A HOLOMORPHIC CHARACTERIZATION OF OPERATOR ALGEBRAS

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Abstract
A necessary and sufficient condition for an operator space to support a multiplication making it completely isometric and isomorphic to a unital operator algebra is proved. The condition involves only the holomorphic structure of the Banach spaces underlying the operator space.

1. Introduction and background

1.1. *Introduction*
If A is an operator algebra, that is, an associative subalgebra of $B(H)$, then $M_n(A) \subset M_n(B(H))$ may be viewed as a subalgebra of $B(\oplus_1^n H)$ and its multiplication is contractive, that is, $\|XY\| \leq \|X\|\|Y\|$ for $X, Y \in M_n(A)$, where XY denotes the matrix or operator product of X and Y. Conversely, if an operator space A (i.e., a closed linear subspace of $B(H)$) is also a unital (not necessarily associative) Banach algebra with respect to a product $x \cdot y$ which is completely contractive in the above sense, then according to [6], it is completely isometric via an algebraic isomorphism to an operator algebra (i.e., an associative subalgebra of some $B(K)$).

Our main result (Theorem 4.8) drops the algebra assumption on A in favor of a holomorphic assumption. Using only natural conditions on holomorphic vector fields on Banach spaces, we are able to construct an algebra product on A which is completely contractive and unital, so that the result of [6] can be applied. Thus we give a holomorphic characterization of operator spaces which are completely isometric to operator algebras. This paper is a companion to [23] where the authors gave holomorphic characterizations of operator spaces that are completely isometric to a C*-algebra or to a ternary ring of operators (TRO). (Holomorphic characterizations of Banach spaces and of Banach algebras which are *isometric* to a C*-algebra were given in [29] and [17], respectively.)

The symmetric part $S(X)$ of a Banach space X is the orbit of the origin under the set of all complete holomorphic vector fields (see subsection 1.2). This holomorphic structure gives rise to a ternary partial triple product $\{\cdot,\cdot,\cdot\}_X$ on the Banach space X:

$$\{\cdot,\cdot,\cdot\}_X : X \times S(X) \times X \to X.$$

We use this ternary product to construct our binary product.

Applications of the symmetric part of a Banach space appeared in [2], [3] where this idea is used to describe the algebraic properties of isometries of certain operator algebras and in [4] to characterize Hilbert spaces. The method was used for the first time in [22] to show that Banach spaces with holomorphically equivalent unit balls are linearly isometric (see [1] for a more detailed exposition of [22]).

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Our technique is to use a variety of elementary isometries on n by n matrices over A (most of the time, $n = 2$) and to exploit the fact that isometries of arbitrary Banach spaces preserve the partial triple product. (See the proofs of Propositions 2.1 and 4.7 and of Lemmas 3.1, 3.3, 3.6, 3.8, 4.1, and 4.3.) The first occurrence of this technique appears in section 2, where for each n a contractive projection P_n on $M_\infty(A)$ (= the closure of the infinite matrices with only finitely many non-zero entries from A, see section 2) with range $M_n(A)$ is constructed as a convex combination of isometries. We define the *completely symmetric part* of A to be the intersection of A (embedded in $M_\infty(A)$) with the symmetric part of $M_\infty(A)$ and show it is the image under P_1 of the symmetric part of $M_\infty(A)$. It follows from [23] that the completely symmetric part of A is completely isometric to a TRO, which is a crucial tool in our work.

Arazy and Solel [3, Cor. 2.9(i)] showed that if A is a subalgebra of $B(H)$ containing the identity operator I, then its symmetric part is the maximal C*-subalgebra $A \cap A^*$ of A. For the same reason, the symmetric part of the operator algebra $M_\infty(A)$ is the maximal C^*-subalgebra of $B(\oplus_1^\infty H)$ contained in $M_\infty(A)$, namely $M_\infty(A) \cap (M_\infty(A))^*$, which shows that the completely symmetric part of A coincides with its symmetric part $A \cap A^*$, and therefore contains I. Moreover, by [3, Cor. 2.9(ii)], the partial triple product in $M_n(A)$ is the restriction of the triple product $(xy^*z+zy^*x)/2$ on $M_n(B(H))$. Thus the conditions (i) and (ii) in our main theorem (Theorem 4.8) which we restate here, hold when A is an operator algebra. In this statement, $\{\cdot,\cdot,\cdot\}_A$ and $\{\cdot,\cdot,\cdot\}_{M_n(A)}$ denote the partial triple products on A and $M_n(A)$ respectively. By Corollary 2.4, $v \in S(A)$ and by Proposition 2.1, $V = \mathrm{diag}(v,\ldots,v) \in S(M_n(A))$, so these partial triple products are defined.

Theorem. An operator space A is completely isometric to a unital operator algebra if and only if there exists an element v of norm one in the completely symmetric part of A such that:

(i): For every $x \in A$, $\{x, v, v\}_A = x$,
(ii): Let $V = \mathrm{diag}(v,\ldots,v) \in M_n(A)$. For all $X \in M_n(A)$,

$$\|\{XVX\}_{M_n(A)}\| \le \|X\|^2.$$

Although we have phrased this theorem in terms of the norm and partial triple product, it should be noted that the two conditions can be restated in holomorphic terms. For any element v in the symmetric part of a Banach space X, h_v will denote the unique complete holomorphic vector field on the open unit ball of X satisfying $h_v(0) = v$. (Complete holomorphic vector fields and the symmetric part of a Banach space are recalled in subsection 1.2.) Then (i) and (ii) become

(i′): For every $x \in A$, $h_v(x+v) - h_v(x) - h_v(v) + v = -2x$
(ii′′): Let $V = \mathrm{diag}(v,\ldots,v) \in M_n(A)$. For all $X \in M_n(A)$,

$$\|V - h_V(X)\| \le \|X\|^2.$$

Let us consider another example. Suppose that A is a TRO, that is, a closed subspace of $B(H)$ closed under the ternary product ab^*c. Since $M_\infty(A)$ is a TRO, hence a JC^*-triple, it is equal to its symmetric part, which shows that the completely symmetric part of A coincides with A.

Now suppose that the TRO A contains an element v satisfying $xv^*v = vv^*x = x$ for all $x \in A$. Then it is trivial that A becomes a unital C*-algebra for the product xv^*y, involution vx^*v, and unit v. By comparison, our main result starts only with an operator space A containing a distinguished element v in its completely symmetric part having a unit-like property. We then construct a binary product from properties of the partial triple product induced by the holomorphic structure. The space A, with this binary product, is then shown to satisfy the hypothesis in [6] and hence is completely isometric to a unital operator algebra. The first assumption is unavoidable since the result of [6] fails in the absence of a unit element. However, it is worth noting that only the first hypothesis in the above theorem is needed to prove the existence and properties of the binary product $x \cdot y$.

According to [7], "The one-sided multipliers of an operator space X are a key to the 'latent operator algebraic structure' in X." The unified approach through multiplier operator algebras developed in [7] leads to simplifications of known results and applications to quantum M-ideal theory. They also state "With the extra structure consisting of the additional matrix norms on an operator algebra, one might expect to not have to rely as heavily on other structure, such as the product." Our result is certainly in the spirit of this statement.

In the rest of this section, a review of operator spaces, Jordan triples, and holomorphy is given. The completely symmetric part of an arbitrary operator space A is defined in section 2. The binary product $x \cdot y$ on A is constructed in section 3 using properties of isometries on 2 by 2 matrices over A and it is shown that the symmetrized product can be expressed in terms of the partial Jordan triple product on A. Section 4 contains a key proposition and the proof of the main theorem.

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1.2. *Background*

In this section, we recall some basic facts that we use about operator spaces, Jordan triples, and holomorphy in Banach spaces. Besides the sources referenced in this section, for more facts and details on the first two topics, see [30],[31],[10] and [11],[24],[25],[8], respectively.

By an **operator space**, sometimes called a quantum Banach space, we mean a closed linear subspace A of $B(H)$ for some complex Hilbert space H, equipped with the matrix norm structure obtained by the identification of $M_n(B(H))$ with $B(H \oplus H \oplus \cdots \oplus H)$. Two operator spaces are **completely isometric** if there is a linear isomorphism between them which, when applied elementwise to the corresponding spaces of n by n matrices, is an isometry for every $n \geq 1$. More generally, a linear map $\phi : A \to B$ between operator spaces is completely bounded if $\|\phi\|_{\mathrm{cb}} = \sup_{n\geq 1} \|\phi_n\|$ is finite, where $\phi_n : M_n(A) \to M_n(B)$ is the map which applies ϕ to each matrix entry of its argument.

By an **operator algebra**, sometimes called a quantum operator algebra, we mean a closed associative subalgebra A of $B(H)$, together with its matrix norm structure as an operator space.

One important example of an operator space is a **ternary ring of operators**, or TRO, which is an operator space in $B(H)$ which contains ab^*c whenever it contains a, b, c.

A TRO is a special case of a JC^***-triple**, that is, a closed subspace of $B(H)$ which contains the symmetrized ternary product $ab^*c + cb^*a$ whenever it contains a, b, c. Important examples of JC*-triples, besides C*-algebras, JC*-algebras, and Hilbert spaces, are the Cartan factors of types 1,2,3, and 4 (see, for example, [14, page 140]).

More generally, a JB^***-triple** is a complex Banach space equipped with a triple product $\{x, y, z\}$ which is linear in the first and third variables, conjugate linear in the second variable, satisfies the algebraic identities

$$\{x, y, z\} = \{z, y, x\}$$

and

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\} \tag{1}$$

and the analytic conditions that for each x, the mapping $y \mapsto \{x, x, y\}$ is hermitian with nonnegative spectrum, and $\|\{x, x, x\}\| = \|x\|^3$.

The following two theorems are fundamental in the theory of JB*-triples.

Theorem 1.1 (Kaup [20]). The class of JB^*-triples coincides with the class of complex Banach spaces whose open unit ball is a bounded symmetric domain.

Theorem 1.2 (Friedman-Russo [13], Kaup [21], Stacho [28]). The class of JB^*-triples is stable under contractive projections. More precisely, if P is a contractive projection on a JB^*-triple E with triple product denoted by $\{x, y, z\}_E$, then $P(E)$ is a JB^*-triple with triple product given by $\{a, b, c\}_{P(E)} = P\{a, b, c\}_E$ for $a, b, c \in P(E)$.

For a JB^*-triple, the following identity is a consequence of the Gelfand Naimark Theorem ([14, Corollary 3]):

$$\|\{xyz\}\| \leq \|x\|\|y\|\|z\|.$$

This suggests Problem 1 at the end of this paper.

The following two theorems, already mentioned above, are instrumental in this work.

Theorem 1.3 (Blecher-Ruan-Sinclair [6]). If an operator space supports a unital Banach algebra structure in which the product (not necessarily associative) is completely contractive, then the operator space is completely isometric to an operator algebra.

Theorem 1.4 (Neal-Russo [23]). If an operator space has the property that the open unit ball of the space of n by n matrices is a bounded symmetric domain for every $n \geq 2$, then the operator space is completely isometric to a TRO.

Finally, we review the construction and properties of the partial Jordan triple product in an arbitrary Banach space. Let X be a complex Banach space with open unit ball X_0. Every holomorphic function $h : X_0 \to X$, also called a holomorphic vector field, is locally integrable, that is, the initial value problem

$$\frac{\partial}{\partial t}\varphi(t, z) = h(\varphi(t, z))\ ,\ \varphi(0, z) = z,$$

has a unique solution for every $z \in X_0$ for t in a maximal open interval J_z containing 0. A **complete holomorphic vector field** is one for which $J_z = \mathsf{R}$ for every $z \in X_0$. In this case, $\varphi(t, \cdot)$ is a holomorphic automorphism of X_0 and $\varphi_t(z) = \varphi(t, z)$ is called the flow of h.

The **symmetric part** of X is the orbit of 0 under the set of complete holomorphic vector fields, and is denoted by $S(X)$. It is a closed subspace of X and is equal to X precisely when X has the structure of a JB^*-triple (by Theorem 1.1).

It is a fact that every complete holomorphic vector field is the sum of the restriction of a skew-Hermitian bounded linear operator on X and a function h_a of the form $h_a(z) = a - Q_a(z)$, where $a \in S(X)$ and Q_a is a quadratic homogeneous polynomial on X.

If $a \in S(X)$, we can obtain a symmetric bilinear form on X, also denoted by Q_a via the polarization formula

$$Q_a(x, y) = \frac{1}{2}\left(Q_a(x + y) - Q_a(x) - Q_a(y)\right)$$

and then the partial Jordan triple product $\{\cdot, \cdot, \cdot\}_X : X \times S(X) \times X \to X$ is defined by $\{x, a, z\}_X = Q_a(x, z)$. The space $S(X)$ becomes a JB^*-triple in this triple product. It is also true that the "main identity" (1) holds whenever $a, y, b \in S(X)$ and $x, z \in X$.

The proof of the following lemma is implicit in [22]. Part (b) is explicitly stated in [9, Lemma 1.8]. A short proof can be based on the deep results of [22] and [9] and the fact that a holomorphic mapping $h : X_0 \to X$ is a complete holomorphic vector field if and only if h has a holomorphic extension to a neighborhood containing the closed unit ball of X and for every $x \in X$ and $f \in X^*$ with $\|x\| = 1 = \|f\| = f(x)$, we have $\Re f(h(x)) = 0$. (See [3, Proposition 2.5] and [4, Lemma 2.8], and for a proof, see [31, Lemma 4.4] or [27].)

Lemma 1.5. If ψ is a linear isometry of a Banach space X onto a Banach space Y, then

- **(a):** For every complete holomorphic vector field h on X_0, $\psi \circ h \circ \psi^{-1}$ is a complete holomorphic vector field on Y_0.
- **(b):** $\psi(S(X)) = S(Y)$ and ψ preserves the partial Jordan triple product:

$$\psi\{x, a, y\}_X = \{\psi(x), \psi(a), \psi(y)\}_Y \text{ for } a \in S(X),\ x, y \in X.$$

In particular, for $a \in S(X)$, $\psi \circ h_a \circ \psi^{-1} = h_{\psi(a)}$.

The symmetric part of a Banach space behaves well under contractive projections, as stated in the next theorem (see [1, 5.2, 5.3, 5.4]).

Theorem 1.6 (Stacho [28]). If P is a contractive projection on a Banach space X and h is a complete holomorphic vector field on X_0, then $P \circ h|_{P(X)_0}$ is a complete holomorphic vector field on $P(X)_0$. In addition $P(S(X)) \subset S(P(X))$ and the partial triple product on $P(X)$ is given by $\{x, y, z\}_{P(X)} = P\{x, y, z\}_X$ for $x, z \in P(X)$ and $y \in P(S(X))$.

Some examples of the symmetric part $S(X)$ of a Banach space X are given in the seminal paper [9].

- $X = L_p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, $p \neq 2$; $S(X) = 0$
- $X =$ (classical) H_p, $1 \leq p < \infty$, $p \neq 2$; $S(X) = 0$
- $X = H_\infty$ (classical) or the disk algebra; $S(X) = \mathbf{C}$
- $X =$ a uniform algebra $A \subset C(K)$; $S(A) = A \cap \overline{A}$

The first example above suggests Problem 2 at the end of this paper. The last example is a commutative predecessor of the example of Arazy and Solel quoted above ([3, Cor. 2.9(i)]).

More examples, due primarily to Stacho [28], and involving Reinhardt domains are recited in [1], along with the following (previously) unpublished example due to Vigué, showing that the symmetric part need not be complemented.

- There exists an equivalent norm on ℓ_∞ so that ℓ_∞ in this norm has symmetric part equal to c_0

2. Completely symmetric part of an operator space

Let $A \subset B(H)$ be an operator space. $M_n(A)$ will denote the Banach space of n by n matrices over A, with the norm $\|x\|_n$ given by the action of the matrix $x = [x_{ij}]$ on $\oplus_1^n H$. Let $M_{\infty,0}(A)$ be the linear space of all infinite matrices over A with only finitely many nonzero entries. We shall identify $M_n(A)$ with

$$\left\{ \begin{bmatrix} M_n(A) & 0 \\ 0 & 0 \end{bmatrix} \in M_{\infty,0}(A) \right\} \subset B(\oplus_1^\infty H).$$

Then $M_{\infty,0}(A)$ is the increasing union $\cup_1^\infty M_n(A)$ and is thus a normed linear space whose completion will be denoted by $M_\infty(A)$. (Although $M_\infty(A)$ may be viewed as an operator space in $B(\oplus_1^\infty H)$, we shall only make use of its Banach space structure.)

A completely bounded map ϕ on A to an operator space B is the same as a bounded map on $M_\infty(A)$ to $M_\infty(B)$ sending $M_n(A)$ into $M_n(B)$ for every n, the norm of the latter being equal to the completely bounded norm of ϕ.

The *completely symmetric part* of A is defined by $CS(A) = A \cap S(M_\infty(A))$. More precisely, if $\psi : A \to M_1(A)$ denotes the complete isometric identification, then $CS(A) = \psi^{-1}(\psi(A) \cap S(M_\infty(A)))$.

These definitions are depicted in the first two rows of the following diagram. The third row is a consequence of Proposition 2.1 below.

$$\begin{array}{ccccc} A & \subset & M_n(A) & \subset & M_\infty(A) \\ \cup & & \cup & & \cup \\ S(A) & & S(M_n(A)) & & S(M_\infty(A)) \\ \cup & & \cup & & \\ CS(A) & \subset & M_n(CS(A)) & & \end{array}$$

For $1 \leq m < N$ let $\psi^N_{1,m} : M_N(A) \to M_N(A)$ and $\psi^N_{2,m} : M_N(A) \to M_N(A)$ be the isometries of order two defined by

$$\psi^N_{j,m} : \left[\begin{array}{cc} M_m(A) & M_{m,N-m}(A) \\ M_{N-m,m}(A) & M_{N-m}(A) \end{array}\right] \to \left[\begin{array}{cc} M_m(A) & M_{m,N-m}(A) \\ M_{N-m,m}(A) & M_{N-m}(A) \end{array}\right]$$

where

$$\psi^N_{1,m} : \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \to \left[\begin{array}{cc} a & -b \\ -c & d \end{array}\right]$$

and

$$\psi^N_{2,m} : \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \to \left[\begin{array}{cc} a & -b \\ c & -d \end{array}\right].$$

These two isometries give rise to two isometries $\tilde{\psi}_{1,m}$ and $\tilde{\psi}_{2,m}$ on $M_{\infty,0}(A)$ as follows. If $x \in M_{\infty,0}(A)$, say $x \in M_N(A)$ where $m < N$, then for $j = 1, 2$, $\tilde{\psi}_{j,m}(x)$ is defined to be $\psi^N_{j,m}(x)$ and is independent of N. We thus have isometries $\psi_{1,m}, \psi_{2,m}$ of $M_\infty(A)$ onto itself, of order 2 which fix $M_m(A)$ elementwise.

The natural projection $\tilde{P}_m$ of $M_{\infty,0}(A)$ onto $M_m(A)$ is thus given by

$$\tilde{P}_m x = \frac{\tilde{\psi}_{2,m}\left(\frac{\tilde{\psi}_{1,m}(x)+x}{2}\right) + \frac{\tilde{\psi}_{1,m}(x)+x}{2}}{2}.$$

The projection $\tilde{P}_m$ on $M_{\infty,0}(A)$ extends to a projection P_m on $M_\infty(A)$, with range $M_m(A)$ given by

$$P_m = \frac{1}{4}(\psi_{2,m}\psi_{1,m} + \psi_{2,m} + \psi_{1,m} + \mathrm{Id}).$$

Proposition 2.1. With the above notation,

(a): $P_n(S(M_\infty(A))) = M_n(CS(A))$;

(b): $M_n(CS(A))$ is a JB*-subtriple of $S(M_\infty(A))$, that is,

$$\{M_n(CS(A)), M_n(CS(A)), M_n(CS(A))\} \subset M_n(CS(A));$$

Moreover,

$$\{M_n(A), M_n(CS(A)), M_n(A)\} \subset M_n(A);$$

(c): $CS(A)$ is completely isometric to a TRO.

Note: In the first displayed formula of (b), the triple product is the one on the JB*-triple $M_n(CS(A))$, namely, $\{xyz\}_{M_n(CS(A))} = P_n(\{xyz\}_{S(M_\infty(A))})$, which, as shown in the proof, is actually the restriction of the triple product of $S(M_\infty(A))$; in the second displayed formula, the triple product is the partial triple product on

$M_\infty(A)$. Hereafter, we shall denote partial triple products simply by $\{\cdot,\cdot,\cdot\}$ if the meaning is clear from the context.

PROOF. Since P_n is a linear combination of isometries of $M_\infty(A)$, and since isometries preserve the symmetric part, $P_n(S(M_\infty(A))) \subset S(M_\infty(A))$.

For any $y = (y_{ij}) \in M_n(A)$, write $y = (R_1, \cdots, R_n)^t = (C_1, \cdots, C_n)$ where R_i, C_j are the (first n) rows and columns of y. Let $\psi_1 = \psi_1^n$ and $\psi_2 = \psi_2^n$ be the isometries on $M_\infty(A)$ whose action is as follows: for $y \in M_n(A)$,

$$\psi_1^n(y) = (R_1, -R_2, \cdots, -R_n)^t \quad , \quad \psi_2^n(y) = (-C_1, \cdots, -C_{n-1}, C_n),$$

and for an arbitrary element $y = [y_{ij}] \in M_{\infty,0}(A)$, say $y \in M_N(A)$, where without loss of generality $N > n$, and for $k = 1, 2$, ψ_k^n maps y into

$$\begin{bmatrix} \psi_k^n[y_{ij}]_{n\times n} & 0 \\ 0 & [y_{ij}]_{(N-n)\times(N-n)} \end{bmatrix}.$$

Suppose now that $x = (x_{ij}) \in P_n(S(M_\infty(A)))$. Then with $a \otimes e_{ij}$ denoting the matrix with a in the (i,j)-entry and zeros elsewhere,

$$x_{1n} \otimes e_{1n} = \frac{\psi_2\left(\frac{\psi_1(x)+x}{2}\right) + \frac{\psi_1(x)+x}{2}}{2} \in S(M_\infty(A)).$$

Now consider the isometry ψ_3 given by $\psi_3(C_1, \cdots, C_n) = (C_n, C_2, \cdots, C_{n-1}, C_1)$. Then $x_{1,n} \otimes e_{11} = \psi_3(x_{1n} \otimes e_{1n}) \in S(M_\infty(A)))$, and by definition, $x_{1n} \in CS(A)$. Continuing in this way, one sees that each $x_{ij} \in CS(A)$, proving that

$$P_n(S(M_\infty(A))) \subset M_n(CS(A)).$$

Conversely, suppose that $x = (x_{ij}) \in M_n(CS(A))$. Since each $x_{ij} \in CS(A)$, then by definition, $x_{ij} \otimes e_{11} \in S(M_\infty(A))$. By using isometries as in the first part of the proof, it follows that $x_{ij} \otimes e_{ij} \in S(M_\infty(A))$, and $x = \sum_{i,j} x_{ij} \otimes e_{ij} \in S(M_\infty(A))$. This proves (a), since

$$M_n(CS(A)) = P_n(M_n(CS(A))) \subset P_n(S(M_\infty(A))) \subset S(M_\infty(A)).$$

As noted above, P_n is a contractive projection on the JB*-triple $S(M_\infty(A))$, so that by Theorem 1.2, the range of P_n, namely $M_n(CS(A))$, is a JB*-triple with triple product

$$\{xyz\}_{M_n(CS(A))} = P_n(\{xyz\}_{S(M_\infty(A))}),$$

for $x, y, z \in M_n(CS(A))$. This proves (c) by Theorem 1.4.

However, P_n is a linear combination of isometries of $M_\infty(A)$ which fix $M_n(A)$ elementwise, and any isometry ψ of $M_\infty(A)$ preserves the partial triple product: $\psi\{abc\} = \{\psi(a)\psi(b)\psi(c)\}$ for $a, c \in M_\infty(A)$ and $b \in S(M_\infty(A))$. This shows that

$$\{xyz\}_{M_n(CS(A))} = \{xyz\}_{S(M_\infty(A))}$$

for $x, y, z \in M_n(CS(A))$, proving the first part of (b). To prove the second part of (b), just note that if $x, z \in M_n(A)$ and $y \in M_n(CS(A))$, then P_n fixes $\{xyz\}$.

Corollary 2.2. $CS(A) = M_1(CS(A)) = P_1(S(M_\infty(A)))$.

Corollary 2.3. For every $u, v, w \in CS(A)$,

$$\{u \otimes e_{ij}, v \otimes e_{kl}, w \otimes e_{pq}\} = \{u, v, w\} \otimes (e_{ij}e_{lk}e_{pq} + e_{pq}e_{lk}e_{ij})/2.$$

In particular,

$$\left\{\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix}\right\} = 0 = \left\{\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}\right\},$$

$$\left\{\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & w \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} \{u, v, w\} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\left\{\begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ \{u,v,w\} & 0 \end{bmatrix},$$

and

$$\left\{\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & \{u,v,w\} \end{bmatrix}.$$

Proof. By Proposition 2.1(c), there is a complete isometry ϕ from $CS(A)$ onto a TRO T. The space $M_2(T)$ is a TRO with product $ab^*c = (a_{ij})(b_{kl})^*(c_{pq}) = (a_{ij})(b^*_{lk})(c_{pq})$, and hence a JB*-triple for the product

$$\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)\ (a,b,c \in M_2(T)).$$

Since the identity $\{u \otimes e_{ij}, v \otimes e_{kl}, w \otimes e_{pq}\} = \{u,v,w\} \otimes (e_{ij}e_{lk}e_{pq} + e_{pq}e_{lk}e_{ij})/2$ trivially holds in $M_2(T)$, and $\phi_2 : M_2(CS(A)) \to M_2(T)$ is an isometry, and hence a JB*-triple isomorphism, the statements of the lemma are clear.

Corollary 2.4. $CS(A) \subset S(A)$ and $P_n\{yxy\} = \{yxy\}$ for $x \in M_n(CS(A))$ and $y \in M_n(A)$.

Proof. For $x \in CS(A)$, let $\tilde{x} = x \otimes e_{11}$. Then $\tilde{x} \in S(M_\infty(A))$ and so there exists a complete holomorphic vector field $h_{\tilde{x}}$ on $(M_\infty(A))_0$ satisfying $h_{\tilde{x}}(0) = \tilde{x}$. Since P_1 is a contractive projection of $M_\infty(A)$ onto A, by Theorem 1.6, $P_1 \circ h_{\tilde{x}}|_{A_0}$ is a complete holomorphic vector field on A_0. But $P_1 \circ h_{\tilde{x}}|_{A_0}(0) = P_1 \circ h_{\tilde{x}}(0) = P_1(\tilde{x}) = x$, proving that $x \in S(A)$.

Recall from the proof of the second part of Proposition 2.1(b) that if $x, z \in M_n(A)$ and $y \in M_n(CS(A))$, then P_n fixes $\{xyz\}$.

The symmetric part of a JC*-triple coincides with the triple. The Cartan factors of type 1 are TROs, which we have already observed are equal to their completely symmetric parts. If A is a Cartan factor of type 2,3, or 4, then A is not a TRO since it is not closed under xy^*z, but it is possible that it is completely isometric to a TRO. (See Problems 3 and 5 at the end of this paper.) The next proposition rules this out in the finite dimensional case.

Proposition 2.5. If A is a finite dimensional Cartan factor of type 2,3 or 4 (and dimension at least two), then $CS(A) = 0$.

Proof. It is known [15] that the surjective linear isometries of the Cartan factors of types 2 and 3 are given by multiplication on the left and right by a unitary operator, and hence they are complete isometries. The same is true for finite dimensional Cartan factors of type 4 by [32]. It is also known that set of inner automorphisms (hence isometries) of any Cartan factor acts transitively on the set of minimal partial isometries (and hence on finite rank partial isometries of the same rank, [16]). It follows that if the completely symmetric part of a finite dimensional Cartan factor of type 2, 3 or 4 is not zero, then it must contain any finite rank partial isometry, and hence a generating set so it coincides with the Cartan factor. Thus our Cartan factor A is completely isometric to a TRO. However, by [12, Prop. A1], every finite dimensional TRO is an injective operator space, and by [26, Theorem], every injective finite dimensional operator space is completely isometric to a direct sum of rectangular matrix algebras. Now the fact that A is merely isometric to a TRO shows that that A is a Cartan factor of type 1, a contradiction.

3. Definition of the algebra product

From this point on, we shall tacitly assume the first hypothesis in our main theorem, namely that A is an operator space and $v \in CS(A)$ is an element of norm 1 which satisfies $\{xvv\} = x$ for every $x \in A$. With this assumption alone, we are able to construct and develop properties of the a binary product. It is not until the last step in the proof of Theorem 4.8 that we need to invoke the second hypothesis.

If not explicitly stated, $a, b, c, d, x, y, z, \alpha, \beta, \gamma$, etc. denote elements of A. In what follows, we work almost exclusively with $M_2(A)$, which it turns out to be sufficient for our result.

Let $\{\cdot,\cdot,\cdot\} : M_\infty(A) \times S(M_\infty(A)) \times M_\infty(A)$ denote the partial triple product defined by the symmetric part of $M_\infty(A)$. By the properties established in the previous section, for each natural n, we can restrict the above product to a mapping from $M_n(A) \times M_n(CS(A)) \times M_n(A)$ to $M_n(A)$. This triple product will be used throughout the whole paper.

Lemma 3.1. $\left\{\begin{bmatrix} x & \pm x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & \pm v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & \pm x \\ 0 & 0 \end{bmatrix}\right\} = 2\begin{bmatrix} \{xvx\} & \pm\{xvx\} \\ 0 & 0 \end{bmatrix}$

Proof. Let $X = M_\infty(A)$ and consider projections Q_1 and Q_2 on X defined by $Q_1 = P_{11}P_2$, $Q_2 = SRP_2$ where P_{11} maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},$$

S maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix},$$

and R maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \frac{1}{2}\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}.$$

Let $A' = \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in A\right\} = Q_1X$ and $A'' = \left\{\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} : a \in A\right\} = Q_2X$, and let $\psi : A' \to A''$ be the isometry defined by $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a/\sqrt{2} & a/\sqrt{2} \\ 0 & 0 \end{bmatrix}$. Finally, let $v' = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$ and $v'' = \begin{bmatrix} v/\sqrt{2} & v/\sqrt{2} \\ 0 & 0 \end{bmatrix}$, and more generally $a' = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $a'' = \psi(a') = \begin{bmatrix} a/\sqrt{2} & a/\sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Since a surjective isometry preserves partial triple products (Lemma 1.5) and the partial triple product on the range of a contractive projection is equal to the projection acting on the partial triple product of the original space (Theorem 1.6), we have

$$\psi\{a'v'b'\}_{Q_1X} = \{a''v''b''\}_{Q_2X}.$$

We unravel both sides of this equation. In the first place

$$\begin{aligned} \{a'v'b'\}_{Q_1X} &= Q_1\{a'v'b'\}_X \\ &= P_{11}P_2\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}\right\}_X \\ &= P_{11}P_2\begin{bmatrix} \{avb\} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \{avb\} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\psi\{a'v'b'\}_{Q_1X} = \begin{bmatrix} \{avb\}/\sqrt{2} & \{avb\}/\sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

Next, R and S are convex combinations of isometries that fix the elements of the product, so that $\{a''v''b''\}_X$ is fixed by R and by S. Hence, $\{a''v''b''\}_{Q_2X} = Q_2\{a''v''b''\}_X = SRP_2\{a''v''b''\}_X = \{a''v''b''\}_X$, so that

$$\{a''v''b''\}_{Q_2X} = \left\{\begin{bmatrix} a/\sqrt{2} & a/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v/\sqrt{2} & v/\sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} b/\sqrt{2} & b/\sqrt{2} \\ 0 & 0 \end{bmatrix}\right\}.$$

This proves the lemma in the case of the plus sign. The proof in the remaining case is identical, with R replaced by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix} a-b & b-a \\ c-d & d-c \end{bmatrix},$$

A'' replaced by $\left\{\begin{bmatrix} a & -a \\ 0 & 0 \end{bmatrix} : a \in A\right\}$, and ψ replaced by $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a/\sqrt{2} & -a/\sqrt{2} \\ 0 & 0 \end{bmatrix}$.

Lemma 3.2.

$$\begin{aligned}\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix} &= \left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \\ &+ 2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}\end{aligned}$$

Proof. By Lemma 3.1

$$\begin{aligned}4\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix} &= 2\begin{bmatrix} \{xvx\} & \{xvx\} \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} \{xvx\} & -\{xvx\} \\ 0 & 0 \end{bmatrix} \\ &= \left\{\begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & x \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} x & -x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & -v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & -x \\ 0 & 0 \end{bmatrix}\right\}.\end{aligned}$$

By expanding the right hand side of the last equation, one obtains 16 terms of which 8 cancel in pairs. The surviving 8 terms are

$$2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\},$$

$$4\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}$$

and

$$2\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}.$$

Since the first surviving term above is equal to $2\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix}$, the lemma is proved.

The following two lemmas, and their proofs parallel the previous two lemmas.

Lemma 3.3. $\left\{\begin{bmatrix} a & 0 \\ 0 & \pm a \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & \pm v \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & \pm b \end{bmatrix}\right\} = \begin{bmatrix} \{avb\} & 0 \\ 0 & \pm\{avb\} \end{bmatrix}$

Proof. Let $X = M_\infty(A)$ and consider projections Q_1 and Q_2 on X defined by $Q_1 = P_{11}P_2$, $Q_2 = SRP_2$ where P_{11} maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},$$

S maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix},$$

and R maps

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \frac{1}{2}\begin{bmatrix} a+d & b+c \\ b+c & a+d \end{bmatrix}.$$

Let $A' = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in A \right\} = Q_1 X$ and $A'' = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in A \right\} = Q_2 X$, and let $\psi : A' \to A''$ be the isometry defined by $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. Finally, let $v' = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$ and $v'' = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$, and more generally $a' = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $a'' = \psi(a') = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$.

Again by Lemma 1.5 and Theorem 1.6, we have

$$\psi\{a'v'b'\}_{Q_1X} = \{a''v''b''\}_{Q_2X}.$$

We unravel both sides of this equation. In the first place

$$\begin{aligned} \{a'v'b'\}_{Q_1X} &= Q_1\{a'v'b'\}_X \\ &= P_{11}P_2 \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \right\}_X \\ &= P_{11}P_2 \begin{bmatrix} \{avb\} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \{avb\} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\psi\{a'v'b'\}_{Q_1X} = \begin{bmatrix} \{avb\} & 0 \\ 0 & \{avb\} \end{bmatrix}.$$

Next, by using appropriate isometries, for example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ to } \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

$\{a''v''b''\}_X$ is fixed by R and by S. Hence, $\{a''v''b''\}_{Q_2X} = Q_2\{a''v''b''\}_X = SRP_2\{a''v''b''\}_X = \{a''v''b''\}_X$, so that

$$\{a''v''b''\}_{Q_2X} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \right\}.$$

This proves the lemma in the case of the plus sign. The proof in the remaining case is identical, with R replaced by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} a-d & b-c \\ b-c & a-d \end{bmatrix},$$

A'' replaced by $\left\{ \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} : a \in A \right\}$, and ψ replaced by $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$.

Lemma 3.4.

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right\} + \left\{ \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \right\} = 0$$

and

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right\} = 0.$$

Proof. By Lemma 3.3

$$\begin{aligned} 2\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} \{xvx\} & 0 \\ 0 & \{xvx\} \end{bmatrix} + \begin{bmatrix} \{xvx\} & 0 \\ 0 & -\{xvx\} \end{bmatrix} \\ &= \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right\} \\ &+ \left\{ \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & -v \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \right\}. \end{aligned}$$

By expanding the right hand side of the last equation, one obtains 16 terms of which 8 cancel in pairs. The surviving 8 terms are

$$2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\},$$

$$4\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}$$

and

$$2\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}.$$

Since the first surviving term above is equal to $2\begin{bmatrix} \{xvx\} & 0 \\ 0 & 0 \end{bmatrix}$, we have

$$2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\} = -\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}.$$

Replacing x by $x + y$ in this last equation results in

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\} + \left\{\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}$$

$$= -\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}$$

Using the isometry of multiplication by the imaginary unit on the second row of this equation and adding then shows that both sides are zero.

Lemma 3.5.

$$\left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\} = \begin{bmatrix} \{xvx\} & 0 \\ 0 & \{yvy\} \end{bmatrix} \tag{2}$$

Proof. By Lemma 3.1, the left hand side of (2) expands into the sum of the right side of (2) and

$$\Delta = 2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\} + 2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}$$

$$+\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}.$$

The last two terms are zero by Lemma 3.4 and Δ has the form $\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$.

We shall prove (2) and hence that $\Delta = 0$ by using Theorem 1.6. To this end, let E denote the Banach space $\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in A\right\} \subset M_2(A)$. Since $v \in CS(A) \subset S(A)$, the function $h(x) = v - \{xvx\}$ is a complete holomorphic vector field on A_0. Denote its flow by $\varphi_t(x)$ so that $\varphi_0(x) = x$ and for all $t \in \mathsf{R}$,

$$\frac{\partial}{\partial t}\varphi_t(x) = h(\varphi_t(x)).$$

Recall that φ_t is a holomorphic automorphism of the open unit ball of A. Now define the holomorphic function

$$H_1\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} h(x) & 0 \\ 0 & h(y) \end{bmatrix},$$

so that $H_1(0) = V = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$. The function H_1 is a complete holomorphic vector field on E_0 with flow

$$\Phi_t\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \begin{bmatrix} \varphi_t(x) & 0 \\ 0 & \varphi_t(y) \end{bmatrix},$$

since $\Phi_t(E_0) \subset E_0$ for all $t \in \mathsf{R}$.

On the other hand, by Proposition 2.1 and Theorem 1.6, $V \in M_2(CS(A)) = P_2(S(M_\infty(A))) \subset S(M_2(A))$ so that $H_2(X) = V - \{XVX\}$ is a complete holomorphic vector field on $M_2(A)$. Let Q be the projection of $M_2(A)$ onto E. Then by Theorem 1.6, $H_3 := Q \circ H_2|_{E_0}$ is a complete holomorphic vector field in E.

Since, as noted above, the left side of (2) is a diagonal matrix, we have

$$H_3\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = Q(H_2\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right)) = V - \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right\}$$

Since $H_3(0) = H_1(0) = V$, and complete holomorphic vector fields are determined by their value at the origin, $H_1 = H_3$ and the proof is complete.

The following lemma asserts the orthogonality of $\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$ with respect to the partial triple product of $M_2(A)$.

Lemma 3.6.

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} = 0$$

Equivalently,

$$\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} = 0$$

Proof. The second statement follows from the first by using the isometry

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

As noted in the proof of Lemma 3.5, Lemma 3.4 and an appropriate isometry (interchange both rows and columns simultaneously) yields

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right\} = 0.$$

Next, the isometry

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} -a & -b \\ c & d \end{bmatrix}.$$

shows that

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix},$$

for some $\gamma, \delta \in A$. Similarly, the isometry

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}$$

shows that

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}.$$

Applying the isometry of multiplication of the second row by the imaginary unit shows that $\gamma = 0$. Hence

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right\} = 0.$$

By appropriate use of isometries as above,

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$$

for some $\beta \in A$. Applying the isometry of multiplication of the second column by the imaginary unit shows that $\beta = 0$. Hence

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}\right\} = 0.$$

Finally, by what is proved above and the fact that $\Delta = 0$ in the proof of Lemma 3.5, we have

$$0 = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\} + \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}$$

and each term is zero since one is of the form $\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$ and the other is of the form $\begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}$.

Lemma 3.7.

$$\begin{aligned} \begin{bmatrix} \{xvy\} & 0 \\ 0 & 0 \end{bmatrix} &= \left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\right\} \end{aligned}$$

Proof. Replace x in Lemma 3.2 by $x + y$.

We can repeat some of the preceding arguments to obtain the following three lemmas, which will be used in the proof of Lemma 3.11. The proof of the following lemma is, except for notation, identical to those of Lemma 3.1 and Lemma 3.3. On the other hand, it also follows from Lemma 3.1 via an isometry.

Lemma 3.8. $\left\{\begin{bmatrix} 0 & x \\ 0 & \pm x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & \pm v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & \pm x \end{bmatrix}\right\} = \begin{bmatrix} 0 & 2\{xvx\} \\ 0 & \pm 2\{xvx\} \end{bmatrix}$

The proof of the following lemma parallels exactly the proof of Lemma 3.2, using Lemma 3.8 in place of Lemma 3.1.

Lemma 3.9.

$$\begin{aligned} \begin{bmatrix} 0 & \{xvx\} \\ 0 & 0 \end{bmatrix} &= \left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\} \\ &+ 2\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \end{aligned}$$

As in Lemma 3.7, polarization of Lemma 3.9 yields the following lemma.

Lemma 3.10.

$$\begin{aligned} \begin{bmatrix} 0 & \{xvy\} \\ 0 & 0 \end{bmatrix} &= \left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \end{aligned}$$

Lemma 3.11.

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}=0.$$

PROOF. Set $y=v$ in Lemma 3.10 to obtain

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}=\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}$$

$$+\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}+\left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}$$

Then

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \tag{3}$$

$$=\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\right\}$$

$$+\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}\right\} \tag{4}$$

$$+\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}\right\}.$$

The three terms on the right each vanish, as is seen by applying the main identity to each term and making use of Lemma 3.6, and Corollary 2.3. For the sake of clarity, we again include the details of the proof. Explicitly,

By the main identity, the first term in (4) equals

$$\left\{\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}$$

$$-\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\} \tag{5}$$

$$+\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\right\}.$$

The first term in (5) is zero by Lemma 3.6. The third term is zero by Corollary 2.3. The middle term is zero by Lemma 3.6 since by Corollary 2.3,

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}.$$

Again by the main identity, the second term in (5) equals

$$\left\{\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}$$

$$-\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}$$

$$+\left\{\begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}\right\}.$$

The first term is zero by Lemma 3.6 and the third term is zero by Corollary 2.3.

The middle term is also zero by Lemma 3.6 since, by Corollary 2.3,

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}.$$

Finally, again by the main identity, the third term in (5) equals

$$\left\{\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}$$
$$-\left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}$$
$$+\left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}\right\}.$$

The first term is zero by Corollary 2.3. The third term is of the form

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

so it is zero by Lemma 3.6. The middle term is zero by Lemma 3.6 since by Corollary 2.3,

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}.$$

Thus (3) is zero.

Lemma 3.12.
$$\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} = 0.$$

Proof. By applying the isometries of multiplication of the second column and second row by -1, we see that for some $a \in A$,

$$\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

and therefore that

$$\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\right\} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

By Lemma 3.7

$$\begin{aligned} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \{avv\} & 0 \\ 0 & 0 \end{bmatrix} &= \left\{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\} \\ &+ \left\{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\right\} \end{aligned} \tag{7}$$

The first term on the right side of (7) is zero by Lemma 3.11.

Using (6), the second term on the right side of (7) is equal to

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}\right\} \tag{8}$$

to which we apply the main identity, obtaining three terms, the first of which is equal to

$$\left\{\begin{bmatrix} 0 & \{vvx\} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}.$$

The second term is equal, by Corollary 2.3 to

$$-\frac{1}{2}\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}.$$

The third term is equal to

$$\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}\right\}$$

which equals

$$\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}.$$

The second term on the right side of (7) is therefore equal to

$$\frac{3}{2}\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}.$$

Let us now write the third term on the right side of (7) as

$$\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\right\}\right\}$$

and apply the main Jordan identity to it, to obtain three terms which each vanish, the first and third by Lemma 3.11 and the second by Corollary 2.3. We have thus shown that

$$\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} = \frac{3}{2}\left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\},$$

proving the lemma.

Definition 3.13. Let us now define a product $y \cdot x$ by

$$\begin{bmatrix} y \cdot x & 0 \\ 0 & 0 \end{bmatrix} = 2\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}$$

and denote the corresponding matrix product by $X \cdot Y$. That is, if $X = [x_{ij}]$ and $Y = [y_{ij}]$, then $X \cdot Y = [z_{ij}]$ where

$$z_{ij} = \sum_k x_{ik} \cdot y_{kj}.$$

Note that

$$\{xvy\} = \frac{1}{2}(y \cdot x + x \cdot y). \tag{9}$$

since by Lemmas 3.7 and 3.12 and symmetry of the partial triple product in the outer variables, we can write

$$\begin{bmatrix} \{xvy\} & 0 \\ 0 & 0 \end{bmatrix} = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}\right\}.$$

4. Main result

The following lemma, in which the right side is equal to $\frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & x \cdot y \end{bmatrix}$, is needed to prove that v is a unit element for the product $x \cdot y$ (Lemma 4.2), and to prove Lemma 4.5 below, which is a key step in the proof of Proposition 4.7 and hence of Theorem 4.8.

Lemma 4.1.

$$\left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}$$

PROOF. Let ψ be the isometry

$$\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix} x & y \\ x & y \end{bmatrix}.$$

As in the proofs of Lemmas 3.1, 3.3 and 3.8, ψ preserves partial triple products. Thus,

$$\begin{aligned}
\frac{1}{2\sqrt{2}} \begin{bmatrix} 0 & x\cdot y \\ 0 & x\cdot y \end{bmatrix} &= \frac{1}{2}\psi\left(\begin{bmatrix} 0 & x\cdot y \\ 0 & 0 \end{bmatrix}\right) \\
&= \psi\left(\left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\}\right) \\
&= \left(\frac{1}{\sqrt{2}}\right)^3 \left\{\begin{bmatrix} x & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & y \end{bmatrix}\right\} \\
&= \left(\frac{1}{\sqrt{2}}\right)^3 \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\
&+ \left(\frac{1}{\sqrt{2}}\right)^3 \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\
&+ \left(\frac{1}{\sqrt{2}}\right)^3 \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\} \\
&+ \left(\frac{1}{\sqrt{2}}\right)^3 \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\},
\end{aligned}$$

(the last step by Lemma 3.6), so that

$$\begin{aligned}
\begin{bmatrix} 0 & x\cdot y \\ 0 & x\cdot y \end{bmatrix} &= \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\
&+ \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} \\
&+ \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\} \\
&+ \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\begin{bmatrix} 0 & x\cdot y \\ 0 & x\cdot y \end{bmatrix} &= \begin{bmatrix} 0 & x\cdot y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & x\cdot y \end{bmatrix} \\
&= 2\left\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\} \\
&+ 2\left\{\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\}
\end{aligned}$$

From the last two displayed equations, we have

$$\begin{aligned}
&\left\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\} \\
&= \left\{\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right\} + \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}.
\end{aligned}$$

The first term on the left of the last displayed equation is of the form $\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and the second is of the form $\begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix}$. Again multiplying rows and columns by

-1 and using the fact that isometries preserve the partial triple product shows that the first term on the right of the last displayed equation is of the form $\begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix}$ and the second is of the form $\begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix}$. Thus

$$\left\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix}\right\}.$$

Lemma 4.2. $x \cdot v = v \cdot x = x$ for every $x \in A$.

PROOF. Apply the main identity to write

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}\right\} = R - S + T$$

where, by Corollary 2.3

$$\begin{aligned} R &= \left\{\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \\ &= \frac{1}{2}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}, \end{aligned}$$

and by the fact that $x \mapsto \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ is an isometry (see the proof of Lemma 3.1),

$$\begin{aligned} S &= \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \\ &= \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} \end{aligned}$$

and

$$\begin{aligned} T &= \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}\right\} \\ &= \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}. \end{aligned}$$

Thus

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}\right\} =$$

$$\frac{1}{2}\left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{4}\begin{bmatrix} x \cdot v & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Apply the main identity again to write

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\}\right\} = R' - S' + T'$$

where by Corollary 2.3,

$$R' = \left\{\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\},$$

$$S' = \left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\right\}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\} = \frac{1}{2}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\},$$

and by Lemma 3.6,

$$T' = \left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\}\right\} = 0.$$

Thus

$$\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\}\right\} = R' - S' + T' =$$

$$\frac{1}{2}\left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\} = \frac{1}{4}\begin{bmatrix} v\cdot x & 0 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

the last step by Lemma 4.1.

By Lemmas 3.7, 3.11 and 4.1

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\}. \tag{12}$$

Adding (10) and (11) and using (12) results in

$$\frac{1}{2}\begin{bmatrix} v\cdot x & 0 \\ 0 & 0 \end{bmatrix} = \left\{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{4}\begin{bmatrix} x\cdot v & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} v\cdot x & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $v\cdot x = x\cdot v$ and since $x\cdot v + v\cdot x = 2\{vvx\} = 2x$, the lemma is proved.

We need yet another lemma, along the lines of Lemmas 3.1, 3.3, 3.8, and 4.1. We omit the by now standard proof, except to point out that the isometry involved is

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}.$$

Lemma 4.3. If $\beta, \delta \in A$ are defined by

$$\left\{\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix}\right\} = \begin{bmatrix} 0 & \beta \\ 0 & \delta \end{bmatrix},$$

then

$$\left\{\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}\right\} = \begin{bmatrix} 0 & 0 \\ \beta & \delta \end{bmatrix},$$

In particular,

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right\}$$

Proposition 4.7 below is critical. To prepare for its proof, we require three more lemmas.

Lemma 4.4. For $X \in M_2(A)$ and $V = \operatorname{diag}(v,v) \in M_2(A)$,

$$\{XVV\} = X.$$

PROOF. Let us write

$$\begin{aligned}
&\left\{\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} = \\
&\qquad \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} + \left\{\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} \\
&\quad + \left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\}.
\end{aligned}$$

The two middle terms on the right side of this equation vanish by Lemma 3.6. The first term can be written (using Lemma 4.3 in the second term) as

$$\begin{aligned}
&\left\{\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right\} = \\
&\qquad\left\{\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}\right\} + \left\{\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & b\\ 0 & 0\end{bmatrix}\right\}\\
&+\quad\left\{\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & 0\\ c & 0\end{bmatrix}\right\} + \left\{\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} v & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & d\end{bmatrix}\right\}\\
&=\quad\begin{bmatrix} \{vva\} & 0\\ 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & v\cdot b\\ 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0\\ v\cdot c & 0\end{bmatrix} + \begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}\\
&=\quad\begin{bmatrix} a & b/2\\ c/2 & 0\end{bmatrix}.
\end{aligned}$$

The last term can be written (using Lemma 4.3 in the second term) as

$$\begin{aligned}
&\left\{\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}\right\} = \\
&\qquad\left\{\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & b\\ 0 & 0\end{bmatrix}\right\}\\
&+\quad\left\{\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ c & 0\end{bmatrix}\right\} + \left\{\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & v\end{bmatrix}\begin{bmatrix} 0 & 0\\ 0 & d\end{bmatrix}\right\}\\
&=\quad\begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & v\cdot b\\ 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0\\ v\cdot c & 0\end{bmatrix} + \begin{bmatrix} 0 & 0\\ 0 & d\end{bmatrix}\\
&=\quad\begin{bmatrix} 0 & b/2\\ c/2 & d\end{bmatrix}.
\end{aligned}$$

Lemma 4.5. For $X, Y \in M_2(A)$ and $V = \operatorname{diag}(v, v) \in M_2(A)$,

$$\begin{bmatrix} 0 & Y\cdot X\\ 0 & 0\end{bmatrix} = 2\left\{\begin{bmatrix} Y & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} V & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & X\\ 0 & 0\end{bmatrix}\right\}.$$

PROOF.[1] The left side expands into 8 terms:

$$\begin{aligned}
\begin{bmatrix} 0 & Y\cdot X\\ 0 & 0\end{bmatrix} &= \begin{bmatrix} 0 & \begin{bmatrix} y_{11}\cdot x_{11} & 0\\ 0 & 0\end{bmatrix}\\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} y_{12}\cdot x_{21} & 0\\ 0 & 0\end{bmatrix}\\ 0 & 0\end{bmatrix}\\
&+ \begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{11}\cdot x_{12}\\ 0 & 0\end{bmatrix}\\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{12}\cdot x_{22}\\ 0 & 0\end{bmatrix}\\ 0 & 0\end{bmatrix}\\
&+ \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0\\ y_{21}\cdot x_{11} & 0\end{bmatrix}\\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0\\ y_{22}\cdot x_{21} & 0\end{bmatrix}\\ 0 & 0\end{bmatrix}\\
&+ \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0\\ 0 & y_{21}\cdot x_{12}\end{bmatrix}\\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0\\ 0 & y_{22}\cdot x_{22}\end{bmatrix}\\ 0 & 0\end{bmatrix}
\end{aligned}$$

For the right side, we have

$$\left\{\begin{bmatrix} \begin{bmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{bmatrix} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} \begin{bmatrix} v & 0\\ 0 & v\end{bmatrix} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}\\ 0 & 0\end{bmatrix}\right\}$$

[1]Although the proof of this lemma is long, it renders the inductive step in the proof of Proposition 4.7 trivial

which is the sum of 32 terms. We show now that 24 of these 32 terms are zero, and each of the other 8 terms is equal to one of the 8 terms in the expansion of the left side. We note first that by changing the signs of the first two columns we have that

$$\left\{\begin{bmatrix}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22}\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix} v & 0 \\ 0 & v\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix} \\ 0 & 0\end{bmatrix}\right\}$$

has the form

$$\begin{bmatrix} 0 & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta\end{bmatrix} \\ 0 & 0\end{bmatrix}.$$

We shall consider eight cases.[2]

Case 1A: $Y = y_{11} \otimes e_{11} = \begin{bmatrix} y_{11} & 0 \\ 0 & 0\end{bmatrix}$, $V = v \otimes e_{11} = \begin{bmatrix} v & 0 \\ 0 & 0\end{bmatrix}$

In this case, further analysis shows that

$$\left\{\begin{bmatrix}\begin{bmatrix} y_{11} & 0 \\ 0 & 0\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & 0 \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}\right\}$$

is of the form

$$\begin{bmatrix} 0 & \begin{bmatrix} \alpha & 0 \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}.$$

and hence is unchanged by applying the isometry C_{14} which interchanges the first and fourth columns. The resulting (form of the) triple product we started with is therefore

$$\left\{\begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{11} \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} 0 & v \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & 0 \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}\right\}$$

which equals (isometries preserve the partial triple product)

$$\begin{bmatrix} 0 & \left\{\begin{bmatrix} 0 & y_{11} \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & v \\ 0 & 0\end{bmatrix}\begin{bmatrix} x_{11} & 0 \\ 0 & 0\end{bmatrix}\right\} \\ 0 & 0\end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} y_{11}\cdot x_{11} & 0 \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}$$

as required. An identical argument, using C_{13} instead of C_{14} shows that

$$\left\{\begin{bmatrix}\begin{bmatrix} y_{11} & 0 \\ 0 & 0\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} 0 & x_{12} \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{11}\cdot x_{12} \\ 0 & 0\end{bmatrix} \\ 0 & 0\end{bmatrix}.$$

To finish case 1A, use the isometry R_{14} which interchanges the first and fourth rows on

$$\left\{\begin{bmatrix}\begin{bmatrix} y_{11} & 0 \\ 0 & 0\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\begin{bmatrix} v & 0 \\ 0 & 0\end{bmatrix} & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ x_{21} & x_{22}\end{bmatrix} \\ 0 & 0\end{bmatrix}\right\}$$

to obtain

$$\left\{\begin{bmatrix} 0 & 0 \\ \begin{bmatrix} 0 & 0 \\ y_{11} & 0\end{bmatrix} & 0\end{bmatrix}\begin{bmatrix} 0 & 0 \\ \begin{bmatrix} 0 & 0 \\ v & 0\end{bmatrix} & 0\end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ x_{21} & x_{22}\end{bmatrix} \\ 0 & 0\end{bmatrix}\right\}$$

which is zero by Lemma 3.6, which is valid for $M_2(A)$. Hence, the original triple product is zero.

[2]Since we only apply this proposition in the case where $x_{11} = x_{22} = y_{11} = y_{22} = 0$ (see the proof of Theorem 4.8), most of these cases become easier. Nevertheless, we prove the more general statement

Case 1B: $Y = y_{11} \otimes e_{11} = \begin{bmatrix} y_{11} & 0 \\ 0 & 0 \end{bmatrix}$, $V = v \otimes e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$

Using the isometry $C_2(i)$ of multiplication of the second column by the imaginary unit i we have that

$$\left\{ \begin{bmatrix} \begin{bmatrix} y_{11} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right\}$$

which is of the form

$$\begin{bmatrix} 0 & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \\ 0 & 0 \end{bmatrix},$$

is equal to a non-zero multiple of its negative, and is thus zero.

Case 2A: $Y = y_{12} \otimes e_{12} = \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix}$, $V = v \otimes e_{11} = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$

Using the isometry $C_1(i)$ of multiplication of the first column by the imaginary unit i we have that

$$\left\{ \begin{bmatrix} \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right\}$$

which is of the form

$$\begin{bmatrix} 0 & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \\ 0 & 0 \end{bmatrix},$$

is equal to a non-zero multiple of its negative, and is thus zero.

Case 2B: $Y = y_{12} \otimes e_{12} = \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix}$, $V = v \otimes e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$

Using the isometry R_{23} which interchanges rows 2 and 3 we have that

$$\left\{ \begin{bmatrix} \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right\} = 0$$

by Lemma 3.6, which is valid for $M_2(A)$.

Using the isometry C_{24} and Lemma 4.1, we have that

$$\left\{ \begin{bmatrix} \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ x_{21} & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} 0 & \begin{bmatrix} y_{12} \cdot x_{21} & 0 \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

Using the isometry C_{23} and Lemma 4.1, we have that

$$\left\{ \begin{bmatrix} \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ 0 & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right\} = \frac{1}{2} \begin{bmatrix} 0 & \begin{bmatrix} 0 & y_{12} \cdot x_{22} \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}.$$

Case 3A: $Y = y_{21} \otimes e_{21} = \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix}$, $V = v \otimes e_{11} = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$

Using the isometry R_{13} which interchanges rows 1 and 3 we have that

$$\left\{ \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ x_{21} & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix} \right\} = 0$$

by Lemma 3.6, which is valid for $M_2(A)$.

Using the isometry C_{14} and Lemma 4.1, we have that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} x_{11} & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ y_{21}\cdot x_{11} & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

Using the isometry C_{13} and Lemma 4.1, we have that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ 0 & y_{21}\cdot x_{11} \end{bmatrix} \\ 0 & 0 \end{bmatrix}.$$

Case 3B: $Y = y_{21}\otimes e_{21} = \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix}$, $V = v\otimes e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$

Using the isometry $C_2(i)$ of multiplication of the second column by the imaginary unit i we have that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ y_{21} & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix}\right\}$$

which is of the form

$$\begin{bmatrix} 0 & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \\ 0 & 0 \end{bmatrix},$$

is equal to a non-zero multiple of its negative, and is thus zero.

Case 4A: $Y = y_{22}\otimes e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \end{bmatrix}$, $V = v\otimes e_{11} = \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix}$

Using the isometry $C_1(i)$ of multiplication of the first column by the imaginary unit i we have that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix}\right\}$$

which is of the form

$$\begin{bmatrix} 0 & \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \\ 0 & 0 \end{bmatrix},$$

is equal to a non-zero multiple of its negative, and is thus zero.

Case 4B: $Y = y_{22}\otimes e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \end{bmatrix}$, $V = v\otimes e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$

Using the isometry R_{23} shows that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \begin{bmatrix} x_{11} & x_{12} \\ 0 & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}\right\} = 0.$$

Using the isometry C_{24} shows that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ x_{21} & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ y_{22}\cdot x_{21} & 0 \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

Using the isometry C_{23} shows that

$$\left\{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & y_{22} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & x_{22} \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & \begin{bmatrix} 0 & 0 \\ 0 & y_{22}\cdot x_{22} \end{bmatrix} \\ 0 & 0 \end{bmatrix}$$

This completes the proof of the lemma.

Lemma 4.6. For $X, Y \in M_2(A)$ and $V = \operatorname{diag}(v, v) \in M_2(A)$,

$$X \cdot Y + Y \cdot X = 2\{XVY\}.$$

PROOF. Lemmas 3.1 to 3.12 and 4.1 to 4.3 now follow automatically for elements of $M_2(A)$, since the proofs for $M_2(A)$ are the same as those for A once you have Lemma 4.4. The lemma follows from Lemma 4.5 in the same way as (9) from the fact that Lemmas 3.7 and 3.12 are valid for $M_2(A)$.

Proposition 4.7. For $X, Y \in M_n(A)$, and $V = \operatorname{diag}(v, v, \ldots, v) \in M_n(A)$,

(a): $\{XVV\} = X$

(b): $\left[\begin{array}{cc} 0 & Y \cdot X \\ 0 & 0 \end{array}\right] = 2\left\{\left[\begin{array}{cc} Y & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} V & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} 0 & X \\ 0 & 0 \end{array}\right]\right\}.$

(c): $X \cdot Y + Y \cdot X = 2\{XVY\}$.

PROOF.We shall prove by induction on k that the proposition holds for $n = 2k-1$ and $2k$. Let $k = 1$ so that $n = 1$ and 2.

If $n = 1$, (a) is the first assumption in Theorem 4.8, (b) is Definition 3.13, and (c) has been noted in (9) as a consequence of Lemmas 3.7 and 3.12. If $n = 2$, (a), (b) and (c) have been proved in Lemmas 4.4,4.5 and 4.6 respectively.

We now assume the the proposition holds for $n = 1, 2, \ldots, 2k$. We shall show that it holds for $n = 2k+1$ and for $n = 2k+2$.

First, for any $X \in M_{2k+1}(A)$, and $V_m = \operatorname{diag}(v, \ldots, v) \in M_m(A)$, let us write

$$\tilde{X} = \left[\begin{array}{cc} X & 0 \\ 0 & 0 \end{array}\right] \in M_{2k+2}(A)$$

and

$$\tilde{V} = \left[\begin{array}{cc} V_{2k+1} & 0 \\ 0 & 0 \end{array}\right] \in M_{2k+2}(A).$$

We then write

$$\tilde{X} = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array}\right]$$

and

$$\tilde{V} = \left[\begin{array}{cc} V_{k+1} & 0 \\ 0 & \left[\begin{array}{cc} V_k & 0 \\ 0 & 0 \end{array}\right] \end{array}\right],$$

where $X_{ij} \in M_{k+1}(A)$. Since $k+1 \le 2k$, the induction proceeds by simply repeating the proofs of Lemmas 4.4,4.5 and 4.6 for the case $n = 2$, with X, Y, V replaced by $\tilde{X}, \tilde{Y}, \tilde{V}$.

Now suppose $n = 2k+2$. For any $X \in M_{2k+2}(A)$, let us write

$$\tilde{X} = \left[\begin{array}{cc} X_{11} & X_{12} \\ X_{21} & X_{22} \end{array}\right]$$

and

$$\tilde{V} = \left[\begin{array}{cc} V_{k+1} & 0 \\ 0 & V_{k+1} \end{array}\right]$$

where $X_{ij} \in M_{k+1}(A)$. Since $k+1 \le 2k$, the induction proceeds, as above, by simply repeating the proofs of Lemmas 4.4,4.5 and 4.6 for the case $n = 2$, with X, Y, V replaced by $\tilde{X}, \tilde{Y}, \tilde{V}$.

We can now complete the proof of our main result which is restated here.

Theorem 4.8. An operator space A is completely isometric to a unital operator algebra if and only there exists $v \in CS(A)$ such that:

(i): $\{xvv\} = x$ for all $x \in A$

(ii): Let $V = \mathrm{diag}(v, \ldots, v) \in M_n(A)$. For all $X \in M_n(A)$

$$\| \{XVX\} \| \leq \|X\|^2.$$

PROOF. We have already noted in the introduction that the conditions are necessary. Conversely, by the first assumption, all the machinery developed so far is available. In particular, v is a unit element for the product $x \cdot y$ and by Lemma 4.6, for every $X \in M_2(A)$, $X \cdot X = \{XVX\}$.

With $X = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ for elements $x, y \in A$ of norm 1, we have

$$\begin{aligned} \max(\|x \cdot y\|, \|y \cdot x\|) &= \left\| \begin{bmatrix} x \cdot y & 0 \\ 0 & y \cdot x \end{bmatrix} \right\| = \|X \cdot X\| = \|\{XVX\}\| \\ &\leq \|X\|^2 = \left\| \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right\|^2 = \max(\|x\|, \|y\|)^2 = 1 \end{aligned}$$

so the multiplication on A is contractive. The same argument, using Proposition 4.7, shows that if $X, Y \in M_n(A)$, then $\|X \cdot Y\| \leq \|X\|\|Y\|$ so the multiplication is completely contractive. The result now follows from [6].

For the sake of completeness, we include the detail of the last inequality:

$$\begin{aligned} \max(\|X \cdot Y\|, \|Y \cdot X\|) &= \left\| \begin{bmatrix} X \cdot Y & 0 \\ 0 & Y \cdot X \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right\| \\ &= \left\| \left\{ \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right\} \right\| \\ &\leq \left\| \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right\|^2 = \max(\|X\|, \|Y\|)^2. \end{aligned}$$

Remark 4.9. The second condition in Theorem 4.8 can be replaced by[3]

$$\left\| \left\{ \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \right\} \right\| \leq \frac{1}{2} \|X\|\|Y\|, \tag{13}$$

so that by Proposition 4.7(b),

$$\|Y \cdot X\| \leq \|X\|\|Y\|.$$

Indeed, by Lemma 4.2 and the first condition, A is a unital (with a unit of norm 1 and not necessarily associative) algebra. Remark 4.9 now follows from [6].

The condition (13) can be restated in holomorphic terms as follows. Let $\tilde{V}$ denote the $2n$ by $2n$ matrix $\begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix}$, where $V = \mathrm{diag}(v, \ldots, v) \in M_n(A)$. For all $X, Y \in M_n(A)$

$$\left\| h_{\tilde{V}}(\begin{bmatrix} Y & X \\ 0 & 0 \end{bmatrix}) - h_{\tilde{V}}(\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}) - h_{\tilde{V}}(\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix} + \tilde{V} \right\| \leq \|X\|\|Y\|.$$

We close by stating some problems which arose in connection with this paper.

Problem 1. Is there a Banach space with partial triple product $\{x, a, y\}$ for which the inequality

$$\|\{x, a, y\}\| \leq \|x\|\|a\|\|y\|$$

does not hold?

[3]Although the 1/2 in (13) conveniently cancels the 2 in Proposition 4.7(b), its presence is justified by the fact that (13) holds in case A is an operator algebra

Problem 2. Is the symmetric part of the predual of a von Neumann algebra equal to 0? What about the predual of a JBW^*-triple which does not contain a Hilbert space as a direct summand?

Problem 3. Is the completely symmetric part of an infinite dimensional Cartan factor of type 2, 3 or 4 zero, as in the finite dimensional case?

In [18], it is proved that all operator algebra products on an operator space A are of the form $x \cdot y = xa^*y$ for an element a which lies in the injective envelope $I(A)$. Here the "quasimultiplier" a lies in the symmetric part of $I(A)$. It is clear that the intersection of an operator space A with the quasimultipliers of $I(A)$ from [18] is a TRO and is contained in the completely symmetric part of A. Our main theorem is that certain elements in the holomorphically defined completely symmetric part induce operator algebra products on A while [18] shows that all operator algebra products on A arise from the more concretely and algebraically defined quasimultpliers. Hence it is natural to ask

Problem 4. Under what conditions does the completely symmetric part of an operator space consist of quasimultipliers?

Of course using direct sums and the discussion in the last two paragraphs of section 1.2, we can construct operator spaces whose completely symmetric part is different from zero and from the symmetric part of the operator space. However it would be more satisfying to answer the following problem.

Problem 5. Is there an operator space whose completely symmetric part is not contractively complemented, different from zero, and different from the symmetric part of the operator space?

The following problem is motivated by the main result.

Problem 6. If we assume that, for each natural n, the matrix $V = \mathrm{diag}\,(v, \ldots, v) \in S(M_n(A))$, and the hypotheses (i) and (ii) hold, does it follow that $v \in CS(A)$?

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