Elementary Analysis Math 140B—Winter 2007
The Chain Rule

THEOREM *Let g be a real valued function defined on an open interval containing $a \in \mathbf{R}$ and suppose that g is differentiable at a with derivative $g'(a)$. Let f be a real valued function defined on an open interval containing $g(a)$ and suppose that f is differentiable at $g(a)$ with derivative $f'(g(a))$. Then $f \circ g$ is differentiable at a with derivative*

$$(f \circ g)'(a) = f'(g(a))\, g'(a). \tag{1}$$

PROOF:

Step 1. $g'(a)$ exists Hence $\forall \epsilon' > 0, \exists \delta' > 0$ such that

$$|g(x) - g(a) - g'(a)(x-a)| < \epsilon'|x-a| \quad \text{if } |x-a| < \delta'. \tag{2}$$

Step 2. $f'(g(a))$ exists Hence $\forall \epsilon'' > 0, \exists \delta'' > 0$ such that

$$|f(y) - f(g(a)) - f'(g(a))(y - g(a))| < \epsilon''|y - g(a)| \quad \text{if } |y - g(a)| < \delta''. \tag{3}$$

Step 3. Wish to show $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|f(g(x)) - f(g(a)) - f'(g(a))g'(a)(x-a)| < \epsilon|x-a| \quad \text{if } |x-a| < \delta. \tag{4}$$

Step 4. g is continuous at a Hence $\exists \delta_c > 0$ such that

$$|g(x) - g(a)| < \delta'' \text{ if } |x-a| < \delta_c. \tag{5}$$

Step 5. Substitute Step 4 in Step 2 Using (5), replace y in (3) by $g(x)$ to obtain

$$|f(g(x)) - f(g(a)) - f'(g(a))(g(x) - g(a))| < \epsilon''|g(x) - g(a)| \quad \text{if } |x-a| < \delta_c. \tag{6}$$

Step 6. Rewrite Step 1 Set $\eta(x) := g(x) - g(a) - g'(a)(x-a)$ so that

$$g(x) - g(a) = g'(a)(x-a) + \eta(x) \tag{7}$$

and by (2),

$$|\eta(x)| < \epsilon'|x-a| \text{ if } |x-a| < \delta'. \tag{8}$$

Step 7. Substitute Step 6 in Step 5 Putting (7) into (6) (in two places!) and setting

$$A(x) := f(g(x)) - f(g(a)) - f'(g(a))[g'(a)(x-a) + \eta(x)] \tag{9}$$

we obtain from (6)

$$|A(x)| < \epsilon''|g'(a)(x-a) + \eta(x)| \quad \text{if } |x-a| < \delta_c. \tag{10}$$

Step 8. Prove Step 3 Given ϵ, choose ϵ' and ϵ'' such that

$$\epsilon''\epsilon' + \epsilon''|g'(a)| + |f'(g(a))|\epsilon' < \epsilon. \tag{11}$$

Then choose δ', δ'', and δ_c as in Steps 1, 2 and 4, and set $\delta = \min(\delta_c, \delta')$. If $|x-a| < \delta$, we have,

$$\begin{aligned}
&|f(g(x)) - f(g(a)) - f'(g(a))g'(a)(x-a)| \\
= \;& |A(x) + f'(g(a))\eta(x)| \qquad \text{(by (9))} \\
\le \;& |A(x)| + |f'(g(a))\eta(x)| \\
\le \;& \epsilon''|g'(a)||x-a| + \epsilon''|\eta(x)| + |f'(g(a))||\eta(x)| \qquad \text{(by (10))} \\
\le \;& [\epsilon''|g'(a)| + \epsilon''\epsilon' + |f'(g(a))|\epsilon']|x-a| \qquad \text{(by (8))} \\
< \;& \epsilon|x-a| \qquad \text{(by (11)}.
\end{aligned}$$

This proves (4). □