Further linear algebra. Chapter V. Bilinear and quadratic forms.

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1 Matrix Representation.

Definition 1.1 *Let V be a vector space over k. A* bilinear form *on V is a function $f : V \times V \to k$ such that*

- $f(u + \lambda v, w) = f(u, w) + \lambda f(v, w)$;
- $f(u, v + \lambda w) = f(u, w) + \lambda f(u, w)$.

I.e. $f(v, w)$ is linear in both v and w.

An obvious example is the following : take $V = \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x, y) = xy$.

Notice here the difference between linear and bilinear : $f(x, y) = x + y$ **is linear,** $f(x, y) = xy$ **is bilinear.**

More generally $f(x, y) = \lambda xy$ is bilinear for any $\lambda \in \mathbb{R}$.

More generally still, given a matrix $A \in M_n(k)$, the following is a bilinear form on k^n:

$$f(v, w) = v^t A w. = \sum_{i,j} v_i a_{i,j} w_j, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

We'll see that in fact all bilenear form are of this form.

Example 1.1 *If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then the corresponding bilinear form is*

$$f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1x_2 + 2x_1y_2 + 3y_1x_2 + 4y_1y_2.$$

Recall that if $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for V and $v = \sum x_i b_i$ then we write $[v]_\mathcal{B}$ for the column vector

$$[v]_\mathcal{B} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Definition 1.2 *If f is a bilinear form on V and $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for V then we define the matrix of f with respect to $\mathcal{B}$ by*

$$[f]_\mathcal{B} = \begin{pmatrix} f(b_1, b_1) & \ldots & f(b_1, b_n) \\ \vdots & & \vdots \\ f(b_n, b_1) & \ldots & f(b_n, b_n) \end{pmatrix}$$

Proposition 1.2 *Let $\mathcal{B}$ be a basis for a finite dimensional vector space V over k, $\dim(V) = n$. Any bilinear form f on V is determined by the matrix $[f]_\mathcal{B}$. Moreover for $v, w \in V$,*

$$f(v, w) \quad = \quad [v]^t_\mathcal{B} [f]_\mathcal{B} [w]_\mathcal{B}.$$

Proof. Let

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n,$$

so

$$[v]_\mathcal{B} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Similarly suppose

$$[w]_\mathcal{B} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then

$$
\begin{aligned}
f(v,w) &= f\left(\sum_{i=1}^{n} x_i b_i, w\right) \\
&= \sum_{i=1}^{n} x_i f\left(b_i, w\right) \\
&= \sum_{i=1}^{n} x_i f\left(b_i, \sum_{j=1}^{n} y_j b_j\right) \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j f\left(b_i, b_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j f\left(b_i, b_j\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i y_j \\
&= [v]_{\mathcal{B}}^t [f]_{\mathcal{B}} [w]_{\mathcal{B}}.
\end{aligned}
$$

$\square$

Let us give some examples.

Suppose that $f\colon \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by

$$
f\left(\left(\begin{array}{c} x_1 \\ y_1 \end{array}\right), \left(\begin{array}{c} x_2 \\ y_2 \end{array}\right)\right) = 2x_1x_2 + 3x_1y_2 + x_2y_1
$$

Let us write the matrix of f in the standard basis.

$$
f(e_1, e_1) = 2, f(e_1, e_2) = 3, f(e_2, e_1) = 1, f(e_2, e_2) = 0
$$

hence the matrix in the standard basis is

$$
\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}
$$

Now suppose $\mathcal{B} = \{b_1, \dots, b_n\}$ and $\mathcal{C} = \{c_1, \dots, c_n\}$ are two bases for V. We may write one basis in terms of the other:

$$
c_i = \sum_{j=1}^{n} \lambda_{j,i} b_j.
$$

The matrix

$$M = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{1,n} \\ \vdots & & \vdots \\ \lambda_{n,1} & \dots & \lambda_{n,n} \end{pmatrix}$$

is called the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. It is always an invertible matrix: its inverse in the transition matrix from $\mathcal{C}$ to $\mathcal{B}$. Recall that for any vector $v \in V$ we have

$$[v]_\mathcal{B} = M[v]_\mathcal{C},$$

and for any linear map $T : V \to V$ we have

$$[T]_\mathcal{C} = M^{-1}[T]_\mathcal{B}M.$$

We'll now describe how bilinear forms behave under change of basis.

Theorem 1.3 (Change of Basis Formula) *Let f be a bilinear form on a finite dimensional vector space V over k. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases for V and let M be the transition matrix from $\mathcal{B}$ to $\mathcal{C}$.*

$$[f]_\mathcal{C} = M^t[f]_\mathcal{B}M.$$

Proof. Let $u, v \in V$ with $[u]_\mathcal{B} = x$, $[v]_\mathcal{B} = y$, $[u]_\mathcal{C} = s$ and $[v]_\mathcal{C} = t$.

Let $A = (a_{i,j})$ be the matrix representing f with respect to $\mathcal{B}$.
Now $x = Ms$ and $y = Mt$ so

$$\begin{aligned} f(u, v) &= (Ms)^t A(Mt) \\ &= (s^t M^t)A(Mt) \\ &= s^t(M^t AM)t. \end{aligned}$$

We have $f(b_i, b_j) = (M^t AM)_{i,j}$. Hence

$$[f]_\mathcal{C} = M^t AM = M^t[f]_\mathcal{B}M.$$

□

For example, let f be the linear form from the previous example. It is given by

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

in the standard basis. We want to write this matrix in the basis

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The transition matrix is :

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

it's transpose is the same. The matrix of f in the new basis is

$$\begin{pmatrix} 6 & 3 \\ 5 & 2 \end{pmatrix}$$

2 Symmetric bilinear forms and quadratic forms.

As before let V be a finite dimensional vector space over a field k.

Definition 2.1 *A bilinear form f on V is called* symmetric *if it satisfies $f(v, w) = f(w, v)$ for all $v, w \in V$.*

Definition 2.2 *Given a symmetric bilinear form f on V, the* associated quadratic form *is the function $q(v) = f(v, v)$.*

Notice that q has the property that $q(\lambda v) = \lambda^2 q(v)$.

For exemple, take the bilinear form f defined by

$$\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}$$

The corresponding quadratic form is

$$q(\begin{pmatrix} x \\ y \end{pmatrix}) = 6x^2 + 5y^2$$

Proposition 2.1 *Let f be a bilinear form on V and let $\mathcal{B}$ be a basis for V. Then f is a symmetric bilinear form if and only if $[f]_{\mathcal{B}}$ is a symmetric matrix (that means $a_{i,j} = a_{j,i}$.).*

Proof. This is because $f(e_i, e_j) = f(e_j, e_i)$. $\square$

Theorem 2.2 (Polarization Theorem) *If $1 + 1 \neq 0$ in k then for any quadratic form q the underlying symmetric bilinear form is unique.*

Proof. If $u, v \in V$ then

$$\begin{aligned} q(u+v) &= f(u+v, u+v) \\ &= f(u,u) + 2f(u,v) + f(v,v) \\ &= q(u) + q(v) + 2f(u,v). \end{aligned}$$

So $f(u,v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$. $\square$

Let's look at an example :
Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

it is a symmetric matrix.

Let f be the corresponding bilinear form. We have

$$f((x_1, y_1), (x_2, y_2)) = 2x_1x_2 + x_1y_2 + x_2y_1$$

and

$$q(x, y) = 2x^2 + 2xy = f((x, y), (x, y))$$

Let $u = (x_1, y_1), v = (x_2, y_2)$ and let us calculate

$$\frac{1}{2}(q(u+v) - q(u) - q(v)) = \frac{1}{2}(2(x_1+x_2)^2 + 2(x_1+x_2)(y_1+y_2) - x_1^2 - 2x_1y_1 - 2x_2^2 - 2x_1y_2)$$

$$= \frac{1}{2}(4x_1x_2 + 2(x_1y_2 + x_2y_1)) = f((x_1, y_1), (x_2, y_2))$$

If $A = (a_{i,j})$ is a symmetric matrix, then the corresponding form is

$$f(x, y) = \sum_i a_{i,i}x_iy_i + \sum_{i<j} a_{i,j}(x_iy_j + x_jy_i)$$

and the corresponding quadratic form is

$$q(x) = \sum_{i=1}^{n} a_{i,i}x_i^2 + 2\sum_{i<j} a_{i,j}x_ix_j$$

then the symmetric matrix $A = (a_{i,j})$ is the matrix representing the underlying bilinear form f.

3 Orthogonality and diagonalisation

Definition 3.1 *Let V be a vector space over k with a symmetric bilinear form f. We call two vectors $v, w \in V$* orthogonal *if $f(v, w) = 0$. It is a good idea to imagine this means that v and w are at right angles to each other. This is written $v \perp w$. If $S \subset V$ be a non-empty subset, then the* orthogonal complement *of S is defined to be*

$$S^{\perp} = \{v \in V : \forall w \in S, w \perp v\}.$$

Proposition 3.1 *$S^{\perp}$ is a subspace of V.*

Proof. Let $v, w \in S^{\perp}$ and $\lambda \in k$. Then for any $u \in S$ we have

$$f(v + \lambda w, u) = f(v, u) + \lambda f(w, u) = 0.$$

Therefore $v + \lambda w \in S^{\perp}$. $\square$

Definition 3.2 *A basis $\mathcal{B}$ is called an* orthogonal basis *if any two distinct basis vectors are orthogonal. Thus $\mathcal{B}$ is an orthogonal basis if and only if $[f]_{\mathcal{B}}$ is diagonal.*

Theorem 3.2 (Diagonalisation Theorem) *Let f be a symmetric bilinear form on a finite dimensional vector space V over a field k in which $1+1 \neq 0$. Then there is an orthogonal basis $\mathcal{B}$ for V; i.e. a basis such that $[f]_{\mathcal{B}}$ is a diagonal matrix.*

Notice that the existence of an orthogonal basis is indeed equivalent to the matrix being diagonal.

Let $B = \{v_1, \ldots, v_n\}$ be an orthogonal basis. By definition $f(v_i, v_j) = 0$ if $i \neq j$ hence the only possible non-zero values are $f(v_i, v_i)$ i.e. on the diagonal.

And of course the converse holds : if the matrix is diagonal, then $f(v_i, v_j) = 0$ of $i \neq j$.

The quadratic form associated to such a bilinear form is

$$q(x_1, \ldots, x_n) = \sum_i \lambda_i x_i^2$$

where λ_is are elements on the diagonal.

Let U, W be two subspaces of V. The sum of U and W is the subspace

$$U + W = \{u + w : u \in U, w \in W\}.$$

We call this a direct sum $U \oplus W$ if $U \cap W = \{0\}$. This is the same as saying that ever element of $U + W$ can be written uniquely as $u + w$ with $u \in U$ and $w \in W$.

Theorem 3.3 (Key Lemma) *Let $v \in V$ and assume that $q(v) \neq 0$. Then*

$$V = \mathrm{Span}\{v\} \oplus \{v\}^{\perp}.$$

Proof. For $w \in V$, let

$$w_1 = \frac{f(v,w)}{f(v,v)}v, \quad w_2 = w - \frac{f(v,w)}{f(v,v)}v.$$

Clearly $w = w_1 + w_2$ and $w_1 \in \mathrm{Span}\{v\}$. Note also that

$$f(w_2, v) = f\left(w - \frac{f(v,w)}{f(v,v)}v, v\right) = f(w,v) - \frac{f(v,w)}{f(v,v)}f(v,v) = 0.$$

Therefore $w_2 \in \{v\}^{\perp}$. It follows that $\mathrm{Span}\{v\} + \{v\}^{\perp} = V$. To prove that the sum is direct, suppose that $w \in \mathrm{Span}\{v\} \cap \{v\}^{\perp}$. Then $w = \lambda v$ for some $\lambda \in k$ and we have $f(w, v) = 0$. Hence $\lambda f(v, v) = 0$. Since $q(v) = f(v, v) \neq 0$ it follows that $\lambda = 0$ so $w = 0$. $\square$

Proof. [of the theorem] We use induction on $\dim(V) = n$. If $n = 1$ then the theorem is true, since any 1×1 matrix is diagonal. So suppose the result holds for vector spaces of dimension less than $n = \dim(V)$.

If $f(v, v) = 0$ for every $v \in V$ then using Theorem 5.3 for any basis $\mathcal{B}$ we have $[f]_{\mathcal{B}} = [0]$, which is diagonal. [This is true since

$$f(e_i, e_j) = \frac{1}{2}(f(e_i + e_j, e_i + e_j) - f(e_i, e_i) - f(e_j, e_j)) = 0.]$$

So we can suppose there exists $v \in V$ such that $f(v, v) \neq 0$. By the Key Lemma we have

$$V = \mathrm{Span}\{v\} \oplus \{v\}^{\perp}.$$

Since $\mathrm{Span}\{v\}$ is 1-dimensional, it follows that $\{v\}^{\perp}$ is $n - 1$-dimensional. Hence by the inductive hypothesis there is an orthonormal basis $\{b_1, \ldots, b_{n-1}\}$ of $\{v\}^{\perp}$.

Now let $\mathcal{B} = \{b_1, \ldots, b_{n-1}, v\}$. This is a basis for V. Any two of the vectors b_i are orthogonal by definition. Furthermore $b_i \in \{v\}^{\perp}$, so $b_i \perp v$. Hence $\mathcal{B}$ is an orthogonal basis. $\square$

4 Examples of Diagonalisation.

Definition 4.1 *Two matrices $A, B \in M_n(k)$ are* congruent *if there is an invertible matrix P such that*

$$B = P^t AP.$$

We have shown that if $\mathcal{B}$ and $\mathcal{C}$ are two bases then for a bilinear form f, the matrices $[f]_{\mathcal{B}}$ and $[f]_{\mathcal{C}}$ are congruent.

Theorem 4.1 *Let $A \in M_n(k)$ be symmetric, where k is a field in which $1 + 1 \neq 0$, then A is congruent to a diagonal matrix.*

Proof. This is just the matrix version of the previous theorem. □

We shall next find out how to calculate the diagonal matrix congruent to a given symmetric matrix.

There are three kinds of row operation:

- swap rows i and j;
- multiply $row(i)$ by $\lambda \neq 0$;
- add $\lambda \times row(i)$ to $row(j)$.

To each row operation there is a corresponding elementary matrix E; the matrix E is the result of doing the row operation to I_n. The row operation transforms a matrix A into EA.

We may also define three corresponding column operations:

- swap columns i and j;
- multiply $column(i)$ by $\lambda \neq 0$;
- add $\lambda \times column(i)$ to $column(j)$.

Doing a column operation to A is the same a doing the corresponding row operation to A^t. We therefore obtain $(EA^t)^t = AE^t$.

Definition 4.2 *By a* double operation *we shall mean a row operation followed by the corresponding column operation.*

If E is the corresponging elementary matrix then the double operation transforms a matrix A into EAE^t.

Lemma 4.2 *If we do a double operation to A then we obtain a matrix congruent to A.*

Proof. EAE^t is congruent to A. $\square$

Recall that a symmetric bilinear forms are represented by symmetric matrices. If we change the basis then we will obtain a congruent matrix. We've seen that if we do a double operation to matrix A then we obtain a congruent matrix. This corresponds to the same quadratic form with respect to a different basis. We can always do a sequence of double operations to transform any symmetric matrix into a diagonal matrix.

Example 4.3 *Consider the quadratic form $q(x,y)^t = x^2 + 4xy + 3y^2$*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This shows that there is a basis $\mathcal{B} = \{b_1, b_2\}$ such that

$$q(xb_1 + yb_2) = x^2 - y^2.$$

Notice that when we have done the first operation, we have multiplied A on the left by $E_{2,1}(-2) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ and when we have done the second, we have multiplied on the right by $E_{2,1}(-2)^t = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

We find that

$$E_{2,1}(-2)AE_{2,1}(-2)^t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence in the basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

the matrix of the corresponding quadratic form is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example 4.4 *Consider the quadratic form* $q(x, y)^t = 4xy + y^2$

$$\begin{aligned}
\begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \quad \rightarrow & \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \\
\rightarrow & \quad \begin{pmatrix} 1 & 2 \\ 0 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -4 \end{pmatrix} \\
\rightarrow & \quad \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

This shows that there is a basis $\{b_1, b_2\}$ *such that*

$$q(xb_1 + yb_2) = x^2 - y^2.$$

The last step in the previous example transformed the -4 into a -1. In general, once we have a diagonal matrix we are free to multiply or divide the diagonal entries by squares:

Lemma 4.5 *For* $\mu_1, \dots, \mu_n \in k^\times = k \setminus \{0\}$ *and* $\lambda_1, \dots, \lambda_n \in k$

$$D(\lambda_1, \dots, \lambda_n) \quad \textit{is congruent to} \quad D(\mu_1^2\lambda_1, \dots \mu_n^2\lambda_n).$$

Proof. Since $\mu_1, \dots, \mu_n \in k \setminus \{0\}$ then $\mu_1 \cdots \mu_n \neq 0$. So

$$P = D(\mu_1, \dots, \mu_n)$$

is invertible. Then

$$\begin{aligned}
P^t D(\lambda_1, \dots, \lambda_n) P &= D(\mu_1, \dots, \mu_n) D(\lambda_1, \dots, \lambda_n) D(\mu_1, \dots, \mu_n) \\
&= D(\mu_1^2\lambda_1, \dots, \mu_n^2\lambda_n).
\end{aligned}$$

□

Definition 4.3 *Two bilinear forms* f, f' *are* equivalent *if they are the same up to a change of basis.*

Definition 4.4 *The rank of a bilinear form* f *is the rank* $[f]_\mathcal{B}$ *for any basis* $\mathcal{B}$.

Clearly if f and f' have different rank then they are not equivalent.

5 Canonical forms over $\mathbb{C}$

Definition 5.1 *Let q be a quadratic form on vector space V over $\mathbb{C}$, and suppose there is a basis $\mathcal{B}$ of V such that*

$$[q]_{\mathcal{B}} = \begin{pmatrix} I_r & \\ & 0 \end{pmatrix}.$$

We call the matrix $\begin{pmatrix} I_r & \\ & 0 \end{pmatrix}$ a canonical form of q (over $\mathbb{C}$).

Theorem 5.1 (Canonical forms over $\mathbb{C}$) *Let V be a finite dimensional vector space over $\mathbb{C}$ and let q be a quadratic form on V. Then q has exactly one canonical form.*

Proof. (Existence) We first choose an orthogonal basis $\mathcal{B} = \{b_1, \dots, b_n\}$. After reordering the basis we may assume that $q(b_1), \dots, q(b_r) \neq 0$ and $q(b_{r+1}), \dots, q(b_n) = 0$. Since every complex number has a square root in $\mathbb{C}$, we may divide b_i by $\sqrt{q(b_i)}$ if $i \leq r$.

(Uniqueness) Change of basis does not change the rank. $\square$

Corollary 5.2 *Two quadratic forms over $\mathbb{C}$ are equivalent iff they have the same canonical form.*

6 Canonical forms over $\mathbb{R}$

Definition 6.1 *Let q be a quadratic form on vector space V over $\mathbb{R}$, and suppose there is a basis $\mathcal{B}$ of V such that*

$$[q]_{\mathcal{B}} = \begin{pmatrix} I_r & & \\ & -I_s & \\ & & 0 \end{pmatrix}.$$

We call the matrix $\begin{pmatrix} I_r & & \\ & -I_s & \\ & & 0 \end{pmatrix}$ a canonical form of q (over $\mathbb{R}$).

Theorem 6.1 (Sylvester's Law of Inertia) *Let V be a finite dimensional vector space over $\mathbb{R}$ and let q be a quadratic form on V. Then q has exactly one (real) canonical form.*

Proof. (existence) Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be an orthogonal basis. We can reorder the basis so that

$$q(b_1), \ldots, q(b_r) > 0, \quad q(b_{r+1}), \ldots, q(b_{r+s}) < 0, \quad q(b_{r+s+1}), \ldots, q(b_n) = 0.$$

Then define a new basis by

$$c_i = \begin{cases} \frac{1}{\sqrt{|q(b_i)|}} b_i & i \leq r+s, \\ b_i & i > r+s. \end{cases}$$

The matrix of q with respect to $\mathcal{C}$ is a canonical form.

(uniqueness) Suppose we have two bases $\mathcal{B}$ and $\mathcal{C}$ with

$$[q]_{\mathcal{B}} = \begin{pmatrix} I_r & & \\ & -I_s & \\ & & 0 \end{pmatrix}, \quad [q]_{\mathcal{C}} = \begin{pmatrix} I_{r'} & & \\ & -I_{s'} & \\ & & 0 \end{pmatrix}.$$

By comparing the ranks we know that $r+s = r'+s'$. It's therefore sufficient to prove that $r = r'$. Define two subspaces of V by

$$U = \text{Span}\{b_1, \ldots, b_r\}, \quad W = \text{Span}\{c_{r'+1}, \ldots, c_n\}.$$

If u is a non-zero vector of U then we have $u = x_1 b_1 + \ldots + x_r b_r$. Hence

$$q(u) = x_1^2 + \ldots + x_r^2 > 0.$$

Similarly if $w \in W$ then $w = y_{r'+1} c_{r'+1} + \ldots + y_n c_n$, and

$$q(w) = -y_{r'+1}^2 - \ldots - y_{r'+s'}^2 \leq 0.$$

It follows that $U \cap W = \{0\}$. Therefore

$$U + W = U \oplus W \subset V.$$

From this we have

$$\dim U + \dim W \leq \dim V.$$

Hence

$$r + (n - r') \leq n.$$

This implies $r \leq r'$. A similar argument (consider $U = Span\{c_1, \ldots, c_{r'}\}$ and $W = Span\{b_{r+1}, \ldots, b_n\}$) shows that $r' \leq r$, so we have $r = r'$. $\square$

The **rank** of a quadratic form is the rank of the corresponding matrix. Clearly, in the complex case it is the integer r that appears in the canonical form.

In the real case, it is $r + s$.

For a **real** quadratic form, the signature is the pair (r, s). In this case $q(v) > 0$ for all non-zero vectors v.

A real form q is **positive definite** if its signature is $(r, 0)$, **negative definite** if its signature is $(0, s)$. In this case $q(v) < 0$ for all non-zero vectors v.

There exists a non-zero vector v such that $q(v) = 0$ if and only is the signature is (r, s) with $r > 0$ and $s > 0$.