the subject, *On the Revolution of Heavenly Spheres*, published upon his death, since he apparently did not want to deal with the inevitable backlash from the powerful of his day who held steadfast to the geocentric model. His book provides many mathematical tables for doing astronomical calculations. One of the justifications for Copernicus's work – as he mentioned in a letter to Pope Paul III – was that the ecclesiastical calendar could be improved in accuracy and ease of calculation. For example, using Copernicus' sun-centered model, as opposed to the earth-centered model of Ptolemy, it was easier to calculate when Easter would occur. However, such mathematical arguments did not influence deeply held convictions about the centrality of the Earth in the universe, as the yet later experience of Galileo demonstrates.

Galileo (1564–1642) challenged the still dominant assumption that the earth is at the center of the universe; and it is well known that a long series of popes took a dim view of Galileo's theories. The Inquisition condemned Galileo in 1633 because his teachings were believed (by those in power) to clash with the *Bible*. The thinking of the time is reflected in the following two quotes from experts of the day. First, Scipio Chiaramonti, Professor of Philosophy and

[2] I define the biosphere as does V.I. Vernadsky in [692], a reprint of his 1926 edition.

Mathematics at the University of Pisa, 1633, [77, p. 6]: *Animals, which move, have limbs and muscles; the earth has no limbs and muscles, hence it does not move.*

Earlier, Martin Luther, leader of the Protestant Reformation, said (about 1543): *People give ear to an upstart astrologer [Copernicus] who strove to show that the earth revolves, not the heavens or the firmament, the sun and the moon. Whoever wishes to appear clever must devise some new system, which of all systems is of course the very best. This fool wishes to reverse the entire science of astronomy.*

Galileo was, in fact, only recently pardoned in 1992 by the Pope for his heresies 359 years before.[3]

What is not well understood is the fact that from the point of view of pure mathematics any point or celestial body can be taken to be the "center of the universe" or "the center of the solar system." What distinguishes the mathematical choice of taking the sun to be the center of the solar system is *simplicity*. Taking the earth as the center of the solar system leads to the math of Ptolemy. While the orbits of the planets are simple ellipses ("slightly flattened circles") in Copernicus's model, the planets in Ptolemy's model follow paths called (curtate and/or prolate) epitrochoids and/or hypotrochoids.[4] In order to reconcile the motion of the planets as viewed by Ptolemy with the motion of rocks and apples on earth, little "angels" were introduced at appropriate times to act on the planets.

It certainly was easier for Newton to discover his inverse square law for gravitation while studying the data accompanying the heliocentric model, as opposed to trying to extract that pattern from the trochoids of Ptolemy. In fact, it is not clear that the latter would have been possible then.

Exercise 3.3 The Logic of Civilization

(i) Given the geocentric and heliocentric models of the solar system, speaking as a pure mathematician, is one true and the other false? Is one far simpler and more useful than the other? Why do you think we use the heliocentric model now, whereas we did not for millennia?

(ii) In switching from the geocentric model to the heliocentric model were there political winners and losers? Did the social system evolve (with some lag-time) along with the science?

(iii) Switching between the heliocentric and geocentric models of the solar system is a simple process of "changing co-ordinates" for today's mathematical physicist. Asserting that humans are not the central/dominant/most important species of life on earth is not so easily viewed as a "change of co-ordinates." The Bio-Copernican Axiom involves biology, ecology – among other subjects. Is this Axiom true or false (or a fuzzy truth value between 0 and 1)? Is this Axiom a useful myth, like the Yam God of the Warlpiri, cf. page 11? Is this Axiom a destructive myth? Is this Axiom a mixture of morality and science?

(iv) What behavioral changes would you anticipate if we switched from an anthropocentric, i.e., human-centered world view, to an all-life-centered world view? What might

[3]Somewhat quicker, according to *The New York Times*, November 2, 2001, on Halloween, October 31, 2001, the state of Massachusetts officially exonerated five women who were tried and hanged as unrepentant witches on Gallows Hill in Salem, Massachusetts over three hundred years before in 1692.

[4]My mathematical analysis of these beautiful objects was deemed sufficiently complicated to be a high school science fair project!

prompt some subcultures to make this change? Would there be political winners and losers? Does your answer to this last question depend on how the change takes place? If this Axiom were adopted, what simplifications might it lead to (in analogy with the simplifications yielded by the heliocentric model)?

(v) Are there any problems we currently face that would become more amenable if all humans adopted the Bio-Copernican Axiom? Pick at least one such problem and its would-be solution in the Bio-Copernican society.

(vi) What cultural manifestations are there which indicate that humans think of themselves as the center of the biosphere, e.g., the "highest ranked, most important species?" Children (should?) learn early on that society does not "revolve" around them, i.e., to not be (totally) self-centered. Is the Bio-Copernican Axiom just an application of this process at the species level?

(vii) If all humans adopted the Bio-Copernican Axiom, what do you think the total human population on earth would be? Would it be greater, less than or equal to what it is at the moment?

(viii) If all humans assumed the Bio-Copernican Axiom, what cultural feedback mechanisms would there be to regulate the total human population? Why?

(ix) Pick a popular religion. Can that religion adapt and become compatible with the Bio-Copernican Axiom?

(x) Discuss the logical implications of *monoculture* in agriculture versus crop diversity. Which is more likely sustainable for long periods of time?

(xi) In what countries is the Precautionary Principle taken seriously? What have been its effects/consequences?

(xii) Find a country that does not take the Precautionary Principle seriously. What have been the effects/consequences?

(xiii) Geoengineering has been proposed to solve the global warming crisis. What are some examples of geoengineering? Are there (might there be) any unintended consequences?

(xiv) How many humans share the assumption that civilization must continue to burn fossil fuels in order to function? Are there widely held assumptions that are incompatible with the long-term survival of human civilization?

(xv) What do you think the implications of the Bio-Copernican Axiom are for the academic discipline of economics, cf., VII? Is material economic growth without bounds consistent with the Bio-Copernican Axiom? If humans accepted the Bio-Copernican Axiom would we "ask" before "taking," i.e, would we at least consider the effects of human actions on other life forms before acting?

(xvi) For a project, come up with a mathematical definition of a complex system and some axioms for a complex system, then try to prove, or at least give a convincing argument, that the Principle of Unintended Consequences is actually a theorem. (A theorem is a statement with a proof that relies on axioms, definitions, traditional logic and other theorems. "Little theorems" that help prove bigger theorems are called *lemmas*, in case you run into that term.)

3.3 Box-Flow Models

One-Box Models in Steady State. One of the simplest and most versatile tools for making models of certain parts of Nature is the *box-flow* model. Boxes, sometimes called *compartments*, are imaginary containers that contain some stuff, usually some type of matter, that is flowing in and flowing out of the container. There are many types of boxes. A few examples are the earth's atmosphere, the airshed over an ecosystem, the stratosphere, a university (with students flowing in and out), a single person, a lake, an anteater, a tree, a house.

Bathtubs. A simple box-flow model is a bathtub with one faucet, through which water flows in, and one drain, through which water flows out. If the rate at which water flows in is equal to the rate at which water flows out of the tub, I will say that our box-flow model is in *steady state.* The amount of stuff in the box, in this case the water in the tub, is called the *stock* of matter in the box.

Steady-State Residence Times. Thus far we have a box with a stock of matter in it and a single rate of flow of matter in/out of the box. There is a third concept associated with this model: *residence time* of the matter in the box. In the case of our bathtub with water flowing in and out at the same rate, I would like to have some idea of how long some bit of water remains in the tub before flowing out. Some bits of water might flow out very soon after having entered the tub, while others might hang around for a very long time before getting close to and then sucked down the drain. Now keeping track of all the individual bits of water is an exercise that is far more challenging than I want to attempt.[5] There is something much easier, but still informative, that we can do. Let's divide the number which represents the amount of water in the tub by the number that represents the rate of flow of the water, either in or out. The result will be a number which represents an amount of time. We can think of this as the average amount of time one might expect a bit of water to remain in the tub before going down the drain.

For a concrete example, suppose our tub has 50 *liters*[6] of water in it and water is flowing into, and out of, the tub at the rate of 5 *liters per minute*, i.e., $\frac{5\ liters}{minute}$. If I divide 50 *liters* by $\frac{5\ liters}{minute}$, viz., $\frac{50\ liters}{\frac{5\ liters}{minute}}$, I get 10 *minutes* for the answer. Note you can perform the last operation mechanically as follows: $\frac{50\ liters}{\frac{5\ liters}{minute}} = \frac{50\ liters}{1}\frac{minute}{5\ liters} = 10\ minutes$, where the *liters* cancel out and 50 is divided by 5.

I can summarize our discussion of this one-box model quite succinctly, if you allow me the use of a few symbols. Suppose I have one box, let M represent the amount of stuff or matter, i.e., the stock, that is in the box. This will be a number with some *units* attached, such as *liters* above. Let F_{in} be the rate of flow of matter into the box; let F_{out} be the rate of flow of matter out of the box. Using these symbols I can make two definitions:

Definition. A single box into which matter flows at rate F_{in} and out of which matter flows at rate F_{out} is in *steady state* if $F_{in} = F_{out}$.

[5] To do this we would have to know a lot about the tub. For example, What is the shape of the tub? Where are the drain and faucet located physically? Is anything stirring the water in the tub? How is the stirring being performed?

[6] A liter is a unit of volume equal to 1000 cubic centimeters, cf., Exercise 3.11 (ii). A liter is also equal to 1.057 U.S. quarts

Definition. If a box is in steady state with a stock M of matter in it and flows $F = F_{in} = F_{out}$, then the *residence time*, T, of matter in the box is $T = \frac{M}{F}$.
Thus for a box in steady state: $M = FT$, where the juxtaposition FT means F multiplied times T.

Boxes are Everywhere. The discussion of one-box models so far might be either so easy it is boring you to tears, or it might be a bit too brisk – or something in between.

If you find what we have done so far to be easy, I recommend you look at [283], [73], and the rest of this book as well. If the box is your house and the flowing matter is radon gas, you are looking at a problem of importance to your health. If the box is the airshed over the northeastern United States and the flowing matter is a certain type of pollution from coal-fired power plants, you are investigating acid rain. These are some of the box models in [283], there are others. Once you become aware of this tool you can find dozens of other box models. We will look at some later.

If you find what we have done so far is a bit overwhelming, using letters to stand for numbers, using units like *liters*, or if you are a bit rusty when it comes to multiplying and dividing, all is not yet lost. In the next section I will begin introducing units and such; and in Part II I will systematically study all these things and more; hopefully this will work for you. If not, don't hesitate to study with friends. When friends spend some of their time studying in small groups, everyone benefits. If you are still lost, there should be a hungry, mathematically talented person nearby who will help you for a fee.

Exercise 3.4 Some One-Box Models: Steady State and Not

(i) Suppose you have a hot tub which holds 1000 liters of water. Suppose water is flowing in, and out, of the tub at the rate of 25 liters per minute. What is the residence time of water in the tub?

(ii) Suppose you are in the situation described in part (i) except that the water in the tub is contaminated with a nasty bacteria floating around in the tub, none sticking to the tub. Suppose that the answer to part (i) is T. If you wait an amount of time T, will the tub be free of bacteria?

(iii) Suppose your hot tub has a capacity of 2000 *liters*, but at noon the tub is half full. Suppose also that water is flowing into the tub at $\frac{25\ liters}{minute}$ and that water is flowing out of the tub at $\frac{3\ liters}{minute}$. When will the tub be full to capacity?

(iv) Suppose a one-box model is in steady state with a given flow rate. If you increase the rate of flow while keeping the stock fixed, is there anything you can say about the residence time?

(v) If a national park biologist estimates that a typical bear in a certain park lives 25 years and that no bears leave the park permanently except by death of which there are 6 deaths per year, how many bears live in the park? Assume that the bear population is in steady state. This might not be true, but this assumption makes the problem easy.

(vi) Suppose your box is the earth's human population. The flow in is the birth rate, the flow out is the death rate. (Note: The number of people leaving the earth at this time by any other means than death is negligible. The number of people arriving on earth by any other means than birth on earth – well if you find such a person let me know immediately.) At the time you read this, is this system in steady state or not? See V.

Multi-Box Models in Steady State. To start with, suppose our university is one box with 25,555 students. If we assume a one-box, steady-state model, i.e., the stock is 25,555 students all of the same type (they all graduate), then I can calculate the residence time of the students (the typical length of time from entering to graduation) if I know the graduation rate. If we assume the graduation rate is 5,555 students per year, then you should be able to check that the residence time for this model is approximately 4.600360036 years, which I round off to 4.6 years.

Now a one-box model of a university, with a fixed population, students entering at a fixed rate and leaving at the same rate is not realistic. I could make this model a little bit more real by assuming that some first-year students flunk out at the end of their first year, but that everyone else graduates – and no one else flunks out at any other time. This leads us to the following:

Exercise 3.5 A Two-Box Model of a University

(i) Suppose our university has 25,555 students and that students leave only by graduating or by flunking out at the end of the first year – no sooner or later. Suppose that 555 students flunk out each year, and that the residence time for all students taken together is 4.6 years. What is the residence time of the students who graduate? Hint: Think of two boxes, a box that contains the students who flunk out and a box that contains the students who eventually graduate. A third box containing all of the students together is also of use.

(ii) In the model of part (i) there are 5,555 students entering each year. (Note: Actually the number is closer to 5555.434783, but fractional people do not exist.) In the first year then we have 5,555 students entering and 555 students leaving. What can you say about the number of students in years 2, 3, 4, and so on? How can you get a total of 25,555 students?

(iii) By using a tool called a spreadsheet, we can do far more complicated models of our university. For example, we could allow students to flunk out at the end of any year or enter at the beginning of any year (by transfer, say). By using a computer we could let each student be in his/her individual box! If you want to start learning how to do some of this immediately see V or [73].

Exercise 3.6 A Cycle with Two or More Boxes

(i) Imagine the following system: Two boxes, A and B, with matter, say water, flowing from box A to box B and vice versa. Suppose both boxes hold the same amount of water, M. If the system is in steady state in the long term, i.e., both boxes continue to hold the same amount of water, M, then what can you say about the flows between the boxes? Can you say anything about the residence times of water in box A relative to box B? This clearly is an example of a *cycle*.

(ii) Again imagine a two box system as in (i) except that Box A holds an amount of water M, and box B holds an amount of water 1000 M. If the system is in steady state, what can you say about the relative rates of flow from one box to another and the relative residence times of water in each box? Related to this exercise is Exercise 5.11.

(iii) Generalize this exercise by having a string of boxes in a circle, each box holding a different amount of water, with a circular flow of water that goes from one box to the next around the circle. If you assume the system is in steady state, i.e., the amount of water in each box does not change in time, can you say anything about the various rates of flow and the various residence times?

(iv) This example is a bit more complex. Suppose two pipes come out of box A, one to box B and another to box C. Then suppose that pipes come out of box B and box C both going to box D. Now assume that a pipe goes from box D to box A. If each box contains some fixed amount of water over time, is there anything you can say about the rates of flow and residence times? Can you make this problem more complicated?

3.4 Cycles and Scales in Nature and Mathematics

Nature's Cycles. Cycles are among the most basic patterns in Nature. Cycles inspired our ancestors to create a good deal of mathematics, some of the earliest mathematics. One of the first cycles of Nature that humans must have observed was that of night and day. Clearly associated with this cycle were the sun, moon, stars and planets. Depending on location, cycles of the seasons with associated weather cycles of rain/drought, heat/cold must have also become obvious if for no other reason than the availability of various foods often correlates with the aforementioned cycles – and humans must eat.[7]

If hunter-gatherers were aware of cycles, agriculturalists[8] were likely even more so. The knowledge of when to plant crops flowed from careful observations of the cycles of heavenly bodies and from concomitant calculations involving rather sophisticated symbols and mathematics. In some early civilizations a lot of social power came to those who were able to precisely deal with astronomical data. Imagine a priesthood that could predict eclipses. Such individuals might convince others that they were closer than most to the gods.[9]

Exercise 3.7 Ancient Mathematics

If you are interested in history, philosophy or cultural anthropology, the stories of how mathematics originated in various places around the world is fascinating. To name a few such cultures consider: the Aztecs and Mayans, in what is now Central America and parts of Mexico; the Incas, in what is now South America, Peru in particular; the Chinese; the Indians; the Greeks; the Sumerians, Babylonians to name only two peoples of the Middle East; the Egyptians, the people of the Great Zimbabwe, to name only two African peoples. Do not limit yourself to this list. For example, likely the oldest of all mathematics with more sophistication than simple counting was created by the Australian Aborigines. I discuss this in some depth in III. See [642], [485], [72], [17].

(i) Investigate the mathematics in two or more ancient cultures. Were there any common natural patterns that inspired the mathematics that you find?

(ii) Does mathematics depend on the culture in which it is developed?

(iii) Is there anything universal about mathematics that transcends the various symbolic systems invented in the various cultures?

(iv) Do you think that universal mathematical properties reflect aspects of the human mind, Nature, both, neither, something else?

(v) If you can find out much information like when to plant crops, when eclipses will occur, by careful observations of the heavens, why not get additional information about

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[7]There once was a group of people, self-named "'Breathairians," in my hometown that claimed they lived on air and water only – until one of the group was seen eating junk food at 2 a.m. at a gas station.

[8]The invention of agriculture, about 12,000 B.C. to 8000 B.C., had both positive and negative effects. *"There is some evidence that human health generally declined with the onset of agriculture."* See [169, p. 237]. In [169, p. 236] it also says: *"However it originated, agriculture started a positive feedback system that put humanity on the road to sociopolitical complexity."*

[9]Can you think of any contemporary "priesthoods"?

your personal life from the same place? This is my (untested) hypothesis of how astrology got started. Does careful observation of Nature confirm or contradict the accuracy of astrological predictions, such as occur in the daily horoscopes found in many newspapers?

The Biosphere and Matter Cycles. There are many other natural cycles of importance, and I will discuss some of them in this book. The *hydrologic cycle*, i.e., *water cycle*, consists of the flow of water among three main boxes of water and their subboxes, viz., the atmosphere (gas) box containing water vapor, the ice box containing water in its solid state, and the liquid box consisting of subboxes such as oceans, lakes, groundwater aquifers.

Though he was not the first to use the term, V.I. Vernadsky in the 1926 edition of [692] defined and described the "life box" or *biosphere* as that term is used today. Briefly, the biosphere is the box that contains all of earth's life forms and the parts of the earth that interact with life forms. Here the term earth includes its atmosphere. Vernadsky's book, available in Russian, French and now English is of current as well as historical interest. Realizing that there are some exceptions[10] to the idea that the biosphere is totally self-contained, I present an axiom:

Axiom on Matter Cycles. *Matter in the biosphere tends to flow in cycles that stay within the biosphere.*

Water cycles within the biosphere. Ecologists study how matter such as carbon cycles in the biosphere, and it is naturally called the *carbon cycle*. I will study the concept of *recycling*, which involves matter in forms referred to as resources and garbage, cf., V, [580], [567], [249]. Mathematics also deals with more abstract cycles that occur in the economy, history, electronics, sociology, politics and more. For example, is there any mathematical structure in [604]?

Cycles and Time. Cycles lead us to different ways of measuring time. First in history was *astronomical time*, which was the standard before 1971. The time it takes for one rotation of the earth about its axis is *one day*, the time for the earth to make one trip around the sun is *one year*. It is interesting to note that the convention of dividing hours into 60 minutes and a minute into 60 seconds goes back to ancient times, and the choice of the number 60 was mathematically motivated. The Babylonians, for example, probably chose the number 60 since they did many astronomical calculations; and the number 60 admits many divisors, minimizing the appearance of fractions.

Exercise 3.8 Astronomical Time

(i) Can you list all of the whole numbers bigger than zero that divide 60 without leaving a remainder? For example, 2 is such a number, since 60 divided by 2 is 30 without remainder.

(ii) Can you think of any real-life situations where astronomical time, i.e., dividing a year into 365 days, a day into 24 hours and so on, is not adequate?

(iii) An anthropologist once told me that the Innuit people, the indigenous people of Alaska and northern Canada, do not have a word for time. In Ethiopia the native time is

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[10] There are exceptions such as: astronauts leaving things on the moon; space probes which never return to earth; asteroids which come from outer space and collide with earth (some believe that life may have been initialized on earth in this way).

found by dividing the time from dawn to dusk into 12 equal parts, which works since they live near the equator. Can you find any other examples of cultures which treat or measure time differently from the way Americans do now?

Since 1971 there has been a new standard of time measurement, called *atomic time*.[11] In the early 1990s the standard became based on cycles or oscillations associated with the substance cesium. The smallest unit of cesium is called an atom of cesium; and the the fundamental unit of time measurement, *the international second*, was defined to be 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom. What this means in practice is that you can actually buy a cesium clock that counts the oscillations (periods) of the cesium-atom "pendulum," and it ticks off one second for each 9,192,631,770 oscillations.

In 1997 scientists at N.I.S.T. announced a new design for the atomic clock called the fountain clock, which is about 10 to 100 times more accurate than the cesium clock. In the journal *Science*, on July 12, 2001, a team of researchers from N.I.S.T. and Germany announced a mercury clock which is 1,000 times more accurate than the cesium clock. This clock makes use of laser technology to trap and cool a single mercury ion and count its 1,000 trillion oscillations per second. It is estimated that it will take until 2015 for this new clock to replace the cesium clock as the accepted standard.

Exercise 3.9 Atomic Clocks

(i) How much more accurate is the cesium clock than the quartz-crystal[12] clock ?
(ii) How much more accurate is the mercury clock than the quartz-crystal clock?

Measuring Length Using Time. The definition of *to measure* is "to compare to a standard." So, for example, not long ago the *meter*, a unit of length, was defined to be one ten-millionth part of the distance along a meridian of the distance between the equator to the pole.[13] For the record a $meter = 39.37\ inches$.[14] Today, the meter is defined to be the distance light travels in a vacuum in $\frac{1}{299{,}792{,}458}\ seconds$. This definition was adopted in October 1983, it is implemented in practice using laser technology.

The meter and the second are on what might be called a human scale, but their modern hyperaccurate definitions have left the realm of experience

[11] If you ever want to know the latest regarding measurements of time (or length) contact the National Institute of Standards and Technology, N.I.S.T.,(formerly the National Bureau of Standards) in Boulder, Colorado, cf., http://www.nist.gov.

[12] Note that the typical quartz-crystal wristwatch ticks off one second for about each 32,000 oscillations that it counts.

[13] *Webster's New Twentieth Century Dictionary*, Simon & Schuster, a Division of Gulf & Western Corp., 1979.

[14] The "standard" inch at one point was the length of the king's thumb, below the knuckle; the "standard" yard was the distance from the king's nose to the tip of the longest finger on his horizontally outstretched – to the side–arm and hand. Today, 1 $yard = 36\ inches$, and the inch has just been defined in terms of a meter.

for most of us. None of this would make sense were it not for the theory of relativity introduced by Albert Einstein in 1905. This theory postulates, and it is experimentally verifiable, that the speed of light in a vacuum is a universal constant. The speed of light in a vacuum is also independent of the frequency (color) of the light, as has been verified by astronomers who looked at the red and blue light coming from a distant binary star.

Starting in 1977 satellites were launched that became part of the *geographical information system*, i.e., GIS. A colleague of mine in the Physics Department, Prof. Neil Ashby, was a consultant to that project and had a difficult time convincing the government and private leaders that the system should be designed with relativity theory built in – they did not believe relativity was relevant to GIS. Fortunately they agreed to launch the satellites with a "switch" so that relativity could be turned on if necessary. The satellites were launched without relativity. The system did not work. Relativity was switched on and the system worked with amazing accuracy.

An example of the capabilities of the satellite systems is the work of another colleague of mine, Professor Kristine Larson, of Aerospace Engineering. She determined in collaboration with Stanford University Professor Jerry Freymueller, that the continents of Antarctica and Australia are moving apart 2 to 3 inches a year.[15]

I conclude this subsection on a personal note. One autumn I was climbing a small mountain in a national forest not far from where I am writing this. On top was a not-yet-completed communications facility, to be used for either cell phones, TV or GPS. Next to the modern facility was a taller structure, an old fire lookout tower, the pieces of which had been carried in on a narrow, minimally invasive trail built by the Civilian Conservation Corps many years ago. The pieces were then bolted together on site like a large version of a toy erector set. Unfortunately, the new building and our current culture caused a wide road to be bulldozed to the top of the peak to facilitate construction. Technology can be great, but we must be careful not to implement an electronic version of the following words of poet, Kahlil Gibran:

Trees are a poem the Earth writes across the Sky. Humanity cuts them down for paper so we may record our emptiness.

Scales, Units and Powers of 10. The meter and the minute are on a human scale. Soon we will develop other words and symbols to deal with very large and very small scales. But first I want to mention some facts about what I will call the American/English system of measurement (used only in the United States).

The American/English System of Units and Measurement. This system was developed with human dimensions in mind, as noted in the approximate

[15]The U.S. Department of Defense and the U.S. Department of Transportation operate what is known as the Global Positioning System, GPS, a system of 25 satellites coupled with thousands of receivers around the world. This system is used by a number of groups to better understand the earth, including, for example, earthquakes.

definition of the inch in terms of a thumb. The next scale up is that of the foot; it takes a counting cycle of 12 inches to get a foot, i.e., 1 $foot$ $indexfoot = 12$ *inches*. It takes a counting cycle of 3 to pass from the scale of one foot to that of a yard, i.e., 1 *yard* = 3 $feet$. There are 5280 $feet$ in a *mile*. Scales below that of the inch have a counting cycle of 2, i.e., $half$*-inch*, *quarter-inch*, *eighth-inch*, *sixteenth-inch*, and so on.

Human stomach size presumably led to the unit of volume called a *pint*. The next smaller scale is the *cup*, 2 *cups* = 1 *pint*. The next smaller scale is that of an *ounce*, i.e., 8 *ounces* = 1 *cup*, 16 *ounces* = 1 *pint*. Going up in scale, 2 *pints* = 1 *quart*, 4 *quarts* = 1 *gallon*.

Powers of 10. Before discussing the *metric system* of measurement, based on one counting cycle, namely 10, and used throughout the world, I need to give names to the various powers of 10. I assume you know what the number 10 means; and the fact that we are going to base our discussions in this section on the number 10 is due, perhaps, to the fact that most people have 10 fingers (and 10 toes). If we agree to use counting cycles of length 10 in going from one scale to the next, then the next scale up would be 10 *times* 10, or 100. This is written as 10^2 in mathematics. The 2 in 10^2 is called an *exponent*, and we say[16] that we "raised 10 to the power 2" to get 10^2. The exponent 2 in 10^2 tells us how many 10s we need to multiply together to get $10^2 = 100$, i.e., $10^2 = (10)(10) =$ *ten times ten* $= 100 =$ *one hundred*.

The next scale up would be (10)(100). Other ways to write this are: $(10)(100) = (10)(10)(10) = 10^3 =$ *one thousand* $= 10$ *raised to the power* 3. We multiply by 10 again to get the next scale up: $(10)(1000) = (10)(10)(10)(10) = 10^4 =$ *ten thousand* $= 10$ *raised to the power* 4. We could keep going, and we will, which leads mathematicians to write 10^n, where $n = 1, 2, 3, 4, 5, 6, ...$[17] This last step should not bother you if you realize that 10^n is just a shorthand way to talk about the various "powers of 10" without telling you which one in particular we are talking about. At first this may seem silly, but it turns out to be quite necessary. I will have a lot more to say about this notation in II. For now, please accept for a while that this way of writing/talking is essential. By the way, $10^1 = 10$. Thus the notation 10^n means "n tens multiplied together." I can also think of 10^n in another way. I write "1." to mean 1 followed by a decimal point. Then 10^n means "1. with the decimal point moved n places to the right, filling in with n zeros."

Now just as medical doctors use Greek and Latin words to refer to body parts and diseases, scientists (including mathematicians) use Greek words, actually Greek prefixes, to refer to the powers of 10 – not English. Thus

[16] Another way this operation is referred to in English is this: "we *exponentiated* 2 to base 10" to get 10^2. Thus "to exponentiate" means "to make an exponent out of."

[17] Note that 1,2,3,4,5,6, ... means 1,2,3,4,5,6, and so on. Three of the dots following the 6 means "and so on" or "keep going" or "keep counting with whole numbers." The last dot is the period for the sentence.

we will not refer to one thousand meters, but rather to *one kilometer*, i.e., 10^3 *meters*. See Table Greek Prefixes below.

10^1	deka (da)	10^{-1}	deci (d)
10^2	hecto (h)	10^{-2}	centi (c)
10^3	kilo (k)	10^{-3}	milli (m)
10^6	mega (M)	10^{-6}	micro (μ)
10^9	giga (G)	10^{-9}	nano (n)
10^{12}	tera (T)	10^{-12}	pico (p)
10^{15}	peta (P)	10^{-15}	femto (f)
10^{18}	exa (E)	10^{-18}	atto (a)

Table Greek Prefixes

Notice in Table Greek Prefixes that negative powers of 10 appear, for example, 10^{-1}. The number 10^{-1} means "one tenth" or "one divided by 10." Thus we can write 10^{-1} in several ways: $10^{-1} = .1 = \frac{1}{10} = \textit{one tenth}$. What is 10^{-2}? I will explain this in more detail in II, but the following are all equal: $10^{-2} = 1 \textit{ divided by } 10^2 = 1 \textit{ divided by } 100 = \textit{one one} - \textit{hundredth} = \frac{1}{100} = (\frac{1}{10})(\frac{1}{10}) = (.1)(.1) = (.1)^2 = .01$.

For 10^{-3} we have: $10^{-3} = 1 \textit{ divided by } 10^3 = 1 \textit{ divided by } 1000 = \textit{one one-thousandth} = \frac{1}{1000} = (\frac{1}{10})(\frac{1}{10})(\frac{1}{10}) = (.1)(.1)(.1) = (.1)^3 = .001$.

The meaning of 10^{-n} is similar to the meaning of 10^n: 10^{-n} means "1 divided by 10^n." It also means n $\frac{1}{10}s$ multiplied together, i.e., $(\frac{1}{10})^n = (.1)^n$. Regarding decimal representations, 10^{-n} is "1. with the decimal moved n places to the left, filling in with $n-1$ zeros." (Question: How many zeros are in the decimal representation of 10^7 versus 10^{-7}?)

Finally, $10^0 = 1$. This is natural if you give it some thought. I will discuss this a bit more in II.

The Metric System. To discuss units and measurement in the metric system we must be familiar with the names of the powers of 10, both positive and negative. Thus, for example, 1 *centimeter* = 1 cm = 10^{-2} *meters*. One *meter* = 1 *m*. One *millimeter* = 1 *mm* = 10^{-3} *m*. One *kilometer* = 1 *km* = 10^3 *m*. One *gigabyte* = 10^9 *bytes*. One *terawatt* = 10^{12} *watts*.

If you know how to measure lengths, then you should be able to measure areas and volumes. For example, a square that is one *meter* on a side has an area of one m^2, i.e., "one square meter" or "one meter squared." This same square is 100 *cm* on a side, so $1\ m^2 = (100\ cm)\ \textit{times}\ (100\ cm) = 10,000\ \textit{square cm} = 10^4\ cm^2$. A cube each edge of which is one *meter* long has a volume of one *cubic meter* = $1\ m^3$. Each edge of this cube is 100 *cm* long, so the volume of this cube in cubic centimeters is (100 *cm*) *times* (100 *cm*) *times* (100 *cm*) =

$1,000,000\ cm^3 = 10^6\ cm^3$. Actually drawing pictures of the relevant squares and cubes might be helpful in visualizing what I have just done.

Measuring Mass in the Metric and American/English Systems. After measuring time, lengths, areas and volumes, there is one basic type of measurement left, viz., that of mass. In the metric system the basic unit of mass is the *gram*$= 1\ g$. In the American/English system the basic units of mass are the *once* $= oz$ or the *pound* $= 1\ lb$. There are 16 oz in 1 lb. One *kilogram* $= 10^3\ g = 1\ kg = 2.2\ lb$.

There is one fact from basic science that we will use now and then. A cube of pure water 1 cm on a side, i.e., 1 cm^3 of water, has a mass of 1 $gram = 1\ g$. In the American/English system this fact becomes 1 *pint* of water has a mass of 1 *pound*. The rhyme: "a pint's a pound the world around" is one way to remember this. My students usually find the American/English system more confusing than the metric system for the purposes of doing mathematics, so I will minimize my use of the former.

Exercise 3.10 How Much Water Can a Pickup Truck Hold?

(i) If a metric tonne is 10^3 kilograms, how much volume is occupied by 1 metric tonne of pure water? Can you express your answer in terms of cubic meters, i.e., m^3? Can you express your answer in terms of pounds?

(ii) Based on your answer to (i) do you think that the back of a full-sized pickup truck could hold a tonne of watermelons?

Exercise 3.11 Units and Powers of 10

(o) Draw a picture as accurately as you can of one square centimeter, that is 1 cm^2. Make a model with paper and sticky tape of one cubic centimeter, i.e., 1 cm^3.

Draw a picture as accurately as you can of a square which is 3 cm on a side; count how many square centimeters are inside of this square.

Draw a cube that is 3 cm on a side; find the number of cubic centimeters inside of this cube.

(i) How many centimeters are in a meter? meters in a kilometer? grams in a kilogram? cm^2 in a m^2 (square centimeters in a square meter)? cm^3 in a m^3 (cubic centimeters in a cubic meter)?

(ii) One liter, denoted l, is $10^3\ cm^3$. How many l are in a m^3? One tonne, also called a metric ton (MT), is 10^3 kilograms. How many grams in one MT?

(iii) Convert the following American English words into powers of 10: one, ten, one hundred, one thousand, one million, one billion, one trillion.

(iv) Convert the following American English words into powers of 10: one tenth, one hundredth, one thousandth, one millionth, one billionth, one trillionth.

(v) Note that an nm is a nanometer (or one billionth of a meter), and a μm is a micrometer, or micron. Find the American English word equivalent to nm, cm, Gm, Tm, mm, μm, pm, dm, km, hm, dam, Em, am.

(vi) Express in decimal form, and in at least one other form, the following: $\frac{1}{10}$, $\frac{1}{10^{-1}}$, $\frac{1}{10^{-5}}$, $\frac{1}{10^8}$, 10^{-9}, 10^7. Hint: $\frac{1}{10^{-n}} = \frac{1}{(\frac{1}{10^n})} = (1)(\frac{10^n}{1}) = 10^n$.

(vii) Express in exponential form, i.e., 10 to some power, the following:

$$(a)\ .0000000000000000001,\ (b)\ 100,000,000,000,000,000,000.$$

and (c)

$$(1,000,000,000,000,000,000,000,000,000.)(.00000000000000000000000000001)$$

Do you see how exponential notation saves time?

(viii) Express as a single power of 10: $(10^2)(10^{-7})$, $\frac{10^{56}}{10^{60}}$, $(10)(10^7)(10^{-3})(\frac{1}{10^{-6}})$.

(ix) What is $(10^{-1})^2$? Review the two paragraphs just before this exercise.

(x) Write as a single power of 10: $(10^3 10^4)^2$, $(\frac{10^4}{10^{-3}})^3$, 1,

$$\frac{(\frac{10^{-2}}{10^3})}{(\frac{10^2}{10^{-90}})}.$$

(xi) What are the following: $\frac{100}{100}$, $\frac{10^5}{10^5}$, $\frac{10^{-7}}{10^{-7}}$, $(10^3)(10^{-3})$, $(10^n)(10^{-n})$, 10^0, $(10^n)^0$?

(xii) Since too many of my students have missed the following in the past I will ask again. How many cm^2 are there in $1\,m^2$? How many cm^3 are there in $1\,m^3$? Draw a picture of a square meter and a cubic meter and really understand what you are doing. Hint: Can you visualize a square meter as an array of little square centimeters, arranged in rows with 100 rows, and 100 little squares in each row? So the total number of little square centimeters would be 100 in the first row plus 100 in the second row plus ... plus 100 in the last, that is, one-hundredth row. Can you similarly visualize a cubic meter as an array of little cubic centimeters arranged in 100 "slices" with 100 times 100 little cubic centimeters in each slice?

Exercise 3.12 Accuracy

I once read on a cereal box an "educational"comment which said that the metic system was superior to the English system of measurement because the metric system was more accurate. Do you agree?

Exercise 3.13 Unit Conversions in America

Note that if you forget some fact, such as there are 12 *inches* in a *foot*, you can find the information you need in a dictionary or on the Web (or you can look in the index of this book to find the relevant page).

(i) How many cubic inches are there in a quart? Hint: $264.2\ U.S.\ gallons\ = 35.31\ cubic\ feet$.

(ii) How many square inches are in a square foot? an acre? a square mile? Hint: $1\ acre = 43,560\ square\ feet$. Also, there are 5280 *feet* in a *mile*.

(iii) How many *grams* of water are there in a *pint* of water?

Exercise 3.14 The Concept of Scale is Important in Science

(i) Is there a natural scale in the design of animals? If you scale up a person's height, thickness and width by a factor of 2, i.e., you double height, thickness, and width, by what factor is that person's mass scaled up? Are there any implications for the relative size of the bones in our person, before and after scaling up by a factor of 2?

(ii) Some mathematical laws of science are unchanged (at least approximately) by a change in scale, others are. Find an example of each, cf., [613, p. 63]. Hint: Look for power laws.

Orders of Magnitude, Numbers in Standard Form, Significant Digits. Before we look at some examples of Nature at various scales, I want to remind you about how scientists write numbers. In this book we will very often but not always write numbers in *standard form.* To write a number in standard form you do the following: move the decimal so that one (nonzero) digit stands to the left of the decimal and then multiply the number by the appropriate power of 10 – so that the resulting number you end up with is the same as the number you started with. In general, if you have a number and you create a new, smaller number by moving the decimal n places to the left, you can recover the original number by multiplying the new number by 10^n. If you have a number and you make a new, bigger number by moving the decimal n places to the right, you can recover the original number by multiplying the new number by 10^{-n}.

Thus given the number 31415.667, we write this in standard form[18] as follows: $3.1415667(10^4)$. Another example: $.0067 = 6.7(10^{-3})$. Finally, $.00093(10^2) = 9.3(10^{-4})(10^2) = 9.3(10^{-2})$. When a number is written in standard form you can immediately see two things. First, you can see the *order of magnitude* of the number; this is *the exponent of the power of ten closest to our number*. This will be the exponent of the power of ten of the number in standard form (plus 1 if the number to the left of the power of 10 is 5.0 or more).[19] For example, $3.9(10^6)$ has order of magnitude 6; $5.0(10^6)$ has order of magnitude 7; and $5.9(10^{-3})$ has order of magnitude -2.

Second, you can see the *significant digits* of the number, i.e., the digits used to the left of the power of ten. Thus $3.1415667(10^4)$ has 8 significant digits and is of order of magnitude 4, $6.7(10^{-3})$ has 2 significant digits and has order of magnitude -2, and $9.3(10^{-2})$ has 2 significant digits and order of magnitude -1. Our definitions of "significant digits" and "order of magnitude" are not complete or perfect; for example, see part (iii) of exercise below. The idea that we are trying to communicate is this: when a scientist writes a number, which comes from an actual measurement, in standard form he/she is not supposed to write a digit if it is not meaningful or significant, i.e., if it is not actually measured. There is an additional complication: all measurements involve error. So a number is not complete unless you are told how much error there might be. For the most part we are going to ignore the complications caused by error in measurements.

Exercise 3.15 Significant Digits and Orders of Magnitude

(i) Write the number π in standard form with 8 significant digits. (Feel free to use a calculator or a book to look up the number.)

(ii) Write the number .0000000985 in standard form. How many significant digits are there? What is the order of magnitude?

(iii) Write the number 102,000 in standard form. How many significant digits are there? Can there be more than one possible correct answer to this question? What additional information do you need in order to know for sure how many digits are really significant? What is the order of magnitude of this number? Does the order of magnitude depend on any of the things you had to consider to determine the correct number of significant digits?

Nature's Smaller Scales. Let's look at some examples of Nature at different scales. First, the human scale is roughly of order of magnitude 0. Most

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[18]Do not forget that multiplication of 3.1415667 by 10^4 can be written in at least three equivalent ways as $3.1415667 * 10^4$, as $3.1415667\ 10^4$ or as $3.1415667(10^4)$.

[19]This is a simple and coarse way to "round off." Many others prefer the more delicate approach. For example, a number between 0.3 and 3 is "rounded" to $10^0 = 1$, a number between 3 and 30 is rounded to 10^1 and so on.

adult humans are from $1(10^0)$ to $2(10^0)$ *meters* tall. Recall that $10^0 = 1$, see Exercise 3.11 (xi). The limit of resolution[20] of the unaided eye is about .1 mm or 10^{-4} *meters*. The compound light microscope can resolve points about 400 times closer than can the naked eye, i.e., the light microscope has a limit of resolution of about $2.5(10^{-7})$ m. Now .1 mm is 100 microns (μm), and 10^{-7} m is .1 microns. At this scale of Nature we find the hvuman egg and the amoeba (size 100 μm), liver cell (20 μm), red blood cell (7 μm), and typhoid bacillus (.2 to .5 μm).[21]

I have a friend, David, who is a cell biologist. He works in the world of the living cell, and he talks about motor molecules and microtubules. He uses an electron microscope with a theoretical limit of resolution of about 2.5 (10^{-12}) m. In practice, however, he works with cell parts from 1 micron to 1 nanometer (10^{-9} m) in size. On this scale of nature we find the typical bacterial virus (80 millimicrons) and the haemoglobin molecule (7 millimicrons).

Exercise 3.16 Millimicrons and Nanometers

(i) What is the relationship between a millimicron and a nanometer?

(ii) What is nanotechnology and what effects might it have on your life? Nanoparticles are likely added to some commercial products, which ones? Is nanotechnology regulated? Might there be (are there) any unintended consequences?[22]

(iii) The smallest known life form is *Mycoplasma*, diameter 200 nm. What is the status of research on *nanobacteria*, ($10 - 200$ nm), at the time you read this? See the Jan. 2010 *Scientific American*, for example. What is the status of research on *prions*, (13 nm), when you read this? While not considered "living" are nanobacteria and prions "on the border" between animate and inamimate? Might there be a fuzzy boundary between animate and inanimate that has made it difficult thus far to give a precise definition of *life*?

At 10^{-9} meters we are getting down to the scale of molecules. A *molecule* of a chemical is the smallest amount of the chemical that has the defining properties of the chemical. As an example there is an experiment that you can actually do to measure the size of an oil molecule, see Exercise 13.12. One order of magnitude less, 10^{-10} m, and we are on the scale of an atom, the building blocks from which molecules are made. The most elementary chemicals are called *elements*. All chemicals are made of elements, but the elements cannot be broken down into simpler constituents by any chemical process. An atom of an element is the smallest amount of that element that can exist. For

[20] The "limit of resolution" is defined to be the minimum distance between two points which can be discerned as separate entities.

[21] The most common waterborne disease in the United States today is Giardiasis, caused by a protozoan *Giardia lamblia*. Giardiasis is common in hikers and backpackers who do not treat their water. This little protozoan is just a "few" microns in diameter. Some water filters can remove all water borne objects larger than a "few" microns. If you rely on such a water filter, it is important to know the size (in microns) of the smallest particle that can pass through it. Just how big is *Giardia lamblia* anyway?

[22] The following paper indicates that nanoparticles already in common household items, e.g., cosmetics, sunscreen, vitamins, toothpaste, and hundreds of other produces, have caused genetic damage in mice: http://www.eurekalert.org/pub_releases/2009-11/uoc–nui111609.php

example, the hydrogen atom, denoted H, the carbon atom, denoted C, and the oxygen atom, denoted O, are all about 10^{-10} m in diameter.[23] At the atomic and molecular scale 10^{-10} m is such a convenient unit that it is given its own name. We define 10^{-10} m to be 1 *angstrom*, which is denoted by the symbol 1 Å. The angstrom is named after Anders Jöns Angström, a Swedish physicist at the University of Uppsala in the 19^{th} century who studied light. The Å unit is used to measure the wavelength of light. It is interesting to note that at the atomic and subatomic scale the distinction between "wave-like" phenomena and "particle-like" phenomena is not as hard and fast as it appears to us at the human scale. In a branch of physics called quantum mechanics one learns that very small particles often act as though they are waves. In fact the electron microscope mentioned above works on the principle that very fast (high energy) electrons act like waves with a very short wavelength (shorter than the wavelengths of visible light for example) and hence this beam of electrons can be used to look at very small things that can not be seen with an ordinary microscope that sees things using light.

Quantum mechanics also tells us that waves, such as light waves, can exhibit particle behavior. Have you ever heard of photons ("particles of light")? In this world of the very small we have to develop an entirely new intuition about how things behave. Mathematics is an invaluable tool for building this intuition.

Exercise 3.17 What are the Smallest Things that Affect Your Life?

(i) What are the smallest measurements of length, area, volume, time and mass that you have ever heard of, seen or used – outside of this book?

(ii) What are the smallest things that have ever had a measurable impact on your life? For example, see Exercise 3.16 and Section 1.6.

(iii) The scale between the size of an atom and the size of a bacteria is called the *mesoscale*. It is a transition zone between classical physics and quantum mechanics, and is the zone of "programmable atoms," which is somewhere between serious physics and science fiction. What are programmable atoms? See [419].

Even in countries that have adopted the metric system[24] the people who build furniture and houses often use systems of measurement that were used before the metric system was introduced.[25] Thus it should not come as a surprise that when Rich and Chris built some cabinets for me they used American/English units of measurement. I learned that fine cabinetry leaves gaps no larger than $\frac{1}{128}th$ of an inch. Rich will say things like: "You can sometimes get away with $\frac{1}{64}th$ of an inch, but I can always see gaps that big and I fill 'em."

[23] See the periodic table of the elements in on page 253 for the various kinds of atoms/elements.

[24] Newspapers reported on October 1, 1995, that England has gone nearly all metric. Pints of beer and milk will still be available. The change over to metric measurement has not been met with universal approval.

[25] Such is the case in Norway, for example, where units of measurement as old as the Vikings still exist.

When I built my own house with Richard, Joe, Lee and Fred, getting the rough framing within $\frac{1}{16}th$ of an inch was good, within $\frac{1}{32}nd$ of an inch was overdoing it; and at the end of a long hard day with the wind howling, a thunderstorm pounding down, and almost in the dark with a warped piece of wood, $\frac{1}{8}th$ of an inch, plus or minus, was looking mighty fine.

Exercise 3.18 Nonmetric Units Persist

(i) Convert the English unit measurements used in cabinetry and carpentry, as mentioned above, to the metric system. Recall that 1 $cm = 10^{-2}$ m and that 1 $in = 2.54$ cm.

(ii) Why did I use American/English units to build my house?

(iii) Why do you think that pubs in England still serve pints of ale, even after officially converting to the metric system in 1995?

(iv) Why do you think that nonmetric units persist in some realms but not in others? Why do we still use 24 hours in a day (60 minutes in an hour) and not 10 (or 100)?

Nature's Larger Scales. The area of the earth is about $5.10(10^{14})$ m^2. The mass of the earth is about $5.98(10^{24})$ kg. The mass of the atmosphere is about $5.14(10^{18})$ kg.[26] The volume of water in the oceans is about $1.35(10^{18})$ m^3. These are the kinds of numbers we need in order to talk about things on the global scale.

Amory Lovins, at the Rocky Mountain Institute in Snowmass, Colorado, often talks about power in terawatts (a terawatt is 10^{12} $W = 1$ TW) and energy in gigajoules (a gigajoule is 10^9 $J = 1$ GJ) when he talks about energy consumption and the economy of the United States.

Between the global and human scales are Nature's ecosystems. The terms ecosystem, ecological system and environmental system will be synonymous for us. These words are not used consistently by everyone. For the sake of at least the appearance of precision I will take Howard T. Odum's, [503], definition of ecosystem: "An organized system of land, water, mineral cycles, living organisms, and their programmatic behavioral control mechanisms."

Forests, seas and the earth's entire biosphere are examples of large ecosystems. A pond, coral head and an aquarium are all examples of small ecosystems. Before passing in the next paragraph to outer space I would like to remark that one of my motivations for writing this book is clearly stated in the following quote from H.T. Odum's book: "..., and there is growing recognition that humans are incomplete without the life support of self-maintaining natural ecosystems."

An astrophysicist friend of mine, Professor Ellen Zweibel, University of Wisconsin, studies cosmology, the sun and the stars among other things. She routinely talks of distances in light-years. A light-year is the distance light travels in one year. The speed of light in metric units is about $3(10^8)$ m/sec.[27] In fact, she even speaks of *parsecs* and *megaparsecs* (a *megaparsec* is 10^6 *parsecs*).

[26] Recall that kg stands for kilograms. How many grams are in a kilogram? Don't forget Table Greek Prefixes as you read this subsection.

[27] More precisely, the speed of light in a vacuum is 299,792,458 m/sec. Where in this book have you see this number before?

The star nearest to our sun is 1.2 *parsecs* away, and 1 *parsec* = 3.26 *light-years*.[28] She tells me that the sun's mass is $1.989(10^{33})$ g. She also studies magnetic fields in space, which are measured in terms of a unit called a *gauss*. Between galaxies the magnetic field is very weak, 10^{-9} *gauss*; and on a neutron star the magnetic field is a *teragauss*.

Exercise 3.19 What are the Largest Things that Affect Your Life?

(i) What is the largest length, area, volume or mass that you have ever heard of, seen or used – outside of this book?

(ii) What is the largest object or process that has ever affected you personally?

(iii) What Greek prefix expresses 10^{-9} *gauss*? How many *gauss* is a *teragauss*?

(iv) How many *kilometers* are in a *light-year*?

(v) Express the speed of light in *miles/sec*.

(vi) How many *miles* are in a *light-year*?

(vii) How many *tonnes* of mass does the sun have? A *tonne* = 10^3 kg. Can you express the sun's mass using some Greek prefixes?

(viii) This exercise compares the global scale to the atomic scale. Calculate the surface area of the earth in square *angstroms*, $Å^2$. Calculate the volume of the earth in cubic *angstroms*, i.e., $Å^3$, and in cm^3. Hint: The earth is (not quite) a sphere, with a radius at the equator of $6.38(10^6)$ m and a radius at the poles of $6.36(10^6)$ m. The volume of a sphere of radius R is $\frac{4}{3}\pi R^3$. What is the area of a sphere of radius R?

3.5 The Art of Estimating

Sometimes you would like to count or otherwise measure something which at first seems "unknowable," or at least difficult or impossible to look up. If you stop and think about what you do know, perhaps even do a little data gathering and/or make some educated guesses, you often can give a rather solid justification for an estimate of the number or measure you are looking for. The following exercise was once suggested to me by the late David Brower.

Our First Estimate: Something to Chew on. Suppose you wanted to know how many times your tongue "gets out of the way of your teeth" while you are eating in, say, one day. This may not be the most pressing issue in your (my) life, but it is certainly a number that is not easily looked up. And I want to know it.

Let's begin. I will suppose that today has been a "serious" eating day, with three full meals reasonably leisurely eaten. For breakfast I had three bowls of cereal and some fruit. I spent 15 minutes at breakfast, with 10 minutes devoted to chewing at the rate of about 50 chews per minute. That gives me 500 chews for breakfast. Lunch was moderate, lasting about 20 minutes, with about 15 minutes devoted to chewing. That gives 750 chews for lunch. Dinner lasted about 35 minutes with four courses of Chinese mixed vegetables, rice

[28] A *light-year* is the distance light travels in one year.

and dessert. About 25 minutes were devoted to chewing. This gives 1250 chews for a total of $1250 + 750 + 500 = 2500$ chews. I did not bite my tongue at all today, so my tongue successfully got out of the way of my chewing teeth 2500 times today.

Now suppose I missed a meal, or one meal became an extended, ravenous feast. It would be easy to adjust the estimates. The main point is that these estimates can be made, and they are not devoid of meaning. In fact, a series of estimates based on slightly altered assumptions and educated guesses will yield *an interval of numbers.* Such a set of numbers is probably more realistic than any one number might be. Finally, you do not want to ascribe more meaning to these estimates than is really there. For example, in the above example, at most two digits are significant, possibly only one is.

Estimating is a Mixture of Art and Science. The process of estimating is not carefully defined and the meaning of your estimate depends entirely on the amount and quality of thought you put into making the estimate. The least meaningful estimate is a wild guess, with nothing to substantiate it. An estimate is not just the number, or interval of numbers, you end up with. It is that, plus the reasoning process you have to back up your estimate(s). Use your critical thinking skills to make the following estimates.

Exercise 3.20 Making Estimates

(i) Estimate the number of people that are within an hour's walk from where you are right now.

(ii) Estimate the number of people who were within an hour's walk of your present location 25 years ago and 100 years ago.

(iii) Estimate the number of medical doctors now in the United States. (Try to do this estimate without directly consulting the Web or the American Medical Association. After you have your own estimate, check it with these two and/or other sources.)

(iv) Estimate the number of television sets in the United States today.

(v) (a) Estimate the number of four-year colleges in the United States today.

(b) Estimate the fraction of the population in the United States that has the opportunity to attend a four-year college – for four years (whether or not they take advantage of it).

(c) Estimate the fraction of the world's (human) population that has the opportunity to attend a four-year college. You may want to look at V for data you can use to estimate the world's human population.

(vi) If you have decided "what you want to be when you grow up," estimate the number of job openings of the type you desire in the year you will be looking for employment. Also estimate the number of people that will be looking for that same type of job.

(vii) Estimate the number of revolutions of your bicycle (or car) tire it takes to wear your tread down by a thickness of 100 Å.

(viii) In a popular book, [551], some frightening afflictions, like the Ebola virus disease, endemic to tropical rain forests are discussed. Estimate the minimum number of hours it would take for an Ebola virus to travel from its native habitat deep in the jungles of Africa to your home. Make two estimates, one before and one after a road is built into the jungle where the Ebola virus lives. I call these estimates *connection times.*

(ix) Estimate the number of people in the world whose birthday falls on the day you happen to be reading this.

(x) Estimate the number of hours Americans spent preparing their income tax forms this year.

(xi) Estimate the number of billionaires in the world, in the United States, in Mexico and India at the time you read this.

(xii) Estimate how much you are paying for each class period where you are studying.

Using Estimation Techniques to Check "Facts." Sometimes you are presented with a "fact" or argument that seems suspicious. Using estimating techniques and/or a little mathematics you can often verify, debunk or otherwise check the reasonableness of the information being presented. The following is a somewhat famous quote from an article titled: "On the American Pet," which appeared in the December 23, 1974 edition of *Time Magazine.*

"[There are] one hundred million dogs and cats in the U.S. . . . Each day across the nation, dogs deposit an estimated four million tons of feces."

You should get in the habit of not taking everything you hear or read for granted, even – or especially – when numbers are involved. Thus it is with the above quote from a major news source. Are the numbers reasonable? Off the top of my head, 10^8 dogs and cats is a reasonable number compared to the U.S. population in 1974 of roughly $2(10^8)$ people. The $4(10^6)$ tons of feces, however, seems a bit high. Is it?

Well, suppose for the moment that all of the 10^8 dogs and cats are dogs. A *ton* is 2000 *pounds* in the American/English system of units, so our 10^8 dogs are (according to the quote) putting out $4(10^6) * 2000 = 8(10^9)$ *pounds* of feces each day. Thus, on average, a single dog is producing $\frac{8(10^9)}{10^8} = 80$ *pounds* of feces per day! Actually, the article is asserting a greater production rate per dog, since not all of the 10^8 pets are dogs.

Exercise 3.21 A Mistake in Major Media Involving Dogs and Cats

(i) If half of the pets in question are dogs and half are cats, what is the average daily production of dog feces per dog according to the above quote from *Time Magazine*?

(ii) Assume that the article had a misprint. Suppose that the last line of the article says: *"Each day across the nation, dogs and cats deposit an estimated four million tons of feces."* Suppose that there are an equal number of cats and dogs in the U.S. and that dogs produce on average three times the feces that cats do on average. What then is the daily feces production per dog?

Well, you get my point. No matter how you look at them, the numbers are ridiculous. Not every mistake or use of spin is so obvious. Accuracy, being one with what is, these are the hallmarks of scholarly journals, which often lose entertainment value in the drive for precision. Most of our media fall short of such scholarly standards, as do even some scholarly works.

Sometimes misleading mathematics is more subtle. Consider the following material, brought to my attention by www.fair.org, that comes from an article titled: "What Price for Good Coffee?" which appeared in the October 5, 2009 edition of *Time.*

The article's lead sentence is: *"Fair Trade practices were created to help small farmers. But they may have hit their limits."* The article then goes on to provide the following information which for the moment I accept as accurate. Fair Trade pays \$1.55 per lb. for coffee from small farmers, almost 10% more than the market price. Then Fair Trade researcher, Christopher Bacon of the University of California, Berkeley, says that (for various reasons) at least \$2 per lb. is needed if small farmers are to rise above subsistence level. It is also noted that if the farmer gets \$1.55 per lb., the retail customer pays \$10 per lb.

Later, the article interviews a coffee drinker and says: *... "The company declined to comment on whether Fair Trade's benefits fall short of its vision or how much it would need to raise prices if **coffee** were to climb to \$2 per lb. Fair Trade 'isn't the only reason I drink Starbucks, but it's a big one,' says Connie Silver, a nurse, sipping a large, \$4.15 Frappuccino outside a Miami store. Asked if she'd pay, say \$4.50 or even \$5 to help absorb higher Fair Trade prices, Silver raises her eyebrows and says, 'Wow, these days, that's a tough one.' "*

My conclusion from a cursory reading of this article is that if the Fair Trade price paid the small farmer goes up \$.45 per lb, the price of 1 Frappuccino will go up \$.35 to \$.85, and this is untennable, especially for the nonrich. But let's take a closer look at the math here. The price paid for a cup of coffee includes many things, only one of which is coffee: for example, milk, sugar or any other ingredients, the paper cup, the rent for the coffee shop, the wages of various people involved in bringing the coffee from the farmer to the shop, and so on. A quick estimate (other details are discussed in the following exercise) goes as follows. Suppose you can get 20 Frappuccinos out of 1 lb. of coffee. Then the intrinsic cost increase in the price for *one* Frappuccino due to the coffee would start at around $\frac{\$.45}{20} \approx \$.02$ *or* \$.03. So depending on the mathematics we get two opposite conclusions!

Exercise 3.22 Is Fair Trade Affordable?

(i) Using the same approach as above, what is the increase in cost of one Frappuccino if you only get 10 from 1 lb. of coffee? Same problem, if you get 25 Frappuccinos from a lb. of coffee? Same problem, if you get 30 Frappuccinos from a lb.?

(ii) Suppose retailers look at the problem as follows. If \$1.55 is paid per lb. wholesale, the retailers charge \$10 per lb. If the same percentage markup is passed on at \$2 per lb. paid to the small farmer, what is the new retail price of 1 lb. of coffee?

(iii) If you can get 20 Frappuccinos from 1 lb. of coffee, and the total price of 20 Frappuccinos is \$83, what percentage of this \$83 is for the coffee, if the coffee retails for \$10 per lb.?

(iv) If 12% of the price of a cup of coffee is for the coffee and the price of coffee goes up 30%, what would be the increase in the \$4.15 price for a cup of coffee? This is the highest estimate, assuming all costs are passed on with the same retail markup rate. Why is this unlikely if most consumers are truly unable to afford such an increase? Do large chain coffee shops pay retail or wholesale prices for coffee? What is the answer to this exercise if only 1.9% of the price of a cup of coffee is for the coffee.

(v) Investigate how much of a pound of coffee is used by your local coffee shop to make a large Frappuccino. Incidentally [521, pp. 8–11] gives a set of numbers concerning coffee independent of the above discussion.

Try your estimation skills on the following.

Exercise 3.23 Do Americans Really Watch That Much TV?

(i) In his book, [441], Jerry Mander states that the typical American sees 21,000 television advertisements a year. Estimates usually are more meaningful when stated as intervals of numbers, say 17,000 to 25,000 advertisements, for example, in this case. Adjusting for the fact that Mander is writing in 1992, do you think his estimate is accurate? What interval of numbers do you get in estimating the number of TV ads seen by a "typical" American in 1 year?

(ii) How many TV ads do you see in 1 year?

(iii) How many Americans do you estimate see between 0 and 100 T.V. ads in 1 year?

The above exercise was relatively easy. The following exercise is more interesting and a bit more difficult. References include [63, 499, 104, 327, 14, 83, 516, 158].

Exercise 3.24 Prisoners and Farmers in America

(i) Estimate the number of Americans incarcerated at various times during the last century up to the present. Be careful to distinguish among inmates of federal prisons, state prisons, jails (city and county), and folks on probation or parole.

(ii) Estimate the number of American farmers at various times from 1900 to the present. Define farmer to be a person who declares farming to be their principal occupation.

(iii) At one point in the last century the number of American farmers was far, far higher than the number of American prisoners, no matter how the latter might have been counted. For some time the number of American farmers has been less than the number of Americans incarcerated. Does this mean that at some time these two numbers were approximately equal? Why? If so, at what time(s) were these two numbers equal?

(iv) Pick a state, estimate the cost per year for that state to maintain one prisoner.

(v) For the same state, estimate the cost per year for one student to attend a state university.

(vi) Estimate the number of people in the United States on death row (executed or not) who were or are innocent. Hint: see "innocence project."

(vii) Compare incarceration rates for various countries around the world, i.e., the number incarcerated for every 100,000 people. Which country has the highest incarceration rate? Why?