THEOREM (Weierstrass Approximation—Ross, Theorem 27.5, page 203) Let $f$ be a continuous function on a closed interval $[a, b]$. Then there is a sequence of polynomials which converges uniformly to $f$ on $[a, b]$.

PROOF\(^1\): It suffices to prove the theorem in the case that $[a, b]$ is of the form $[\beta, 1 - \beta]$, where $0 < \beta < 1/2$. (See Ross, Exercise 27.1, page 204, and the hint on page 333)

Extend $f$ to a continuous function (which we will still call $f$) which vanishes on $(-\infty, 0) \cup [1, \infty)$. For each $n \geq 1$, define a polynomial

$$P_n(x) = c_n \int_0^1 f(t) [1 - (x - t)^2]^n \, dt, \quad \text{where} \quad c_n = \frac{1}{\int_{-1}^{1} (1 - t^2)^n \, dt}.$$  

By the change of variable $s = x - t$, we have

$$P_n(x) = c_n \int_{x-1}^{x} f(x - s)(1 - s^2)^n \, ds$$

and since $f(x - s) = 0$ for $s \geq x$ and for $s \leq x - 1$, we have

$$P_n(x) = c_n \int_{-1}^{1} f(x - s)(1 - s^2)^n \, ds.$$  

Thus

$$P_n(x) - f(x) = c_n \int_{-1}^{1} [f(x - s) - f(x)](1 - s^2)^n \, ds = I_1(x) + I_2(x) + I_3(x)$$  

where $I_1(x)$ is the part of the integral over $[-1, -\delta]$, $I_2(x)$ is the part over $[-\delta, \delta]$ and $I_3(x)$ is the part over $[\delta, 1]$, and $\delta$ is chosen as follows.

Since $f$ is continuous, it is therefore uniformly continuous on the closed interval $[-1, 1]$, so if $\epsilon > 0$ is given, choose $\delta = \delta(\epsilon)$ such that

$$\sup_{x \in [-1,1]} |f(x) - f(x)| < \epsilon/3 \quad \text{whenever} \quad |s| < \delta.$$  

We thus have

$$|I_2(x)| \leq \frac{\epsilon}{3} c_n \int_{-\delta}^{\delta} (1 - s^2)^n \, ds \leq \frac{\epsilon}{3} c_n \int_{-1}^{1} (1 - s^2)^n \, ds = \epsilon/3.$$  

Note that this estimate for $I_2(x)$ holds for every $x \in [-1, 1]$.

Since $f$ is continuous, it is bounded on the closed interval $[-1, 1]$, that is,

$$\sup_{x \in [-1,1]} |f(x)| \leq M,$$  

for some $M > 0$. Note that

$$\frac{1}{c_n} = \int_{-1}^{1} (1 - t^2)^n \, dt \geq \int_{0}^{\sqrt{2}/2} (1 - t^2)^n \, dt \geq \frac{\delta}{2} (1 - \frac{\delta^2}{4})^n$$  

and on the interval $[\delta, 1]$, we have $(1 - s^2)^n \leq (1 - \delta^2)^n$. So

$$|I_3| \leq c_n 2M \int_{\delta}^{1} (1 - s^2)^n \, ds \leq 4M(1 - \delta) \frac{(1 - \frac{\delta^2}{4})^n}{\delta} \rightarrow 0 \text{ as } n \rightarrow \infty.$$  

Thus $|I_3(x)| < \epsilon/3$ for $n > N_1 = N_1(\epsilon)$, and all $x \in [-1, 1]$.

A similar argument shows that $|I_1(x)| \leq \epsilon/3$ for $n < N_2 = N_2(\epsilon)$, independent of $x \in [-1, 1]$.

From (1), we have $\sup_{x \in [-1,1]} |P_n(x) - f(x)| < \epsilon$ for $n > \max\{|N_1, N_2\}$. Thus, $P_n \rightarrow f$ uniformly on $[-1, 1]$ and hence on $[\beta, 1 - \beta]$, and hence on $[a, b]$.

\(^1\)Ralph P. Boas: A Primer of Real Functions, page104