THEOREM Let \( g \) be a real valued function defined on an open interval containing \( a \in \mathbb{R} \) and suppose that \( g \) is differentiable at \( a \) with derivative \( g'(a) \). Let \( f \) be a real valued function defined on an open interval containing \( g(a) \) and suppose that \( f \) is differentiable at \( g(a) \) with derivative \( f'(g(a)) \). Then \( f \circ g \) is differentiable at \( a \) with derivative

\[
(f \circ g)'(a) = f'(g(a)) g'(a). 
\]

PROOF:

Step 1. \( g'(a) \) exists Hence \( \forall \epsilon' > 0, \exists \delta' > 0 \) such that

\[
|g(x) - g(a) - g'(a)(x-a)| < \epsilon' |x-a| \quad \text{if} \ |x-a| < \delta'.
\]

Step 2. \( f'(g(a)) \) exists Hence \( \forall \epsilon'' > 0, \exists \delta'' > 0 \) such that

\[
|f(y) - f(g(a)) - f'(g(a))(y-g(a))| < \epsilon'' |y-g(a)| \quad \text{if} \ |y-g(a)| < \delta''.
\]

Step 3. Wish to show \( \forall \epsilon > 0, \exists \delta > 0 \) such that

\[
|f(g(x)) - f(g(a)) - f'(g(a))g'(a)(x-a)| < \epsilon |x-a| \quad \text{if} \ |x-a| < \delta.
\]

Step 4. \( g \) is continuous at \( a \) Hence \( \exists \delta_c > 0 \) such that

\[
|g(x) - g(a)| < \delta'' \quad \text{if} \ |x-a| < \delta_c.
\]

Step 5. Substitute Step 4 in Step 2 Using (5), replace \( y \) in (3) by \( g(x) \) to obtain

\[
|f(g(x)) - f(g(a)) - f'(g(a))(g(x) - g(a))| < \epsilon'' |g(x) - g(a)| \quad \text{if} \ |x-a| < \delta_c.
\]

Step 6. Rewrite Step 1 Set \( \eta(x) := g(x) - g(a) - g'(a)(x-a) \) so that

\[
g(x) - g(a) = g'(a)(x-a) + \eta(x)
\]

and by (2),

\[
|\eta(x)| < \epsilon' |x-a| \quad \text{if} \ |x-a| < \delta'.
\]

Step 7. Substitute Step 6 in Step 5 Putting (7) into (6) (in two places!) and setting

\[
A(x) := f(g(x)) - f(g(a)) - f'(g(a))(g'(a)(x-a) + \eta(x))
\]

we obtain from (6)

\[
|A(x)| < \epsilon' |g'(a)(x-a) + \eta(x)| \quad \text{if} \ |x-a| < \delta_c.
\]

Step 8. Prove Step 3 Given \( \epsilon \), choose \( \epsilon' \) and \( \epsilon'' \) such that

\[
\epsilon' \epsilon'' + \epsilon'' |g'(a)| + |f'(g(a))| \epsilon' < \epsilon.
\]

Then choose \( \delta', \delta'', \) and \( \delta_c \) as in Steps 1, 2 and 4, and set \( \delta = \min(\delta_c, \delta') \). If \( |x-a| < \delta \), we have,

\[
|f(g(x)) - f(g(a)) - f'(g(a))g'(a)(x-a)|
\]

\[
= |A(x) + f'(g(a))\eta(x)| \quad \text{(by (9))}
\]

\[
\leq |A(x)| + |f'(g(a))\eta(x)|
\]

\[
\leq \epsilon'' |g'(a)||x-a| + \epsilon'|\eta(x)| + |f'(g(a))||\eta(x)| \quad \text{(by (10))}
\]

\[
\leq \epsilon'' |g'(a)| + \epsilon' \epsilon'' + |f'(g(a))| \epsilon' |x-a| \quad \text{(by (8))}
\]

\[
< \epsilon |x-a| \quad \text{(by (11))}
\]

This proves (4).