THEOREM: If $S$ and $T$ are transformations such that $T$ is differentiable at $p_0$ and $S$ is differentiable at $T(p_0)$, then $S \circ T$ is differentiable at $p_0$ and

$$(S \circ T)'(p_0) = S'(T(p_0)) \circ T'(p_0).$$

PROOF: Since $S$ is differentiable at $T(p_0)$, we have

$$S(T(p)) - S(T(p_0)) = S'(T(p_0))(T(p) - T(p_0)) = B(p)$$

where $B(p)/|T(p) - T(p_0)| \to 0$ as $T(p) \to T(p_0)$. Since differentiability implies continuity, $T$ is continuous at $p_0$ so that in fact

$$\frac{B(p)}{|T(p) - T(p_0)|} \to 0 \text{ as } p \to p_0. \tag{2}$$

Since $T$ is differentiable at $p_0$, we have

$$T(p) - T(p_0) = C(p)$$

where

$$\frac{C(p)}{|p - p_0|} \to 0 \text{ as } p \to p_0. \tag{4}$$

Substitute (3) into (1) to get

$$S(T(p)) - S(T(p_0)) = S'(T(p_0))(p - p_0) + C(p) = B(p).$$

and therefore

$$S(T(p)) - S(T(p_0)) = S'(T(p_0))T'(p_0)(p - p_0) = S'(T(p_0))C(p) + B(p). \tag{5}$$

Let $A(p) = S'(T(p_0))C(p) + B(p)$ denote the right side of (5). It remains to show that $A(p)/|p - p_0| \to 0$ as $p \to p_0$.

Since

$$\frac{A(p)}{|p - p_0|} = \frac{S'(T(p_0))C(p)}{|p - p_0|} + \frac{B(p)}{|p - p_0|} \tag{6}$$

and since $S'(T(p_0))$ is continuous, it follows from (4) that the first term on the right side of (6) approaches 0 as $p \to p_0$.

To show that the second term on the right side of (6) approaches 0 as $p \to p_0$, note first that $|T'(p_0)q| \leq M|q|$ for some constant $M$ and all vectors $q$ (continuity of linear transformations again), and therefore from (3)

$$|T(p) - T(p_0)| = |T'(p_0)(p - p_0) + C(p)| \leq |T'(p_0)(p - p_0)| + |C(p)| \leq M|p - p_0| + |C(p)|$$

so that

$$\frac{|B(p)|}{|T(p) - T(p_0)|} \geq \frac{|B(p)|}{M|p - p_0| + |C(p)|} = \frac{|B(p)|}{M + \frac{|C(p)|}{|p - p_0|}} \frac{1}{|p - p_0|},$$

and therefore from (2)

$$\frac{|B(p)|}{|p - p_0|} \leq \frac{|B(p)|}{|T(p) - T(p_0)|} \frac{M}{|p - p_0|} \to M \cdot 0 = 0$$

as $p \to p_0$, as required. QED