Elementary Analysis Math 140B—Winter 2007
Celebrity Theorem 2—Weierstrass' Approximation Theorem

THEOREM (Weierstrass Approximation—Ross, Theorem 27.5, page 203) Let f be a continuous function on a closed interval $[a,b]$. Then there is a sequence of polynomials which converges uniformly to f on $[a,b]$.

PROOF[1]**:** It suffices to prove the theorem in the case that $[a,b]$ is of the form $[\beta, 1-\beta]$, where $0 < \beta < 1/2$. (See Ross, Exercise 27.1, page 204, and the hint on page 333)

Extend f to a continuous function (which we will still call f) which vanishes on $(-\infty,0]\cup[1,\infty)$. For each $n \geq 1$, define a polynomial

$$P_n(x) = c_n \int_0^1 f(t)[1-(x-t)^2]^n\,dt, \quad \text{where} \quad c_n = \frac{1}{\int_{-1}^1 (1-t^2)^n\,dt}.$$

By the change of variable $s = x - t$, we have

$$P_n(x) = c_n \int_{x-1}^{x} f(x-s)(1-s^2)^n\,ds$$

and since $f(x-s) = 0$ for $s \geq x$ and for $s \leq x-1$, we have

$$P_n(x) = c_n \int_{-1}^{1} f(x-s)(1-s^2)^n\,ds.$$

Thus

$$P_n(x) - f(x) = c_n \int_{-1}^{1} [f(x-s) - f(x)](1-s^2)^n\,ds = I_1(x) + I_2(x) + I_3(x) \tag{1}$$

where $I_1(x)$ is the part of the integral over $[-1,-\delta]$, $I_2(x)$ is the part over $[-\delta,\delta]$ and $I_3(x)$ is the part over $[\delta,1]$, and δ is chosen as follows.

Since f is continuous, it is therefore uniformly continuous on the closed interval $[-1,1]$, so if $\epsilon > 0$ is given, choose $\delta = \delta(\epsilon)$ such that

$$\sup_{x\in[-1,1]} |f(x-s) - f(x)| < \epsilon/3 \quad \text{whenever} \quad |s| < \delta.$$

We thus have

$$|I_2(x)| \leq \frac{\epsilon}{3} c_n \int_{-\delta}^{\delta} (1-s^2)^n\,ds \leq \frac{\epsilon}{3} c_n \int_{-1}^{1} (1-s^2)^n\,ds = \epsilon/3.$$

Note that this estimate for $I_2(x)$ holds for every $x \in [-1,1]$.

Since f is continuous, it is bounded on the closed interval $[-1,1]$, that is,

$$\sup_{x\in[-1,1]} |f(x)| \leq M,$$

for some $M > 0$. Note that

$$\frac{1}{c_n} = \int_{-1}^{1} (1-t^2)^n\,dt \geq \int_0^{\delta/2} (1-t^2)^n\,dt \geq \frac{\delta}{2}(1-\frac{\delta^2}{4})^n$$

and on the interval $[\delta,1]$, we have $(1-s^2)^n \leq (1-\delta^2)^n$. So

$$|I_3| \leq c_n 2M \int_{\delta}^{1} (1-s^2)^n\,ds \leq \frac{4M(1-\delta)}{\delta}\left(\frac{1-\delta^2}{1-\frac{\delta^2}{4}}\right)^n \to 0 \text{ as } n \to \infty.$$

Thus $|I_3(x)| < \epsilon/3$ for $n > N_1 = N_1(\epsilon)$, and all $x \in [-1,1]$.

A similar argument shows that $|I_1(x)| \leq \epsilon/3$ for $n < N_2 = N_2(\epsilon)$, independent of $x \in [-1,1]$.

From (1), we have $\sup_{x\in[-1,1]} |P_n(x) - f(x)| < \epsilon$ for $n > \max\{N_1, N_2\}$. Thus, $P_n \to f$ uniformly on $[-1,1]$ and hence on $[\beta, 1-\beta]$, and hence on $[a,b]$. □

[1] Ralph P. Boas: A Primer of Real Functions, page104