Elementary Analysis Math 140B—Winter 2007
Celebrity Theorem 1—Abel's Theorem

THEOREM (Abel's Theorem) Let $\sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R, $0 < R < \infty$ and suppose that the series $\sum_{n=0}^{\infty} a_n R^n$ converges. Then the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, R]$.

PROOF[1]: It suffices to prove the theorem in the case $R = 1$. (See Ross, page 198, Case 2)

For $\epsilon > 0$, pick N such that $|\sum_{k=m}^{n} a_k| < \epsilon$ for all $n \geq m > N = N(\epsilon)$ (Cauchy condition for the convergent series $\sum_{k=0}^{\infty} a_k$)

For notation's sake, let $B_{m,n} := \sum_{k=m}^{n} a_k$, so that $a_k = B_{k,n} - B_{k+1,n}$.

Now suppose $0 \leq x < 1$. Then for $n \geq m$,

$$\begin{aligned}
\sum_{k=m}^{n} a_k x^k &= a_m x^m + a_{m+1}x^{m+1} + \cdots + a_{n-1}x^{n-1} + a_n x^n \\
&= (B_{m,n} - B_{m+1,n})x^m + (B_{m+1,n} - B_{m+2,n})x^{m+1} + \cdots + (B_{n-1,n} - B_{n,n})x^{n-1} + B_{n,n}x^n \\
&= B_{m,n}x^m + B_{m+1,n}(x^{m+1} - x^m) + B_{m+2,n}(x^{m+2} - x^{m+1}) + \cdots \\
&\quad \cdots + B_{n-1,n}(x^{n-1} - x^{n-2}) + B_{n,n}(x^n - x^{n-1}) \\
&= B_{m,n}x^m + (x-1)x^m[B_{m+1,n} + B_{m+2,n}x + \cdots + B_{n-1,n}x^{n-m-2}] + B_{n,n}x^{n-m-1}
\end{aligned}$$

Therefore[2] , since $|x^m| \leq 1$,

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k x^k\right| &\leq \epsilon x^m + (1-x)x^m[\epsilon + \epsilon x + \cdots + \epsilon x^{n-m-1}] \\
&\leq \epsilon + \epsilon(1-x)\frac{1-x^{n-m}}{1-x} < 2\epsilon.
\end{aligned}$$

Thus, by the uniform Cauchy condition for series of functions (Theorem 25.6), the series $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[0, 1)$. Since it is given that the series converges for $x = 1$, the convergence is uniform on [0,1]. □

[1]Buck: Advanced Calculus, page 279

[2]the cancellation of $1 - x$ in the last step is the key point of this proof