Modular Bernstein Algebras*

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In this paper we give a characterization of Bernstein algebras whose lattices of subalgebras are modular. When the ground field is algebraically closed we prove that such algebras must be genetic and give a complete classification up to isomorphism. © 1994 Academic Press, Inc.

0. INTRODUCTION AND PRELIMINARIES

Several authors have worked on the problem of finding out the consequences of imposing classical lattice conditions on the lattice of subalgebras of an algebra. Modularity may be the most popular of these conditions and we can find, for example, several papers on the structure of Lie and Malcev algebras whose lattices of subalgebras are modular of semimodular (see [1–4]) and, recently, on the same problem for Jordan algebras (see [5]) and also for Bernstein algebras (see [6, 7]).

We will suppose that the reader knows the usual terms in lattice theory: sublattice, chain, lattice isomorphism, length of a lattice (the supremum of the lengths of all the chains, where the length of a chain is its cardinality minus one), ...

A lattice \((L, \leq) = (L, \wedge, \vee)\) is said to be modular when one of the three following equivalent conditions holds:

1. If \(x \geq z\) then \(x \wedge (y \vee z) = (x \wedge y) \vee z\).
2. \(x \wedge (y \vee z) = x \wedge [(y \wedge (x \vee z)) \vee z]\) ("shearing identity").
3. \(L\) does not contain a pentagon (as a sublattice).

Given \(a, b\) in a lattice \(L\), we define the closed interval between \(a\) and \(b\) and put \([a, b]\) for the sublattice of \(L\) given by \(\{z \in L \mid a \leq z \leq b\}\).

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In a lattice \((L, \leq) = (L, \land, \lor)\) we put \(x \prec y\) and say \(x, y\) are \textit{consecutive} when \(x \leq y, x \neq y\), and \([x, y] = \{x, y\}\).

A lattice \((L, \leq) = (L, \land, \lor)\) is said to be semimodular if

\[a \prec b \Rightarrow a \lor c \prec b \lor c \quad \text{or} \quad a \lor c = b \lor c \quad \text{for any} \ c \ \text{in} \ L.\]

It is well known that a modular lattice is always semimodular.

(See Grätzer [8], for the terms and results mentioned on lattice theory.)

Let \(A\) be an algebra. We will put \(L(A)\) for its lattice of subalgebras, where \(\leq, \land, \lor\) are naturally defined (\(\leq\) is the inclusion relation). We define the \textit{length of} \(A\) (and put \(l(A)\)) as the length of \(L(A)\).

It is very easy to prove that if \(A\) is a solvable algebra, then its length and its dimension coincide. In general we have obviously that \(l(A) \leq \dim_K A\) (if \(K\) is the ground field).

\textbf{(0.1) Definition.} An algebra \(A\) is called \textit{modular} (semimodular) if \(L(A)\) is modular (semimodular).

Keeping in mind that closed intervals inherit the properties of modularity and semimodularity from the lattice in which they are contained, we have the following result:

\textbf{(0.2) Proposition.} For any algebra \(A\) we have:

(i) If \(A\) is modular then it is semimodular.

(ii) If \(A\) is modular (semimodular), then all of its subalgebras and quotients are modular (semimodular).

Given \(X\) a subset of an algebra \(A\), we will put \((X)\) for the subalgebra of \(A\) generated by \(X\) and \(\langle X \rangle\) for the vector space spanned by \(X\).

\(\leq\) will mean "is a subalgebra of" when we deal with algebras.

\(\lor\) will mean sum of vector subspaces when we deal with vector spaces.

\(\oplus\) will mean direct sum of vector spaces.

Throughout this paper we will deal with algebras over infinite fields of characteristic not two.

\textbf{(0.3) Definition.} A Bernstein \(K\)-algebra \(A\) is a commutative finite-dimensional algebra over \(K\) for which there exists a non-zero algebra homomorphism \(\omega: A \to K\) such that \((x^2)^2 = \omega(x)^2 x^2\), for any \(x\) in \(A\).

The homomorphism \(\omega\) is unique and is called the \textit{weight homomorphism} of \(A\). The Bernstein algebra is usually written \((A, \omega)\).
It is readily seen that, given \((A, \omega)\) a Bernstein algebra, \(\{x^2 \mid x \in A, \omega(x) = 1\}\) is the set of all idempotents in \(A\) and such a set is never vacuous.

Let \(e\) be an idempotent in \(A\), then there exists what is called a Peirce decomposition of \(A\):

\[ A = Ke + \text{Ker} \omega, \quad \text{Ker} \omega = U_e + V_e, \]

where \(U_e = \{x \in A \mid ex = \frac{1}{2}x\} = \{xe \mid x \in \text{Ker} \omega\}\), \(V_e = \{x \in A \mid ex = 0\}\).

It is also known that \(U_e\) and \(V_e\) satisfy

\[ U_e V_e \subseteq U_e, \quad U_e V_e \subseteq V_e, \quad V_e ^2 \subseteq U_e, \quad U_e V_e ^2 = \langle 0 \rangle, \]

\[ u^3 = uv^2 = (uv)^2 = u(uv) = u^2(uv) = u^2v^2 = 0 \quad \text{for all} \quad u \in U_e, v \in V_e. \]

If we define for an element \(x\) in a Bernstein algebra \(x' = x^{i-1} x = xx^{i-1}\) for all \(i \geq 2\) in an inductive way, it is possible to prove that

\[ 2x^i x'^j = \omega(x)^i x'^j + \omega(x'^{i-1}) x'^i \quad \text{for all} \quad i, j \geq 2. \]

Using that expression it is readily seen that in a Bernstein algebra the subalgebra generated by an element \(x\) is spanned as a vector space by \(x, x^2, x^3, \ldots\).

The set of all idempotents can be written in terms of a given idempotent \(e\) in the following way: \(\{e + u + u^2 \mid u \in U_e\}\).

Given two idempotents \(e\) and \(e_1 = e + u + u^2\), with \(u \in U_e\), it is possible to find the relations between Peirce decompositions relatives to \(e\) and \(e_1\),

\[ U_{e_1} = \{w + 2wu \mid w \in U_e\}, \quad V_{e_1} = \{v - 2(u + u^2)v \mid v \in V_e\}. \]

From these relations and the above properties on the product of the components of the Peirce decomposition it is readily shown that the dimensions of such components do not depend on the choice of the idempotent. If \(r = \dim_K U_e, s = \dim_K V_e\), the pair \((r + 1, s)\) is called the type of the Bernstein algebra \(A\). It is also shown that \(\dim_K U_e^2\) and \(\dim_K (U_e V_e + V_e^2)\) do not depend on the choice of the idempotent.

(0.4) Definition. A Bernstein algebra \((A, \omega)\) is called exclusive if \(U_e^2 = 0\) for some (equivalently for all) nonzero idempotent \(e\) in \(A\). A Bernstein algebra \((A, \omega)\) is said to be generic if \(\text{Ker} \omega\) is nilpotent.

The property of “being genetic” is defined for a broader class of algebras in a more general way; for Bernstein algebras this definition is equivalent to the one we give (see Wörz-Busekros [9] for the terms and results mentioned on Bernstein algebras).

As a consequence of the fact that for any Bernstein algebra \((A, \omega)\), \(\text{Ker} \omega\) is solvable, it is possible to see that for any subalgebra of a Bernstein
algebra, length and dimension coincide. This allows us to prove in [6] the following theorem:

(0.5) Theorem. Let \( A \) be a Bernstein algebra. Then the following are equivalent:

(i) \( A \) is modular.

(ii) \( A \) is semimodular.

(iii) Given any pair of subalgebras of \( A \), \( S \), and \( T \), \( S + T \) is always a subalgebra of \( A \).

In [6] we also proved:

(0.6) Proposition. Let \( A \) be a trivial Bernstein algebra. Then \( A \) is modular if and only if \( A \) has type \((n+1, 0)\) or \((1, n)\).

(0.7) Lemma. A modular Bernstein algebra is always exclusive.

The following lemma was proved in [7]:

(0.8) Lemma. Let \( A \) be non-trivial modular Bernstein algebra. Let \( e \) be a non-zero idempotent in \( A \). Then \( v^2 \neq 0 \) for all \( v \neq 0 \) in \( V_e \).

1. Modular Bernstein Algebras over Arbitrary Fields

In this section we will generalize most of the results obtained in [7] for genetic Bernstein algebras (Bernstein algebras such that \( \text{Ker } \omega \) is nilpotent) to Bernstein algebras which are not necessarily genetic. Since modular Bernstein algebras are completely known when they are trivial (see (0.6)) we will begin with the study of some properties of modular non-trivial Bernstein algebras.

Let \((A, \omega)\) be a Bernstein algebra. If \( x \) is an element in \( \text{Ker } \omega \), we can consider \( m(x) \), the largest positive integer such that the elements \( x, x^2, ..., x^{m(x)} \) are linearly independent. Let us fix \( e \), a non-zero idempotent in \( A \). If we take \( 0 \neq v \) in \( V_e \), Lemma (0.8) tells us that \( m(v) \geq 2 \) and moreover, from the definition of \( m(v) \), we get that

\[
(v) = Kv + Kv^2 + \cdots + Kv^{m(v)}.
\]

(1.1) Lemma. Let \( A \) be a non-trivial modular Bernstein algebra. Let \( e \) be a nonzero idempotent in \( A \). Then

\[
U_e = Kv^2 + U_e v, \quad \text{for all } \quad 0 \neq v \text{ in } V_e.
\]
Proof. Take \( u \) in \( U_\varepsilon \), \( 0 \neq v \) in \( V_\varepsilon \). The product \( uv \) satisfies
\[
uv \in [(u) \lor (v)] \cap U_\varepsilon = [(u) + (v)] \cap U_\varepsilon \\
= [Ku + (Kv + Kv^2 + \cdots + Kv^{m(v)})] \cap U_\varepsilon.
\]
The element \( uv \) is in \( U_\varepsilon \) and so has zero component in \( v \). Thus we can write
\[
uv = xu + \beta v^2 + x,
\]
with
\[
x \text{ in } Kv^3 + \cdots + Kv^{m(v)} \subseteq U_\varepsilon \text{ if } x = 0 \text{ if } m(v) = 2),
\]
and \( x, \beta \in K \).

If \( x \neq 0 \), then \( u = x^{-1}(uv - \beta v^2 - x) \in Ku^2 + U_\varepsilon v \).

If \( x = 0 \), then \( uv = \beta v^2 + x \in Kv^2 + \cdots + Kv^{m(v)} \).

Hence \( K(u + v) + (Kv^2 + \cdots + Kv^{m(v)}) \) turns out to be a subalgebra of \( A \), since
\[
(u + v)^2 = v^2 + 2uv \in Kv^2 + \cdots + Kv^{m(v)}
\]
(\( u^2 = 0 \) since \( A \) is exclusive from (0.7)),
\[
(u + v) v^k = v^{k+1} \in Kv^2 + \cdots + Kv^{m(v)},
\]
\[
2 \leq k \leq m(v) \text{ (} uv^k = 0 \text{ since } A \text{ is exclusive as above)}.
\]

Using (0.5), \( Ke + K(u + v) + (Kv^2 + \cdots + Kv^{m(v)}) \) is also a subalgebra of \( A \) and so it has to contain the element \( u = 2e(u + v) \). Hence
\[
u \in [Ke + K(u + v) + (Kv^2 + \cdots + Kv^{m(v)})] \cap U_\varepsilon
\]
\[
= Kv^2 + \cdots + Kv^{m(v)} \subseteq Kv^2 + U_\varepsilon v.
\]

We have proved \( U_\varepsilon \subseteq Kv^2 + U_\varepsilon v \). The reverse containment holds for arbitrary Bernstein algebras from the elementary properties of Peirce decomposition.

(1.2) LEMMA. Let \( A \) be a non-trivial modular Bernstein algebra. Let \( e \) be a nonzero idempotent in \( A \). Then
\[
U_\varepsilon = Kv^2 + \cdots + Kv^{m(v)}, \quad \text{for all } 0 \neq v \in V_\varepsilon.
\]

Proof. From (1.1), it is enough to prove \( U_\varepsilon v \subseteq Kv^2 + \cdots + Kv^{m(v)} \). If \( u \) is in \( U_\varepsilon \), then \( e + u \) is an idempotent, since \( A \) is exclusive and, using (0.5), \( K(e + u) + (v) \) is a subalgebra of \( A \).

Hence \( uv = (e + u)v \in [K(e + u) + (v)] \cap U_\varepsilon = Kv^2 + \cdots + Kv^{m(v)} \).
(1.3) Corollary. Let $A$ be a non-trivial modular Bernstein algebra of type $(r + 1, s)$ and $e$ a non-zero idempotent in $A$. Then for all $0 \neq v$ in $V_e$ we have

$$m(v) = r + 1, \quad U_v = Kv^2 \oplus \cdots \oplus Kv^{r+1}.$$ 

(1.4) Corollary. Let $A$ be a non-trivial Bernstein algebra of type $(2, n-1)$. If $A$ is modular then it is genetic.

Proof. Fix $e$, a non-zero idempotent in $A$. From (1.3) we obtain

$$A = K e \oplus U_e \oplus V_e, \quad \text{where} \quad U_e = Kv^2, \quad \text{for any} \quad 0 \neq v \in V_e.$$ 

Thus $v^3 = xv^2$, for some $x$ in $K$.

If $x \neq 0$, then $(v - \frac{1}{2}x^{-1}v^2)^2 = v^2 - \frac{1}{2}x^{-1}v^2 = 0$, and $K(v - \frac{1}{2}x^{-1}v^2)$ is a subalgebra of $A$. Hence (use (0.5)), $Ke + K(v - \frac{1}{2}x^{-1}v^2)$ is a subalgebra which must contain the element $-\frac{1}{2}x^{-1}v^2 = 2e(v - \frac{1}{2}x^{-1}v^2)$, which is not possible.

We have proved that $U_e V_e = 0$.

Now it is clear that $(\text{Ker} \omega)^3 = (U_e \oplus V_e)^3 \subseteq U_e$, from the properties of the Peirce decomposition and $(\text{Ker} \omega)^3 \subseteq U_e(U_e \oplus V_e) = 0$ since $A$ is exclusive.

(1.5) Lemma. Let $A$ be a non-trivial modular Bernstein algebra of type $(r + 1, s)$ and $e$ a non-zero idempotent in $A$. Then

$$v^{r+2} \in K v^3 \oplus \cdots \oplus K v^{r+1}, \quad \text{for all} \quad v \in V_e.$$ 

Proof. We know, from (1.3), that $v^{r+2} = \sum_{i=1}^{r+1} x_i v^i$, for some $x_i$ in $K$ since $v^{r+2}$ is in $U_e$.

Take $u = \sum_{i=2}^{r} x_{i+1} v^i - v^{r+1}$. It is clear that $0 \neq u \in U_e$ and

$$(2x_3 v + u)^3 = 4x_3^2 v^3 + 4x_3 u v = 4x_3^2 v^3 + 4x_3 \left[ \sum_{i=2}^{r} x_i v^i \right] - 4x_3 v^{r+2}$$

$$= 4x_3 \left[ \sum_{i=2}^{r} x_i v^i \right] - 4x_3 v^{r+2} = 0.$$

Hence $K(2x_3 v + u)$ is a subalgebra of $A$ and, from (0.5), $Ke + K(2x_3 v + u)$ is also a subalgebra of $A$.

Thus $u = 2e(2x_3 v + u) \in Ke + K(2x_3 v + u)$, which is possible only if $x_2 = 0$.

Given $x_3, \ldots, x_{r+1} \in K$, we can define

$$p_1 = x_3 \in K[X_1]$$

$$p_k = X_k + \sum_{i=1}^{k-1} p_i x_{r+2-k+i} \in K[X_1, \ldots, X_k], \quad 2 \leq k \leq r.$$ 

Let us note that $p_k(0, \ldots, 0) = 0, k = 1, \ldots, r$. 

(1.6) **Lemma.** Let $A$ be an exclusive Bernstein algebra over $K$ and $e$ a non-zero idempotent in $A$. Let us suppose that there exists a non-zero element $v$ in $V$, such that $v^r, \ldots, v^{r+1}$ are linearly independent and

$$v^{r+2} = \sum_{i=3}^{r+1} x_i v^i, \quad x_i \in K, \quad i = 3, \ldots, r+1, \quad r \geq 2.$$ 

**Defining the elements**

$$u = x_r v^r + \cdots + x_1 v^{r+1} \in U, \quad \text{where } x_i \in K, \quad i = 1, \ldots, r,$$

$$e_1 = e + u \quad \text{which is a non-zero idempotent in } A,$$

$$v_1 = v - 2uv \in V, \quad e_1,$$

and writing $p_k(x_1, \ldots, x_k) = p_k, \quad 1 \leq k \leq r$, then

$$v_1^2 = v^2 - 4p_2z_3v^3 - 4 \sum_{i=4}^{r+1} (x_{r+4-i} + p_1z_{r+1} + p_2z_i) v^i,$$

$$v_1^k = -4 \sum_{i=3}^{k-1} \left( \sum_{j=3}^{i} p_{k-i+j} z_j \right) v^i + \left[ 1 - 4 \sum_{i=3}^{k} p_i z_i \right] v^k$$

$$- 4 \sum_{i=k+2}^{r+1} \left[ x_{r+2-i+k} + \sum_{j=1}^{k} p_j z_{i-k-j} \right] v^i$$

for $3 \leq k \leq r$ (with the obvious changes when $k = 3$ or $k = r$),

$$v_1^{r+1} = -4 \sum_{i=4}^{r} \left( \sum_{j=3}^{r-i+1} p_{r+1-i+j} z_j \right) v^i + \left[ 1 - 4 \sum_{i=3}^{r} p_i z_i \right] v^{r+1}$$

$$- 4 \left[ \sum_{i=2}^{r} p_i z_{i+1} \right] \left[ \sum_{i=3}^{r+1} z_i v^i \right].$$

**Proof.**

$$v_1 = v - 2uv = v - 2 \sum_{i=2}^{r+1} x_{r+2-i} v^{i+1}$$

$$= v - 2 \sum_{i=2}^{r} x_{r+2-i} v^{i+1} - 2x_1 \sum_{i=3}^{r+1} z_i v^i$$

$$= v - 2 \sum_{i=3}^{r+1} (x_{r+3-i} + p_i z_i) v^i.$$ 

Thus
\[\begin{align*}
v_1^2 &= v^2 - 4 \sum_{i=3}^{r+1} (x_{r+3-i} + p_i x_i) v^{i+1} \\
 &= v^2 - 4 \sum_{i=4}^{r+1} (x_{r+4-i} + p_i x_{i-1}) v^i - 4(x_2 + p_1 x_{r+1}) \sum_{i=3}^{r+1} x_i v^i \\
 &= v^2 - 4p_2 x_3 v^3 - 4 \sum_{i=4}^{r+1} (x_{r+4-i} + p_i x_{i-1} + p_2 x_i) v^i.
\end{align*}\]

\[v_1^3 = v_1^2 v = v^3 - 4p_2 x_3 v^4 - 4 \sum_{i=5}^{r+1} (x_{r+5-i} + p_i x_{i-2} + p_2 x_{i-1} + p_3 x_i) v^i - 4p_2 x_3 v^3 - 4p_3 x_4 v^4\]

\[(1 - 4p_3 x_3) v^3 - 4(p_2 x_3 + p_3 x_4) v^4 - 4 \sum_{i=5}^{r+1} \left[ x_{r+5-i} + \sum_{j=1}^{3} p_j x_{i-j} \right] v^i.\]

The general formula for \(v_k^4, k = 4, \ldots, r - 1\), can be seen by induction on \(k\).

For \(k = 4\),

\[\begin{align*}
v_4^4 &= v_1^3 v = (1 - 4p_3 x_3) v^4 - 4(p_2 x_3 + p_3 x_4) v^5 - 4p_4 x_3 v^3 - 4p_4 x_4 v^4 \\
 &\quad - 4p_4 x_5 v^5 - 4 \sum_{i=6}^{r+1} \left[ x_{r+6-i} + \sum_{j=1}^{4} p_j x_{i-j} + p_4 x_i \right] v^i \\
 &= -4p_4 x_3 v^3 + \left[ 1 - 4 \sum_{i=3}^{4} p_i x_i \right] v^4 - 4 \left[ \sum_{i=2}^{4} p_i x_{i+1} \right] v^5 \\
 &\quad - 4 \sum_{i=6}^{r+1} \left[ x_{r+6-i} + \sum_{j=1}^{4} p_j x_{i-j} \right] v^i.\end{align*}\]

Suppose the formula is true until \(k - 1\), where \(5 \leq k \leq r\). Then

\[\begin{align*}
v_k^4 &= v_1^{k-1} v = -4 \sum_{i=3}^{k-2} \left[ \sum_{j=i}^{i+2} p_{k-1-i+j} x_j \right] v^{i+1} \\
 &\quad + \left[ 1 - 4 \sum_{i=3}^{k-1} p_i x_i \right] v^k - 4 \left[ \sum_{i=2}^{k-1} p_i x_{i+1} \right] v^{k+1} \\
 &\quad - 4 \sum_{i=k+1}^{r+1} \left[ x_{r+2-i} + \sum_{j=1}^{k-1} p_j x_{i-j} + x_{i-k+1} \right] v^{i+1} \\
 &= -4 \sum_{i=4}^{k-1} \left[ \sum_{j=3}^{i+1} p_{i+1-j} x_j \right] v^i + \left[ 1 - 4 \sum_{i=3}^{k} p_i x_i \right] v^k - 4 \left[ \sum_{i=2}^{k} p_i x_{i+1} \right] v^{k+1} \\
 &\quad - 4 \sum_{i=k+2}^{r+1} \left[ x_{r+2-i} + \sum_{j=1}^{k-1} p_j x_{i-j} \right] v^i - 4p_k \sum_{i=3}^{r+1} x_i v^i.\end{align*}\]
\[ v^r = -4 \sum_{i=2}^{r} \left[ \sum_{i}^j p_{r-i+j} x_i \right] v^i + \left[ 1 - 4 \sum_{i=3}^k p_{i} x_i \right] v^k - 4 \sum_{i=2}^k p_{i} x_{i+1} v^{r+1}. \]

and

\[ v^r = -4 \sum_{i=2}^{r} \left[ \sum_{i}^j p_{r-1-i+j} x_i \right] v^i + \left[ 1 - 4 \sum_{i=3}^k p_{i} x_i \right] v^k - 4 \sum_{i=2}^k p_{i} x_{i+1} v^{r+1}. \]

Note. The hypothesis of (1.6) are satisfied by all non-trivial modular Bernstein algebras. Moreover, from (1.3), the element \( v_1 \) is such that \( m(v_1) = r + 1 \), i.e., \( v_1, \ldots, v_1^{r+1} \) are linearly independent and this must be true for any choice of \( x_1, \ldots, x_r \) in \( K \).

(1.7) Theorem. Let \( A \) be a non-trivial modular Bernstein algebra of type \( (r+1, s) \) and \( e \) a non-zero idempotent in \( A \). Then, for any \( v \) in \( V_r \), \( v^{r+2} = 0 \).

Proof. We have \( r \geq 1 \) since \( A \) is non-trivial. The case \( r = 1 \) can be seen in the proof of (1.4). Let us use the cases \( r = 2 \) and \( r = 3 \) independently.

Suppose \( r = 2 \). From (1.3) and (1.5), \( U_r = K v^2 + K v^3 \), \( v^4 = x_3 v^3 \), \( x_3 \in K \), for any \( 0 \neq v \) in \( V_r \). With the same notation as in (1.6),

\[ u = x_2 v^2 + x_1 v^3, \quad x_1, x_2 \in K, \]

\[ v_1 = v - 2uv = v - 2x_2 v^3 - 2x_3 v^3 = v - 2p_2 v^3, \]

\[ v_1^2 = v^2 - 4p_2 x_3 v^3, \]

\[ v_1^3 = v^3 - 4p_2 x_2^3 v^3. \]
If $x_3 \neq 0$, taking $x_1 = 0$, $x_2 = \frac{1}{4} x_3^{-2}$, we get

$$p_1 = 0, \quad p_2 = \frac{1}{4} x_3^{-2}, \quad \text{and} \quad v_1^4 = 0,$$

which is a contradiction since $v_1^4, v_1^3$ are linearly independent (see (1.3)).

Suppose $r = 3$. $U_r = K v^2 + K v^3 + K v^4$, $v^3 = x_4 v^3 + x_4 v^4$, $x_3, x_4 \in K$, for any $0 \neq v$ in $V_r$. As in (1.6),

$$u = x_3 v^2 + x_2 v^3 + x_1 v^4, \quad x_1, x_2, x_3, x_4 \in K,$$

$$v_1 = v - 2 u v,$$

$$v_1^2 = v^2 - 4 p_2 x_3 v^3 - 4 p_3 v^4,$$

$$v_1^3 = (1 - 4 p_3 x_3) v^3 - 4 (p_2 x_3 + p_3 x_4) v^4,$$

$$v_1^4 = (1 - 4 p_3 x_3) v^4 - 4 (p_2 x_3 + p_3 x_4) (x_3 v^3 + x_4 v^4),$$

If $x_3 \neq 0$, take $x_1 = 0$, $x_2 = -\frac{1}{4} x_4 x_3^{-2}$, $x_3 = \frac{1}{4} (x_3 + x_4) x_3^{-2}$. We get

$$p_1 = 0, \quad p_2 = x_2 + x_1 x_4 = x_2 = -\frac{1}{4} x_4 x_3^{-2},$$

$$p_3 = x_3 + p_1 x_3 + p_2 x_4 = \frac{1}{4} \left( x_3 + \frac{1}{4} x_4 \right) x_3^{-2} - \frac{1}{4} x_4 x_3^{-2} = \frac{1}{4} x_3^{-1}.$$ 

Hence

$$1 - 4 p_3 x_3 = 0,$$

$$p_2 x_3 + p_3 x_4 = -\frac{1}{4} x_4 x_3^{-1} + \frac{1}{4} x_3^{-1} x_4,$$

$$v_1^4 = 0,$$

which contradicts the fact that $v_1^2, v_1^3, v_1^4$ are linearly independent. Thus $x_3 = 0$ and

$$v_1^2 = v^2 - 4 p_3 v^4,$$

$$v_1^3 = v^3 - 4 p_3 x_4 v^4,$$

$$v_1^4 = v^4 - 4 p_3 x_4 v^4.$$ 

If $x_4 \neq 0$, take $x_1 = x_2 = 0$, $x_3 = \frac{1}{4} x_4 x_3^{-2}$, and we have $p_1 = p_2 = 0$, $4 p_3 x_4^2 = 4 x_3 x_4^2 = 1$. Hence $v_1^4 = 0$, which is a contradiction.

Suppose $r \geq 4$. $U_r = K v^2 + \cdots + K v^r$, $v^r = \sum_{i=3}^{r+1} x_i v^i$, where $x_3, \ldots, x_{r+1} \in K$, for any $0 \neq v$ in $V_r$.

Using Lemma (1.6) we will prove $x_k = 0$, $k = 3, \ldots, r + 1$.

For $k = 3$, if $x_3 \neq 0$, take

$$x_1 = 0, \quad x_2 = -\frac{1}{4} x_4 x_3^{-2}, \quad x_3 = \frac{1}{4} (x_3 + x_4 x_{r+1}) x_3^{-2},$$

$$x_i = \frac{1}{4} (x_4 x_{r+4} - x_3 x_{r+5} \ldots) x_3^{-2}, \quad i = 4, \ldots, r.$$
From (1.6), the coordinates of \( v^i_3 \) with respect to the basis \( v^2, \ldots, v^{r+1} \) are coordinate in \( v^3 \)

\[
1 - 4p_3 x_3 = 1 - 4(x_3 + p_1 x_1 + p_2 x_{r+1}) x_3 \\
= 1 - 4\left[ x_3 + x_1 x_3 + (x_2 + x_1 x_{r+1}) x_{r+1} \right] x_3 \\
= 1 - (x_3^{-1} + x_4 x_{r+1} x_3^{-2} - x_4 x_3^{-2} x_{r+1}) x_3 = 0.
\]

The coordinate in \( v^3 \) is a multiple of \( p_3 x_3 + p_3 x_4 = -\frac{1}{4} x_4 x_3^{-1} + \frac{1}{4} x_3^{-1} x_4 = 0. \)

For \( 5 \leq i \leq r+1 \), the coordinate in \( v^i \) is a multiple of

\[
x_{r+2-i+3} + p_1 x_{r-i} + p_2 x_{r-1} + p_3 x_i \\
= \frac{1}{4}(x_4 x_{r-1} - x_3 x_i) x_3^{-2} - \frac{1}{4} x_4 x_3^{-2} x_{r-1} + \frac{1}{4} x_3^{-1} x_i = 0.
\]

Thus we have \( v^1_3 = 0 \), which is impossible. Hence, \( x_3 = 0. \)

Suppose \( x_3 = \cdots = x_{k-1} = 0 \) for all \( k \) such that \( 3 \leq k-1 \leq r-2 \). Let us see \( x_k = 0. \) If \( x_k \neq 0 \), take \( x_1 = \cdots = x_{k-2} = 0, \)

\[
x_k - 1 = -\frac{1}{4} x_{k+1} x_k^{-2}, \quad x_k = \frac{1}{4}(x_k + x_{r+1} x_{k+1}) x_k^{-2}, \\
x_i = \frac{1}{4}(x_{k+1} x_{r+1+k-i} - x_k x_{r+2+k-i}) x_k^{-2}, \quad k+1 \leq i \leq r.
\]

We obtain

\[
p_1 = \cdots = p_{k-2} = 0, \quad p_{k-1} = -\frac{1}{4} x_{k+1} x_k^{-2}, \\
p_k = \frac{1}{4}(x_k + x_{r+1} x_{k+1}) x_k^{-2} - \frac{1}{4} x_{k+1} x_k^{-2} x_{r+1} = \frac{1}{4} x_k^{-1}.
\]

Let us study the coordinates of \( v^i_k \) (use (1.6)).

The coordinate in \( v^i \) is 0 for \( i = 3, \ldots, k-1, \) since \( x_3 = \cdots = x_{k-1} = 0. \) The coordinate in \( v^i \) is \( 1 - 4 \sum_{j=3}^i p_j x_j = 1 - 4p_k x_k = 0. \) The coordinate in \( v^{k+1} \) is a multiple of

\[
\sum_{j=3}^{k+1} p_j x_{j+1} = p_{k-1} x_k + p_k x_{k+1} = -\frac{1}{4} x_{k+1} x_k^{-1} + \frac{1}{4} x_k^{-1} x_{k+1} = 0.
\]

The coordinate in \( v^i \), for \( k+2 \leq i \leq r+1 \), is a multiple of

\[
x_{r+2-i+k} + \sum_{j=1}^k p_j x_{i-j} \\
= \frac{1}{4}(x_k + x_{r+1} x_k - x_k x_i) x_k^{-2} + p_{k-i} x_{r+i} + p_k x_i \\
= \frac{1}{4}(x_{k+1} x_{r+1+k-i} - x_k x_{r+2+k-i}) x_k^{-2} x_{r+1} + \frac{1}{4} x_k^{-1} x_i = 0.
\]

Hence \( v^i_k = 0 \), and \( k \leq r-1 \), which is a contradiction. Hence \( x_k = 0. \) We have proved \( x_3 = \cdots = x_{r-1} = 0. \)
If \( \varepsilon, \neq 0 \), take \( x_1 = \cdots = x_{r-2} = 0, x_{r-1} = -\frac{1}{4} x_{r+1} x_{r+2} \).

\[
x_r = \frac{1}{4} (x_r + x_{r+1}^2) x_{r+2}^{-2}.
\]

Thus \( p_1 = \cdots = p_{r-2} = 0, p_{r-1} = -\frac{1}{4} x_{r+1} x_{r+2}^{-2}, \)

\[
p_r = x_r + p_{r-1} x_{r+1} = \frac{1}{4} (x_r + x_{r+1}^2) x_{r+2}^{-2} - \frac{1}{4} x_{r+1} x_{r+2}^{-2} = \frac{1}{4} x_{r+1}^{-1}.
\]

Hence, \( v'_1 \) has the following coordinates in the basis \( v^1, \ldots, v'^{r+1} \) in \( v^1 \), for \( i = 3, \ldots, r-1; 1 - 4p_r x_r = 0 \) in \( v^1 \); and the coordinate in \( v'^{r+1} \) is a multiple of \( p_{r-1} x_r + p_r x_{r+1} = -\frac{1}{4} x_{r+1} x_r^{-1} + \frac{1}{4} x_r^{-1} x_{r+1} = 0 \). We get \( v'_1 = 0 \), which is not possible.

We have proved \( x_r = 0 \).

If \( x_{r+1} \neq 0 \), take \( x_1 = \cdots = x_{r-1} = 0, x_r = \frac{1}{4} x_{r+1}^{-1} \), and we obtain

\[
p_1 = \cdots = p_{r-1} = 0, \quad p_r = x_r.
\]

But now, \( v'^{r+2} = v'^{r+1} - 4p_r x_{r+1} v'^{r+1} = 0 \), which is a contradiction.

Hence \( x_{r+1} = 0 \), and \( v'^{r+2} = 0 \).

With these results we have just proved that the characterization of modular Bernstein algebras, given in \([7]\) supposing that the algebras are genetic, is valid without this restriction.

(1.8) Theorem. Let \((A, \omega)\) be a non-trivial Bernstein algebra of type \((r+1, s)\). Then the following are equivalent:

(i) \( A \) is a modular algebra.

(ii) \( A \) is exclusive and for all non-zero idempotent \( e \) in \( A \) and any \( 0 \neq v \) in \( V_e \),

\[
U_e = K v^2 + \cdots + K v^{r+1} \quad \text{and} \quad v^{r+2} = 0.
\]

(iii) \( A \) is exclusive and there exists a non-zero idempotent \( e \) in \( A \) such that for any \( 0 \neq v \) in \( V_e \),

\[
U_e = K v^2 + \cdots + K v^{r+1} \quad \text{and} \quad v^{r+2} = 0.
\]

(iv) \( A \) is exclusive and the subalgebras of \( A \) contained in \( \text{Ker} \, \omega \) are contained in \( U_e \) or contain \( U_e \).

(v) \( A \) is exclusive and the subalgebras of \( A \) contained in \( \text{Ker} \, \omega \) are exactly the subspaces of \( U_e \) and the ideals of \( A \) contained in \( \text{Ker} \, \omega \).

Proof. The difficult part of this proof is (i) \( \Rightarrow \) (ii), which is (0.7), (1.3), and (1.7). The rest of the proof is quite simple and can be found in \([7]\).
2. Modular Bernstein Algebras over Algebraically Closed Fields

The natural question, after proving (1.8) is if all modular Bernstein algebras are necessarily genetic. First of all, let us see that this question has a negative answer over arbitrary fields. Indeed we will construct a modular Bernstein algebra over \( \mathbb{Q} \), the field of rational numbers, that is not genetic.

(2.1) Example. Let \( K \) be \( \mathbb{Q} \) and \( A \) a commutative \( K \)-algebra of dimension 6, given by

\[
A = Ke + Ku_1 + Ku_2 + Ku_3 + Kv + Kw,
\]

where

\[
e^2 = e, \quad eu_i = 1/2u_i, \quad i = 1, 2, 3, \quad ev = ew = 0,
\]

\[
u, u_j = 0, \quad i, j = 1, 2, 3,
\]

\[
u_1 v = u_2, \quad u_2 v = u_3, \quad u_3 v = 0,
\]

\[
u_1 w = 0, \quad u_2 w = u_1, \quad u_3 w = -u_2,
\]

\[
u^2 = u_1, \quad w^2 = u_3, \quad vw = \frac{1}{2}(-u_1 + u_3).
\]

Define the linear map \( \omega: A \rightarrow K \), by \( \omega(e) = 1, \quad \omega(u_i) = 0, \quad i = 1, 2, 3, \quad \omega(v) = \omega(w) = 0 \). It is straightforward to see that \( (A, \omega) \) is a Bernstein algebra.

It is also quite simple to see that \( (A, \omega) \) satisfies the condition given in (1.8)(iii) for the idempotent \( e \). Thus \( (A, \omega) \) is modular. Nevertheless, \( A \) is not genetic since

\[
0 \neq u_1 = u_2 w = (u_1 v) w = ((u_2 w) v) w = \ldots.
\]

(2.2) Theorem. Any modular Bernstein algebra over an algebraically closed field \( K \) of characteristic different from two is genetic.

Proof. Let \( A \) be a modular Bernstein algebra over \( K \). If \( A \) is trivial, then it is obviously genetic. Thus, let us suppose that \( A \) is non-trivial of type \((r + 1, s)\).

The case \( r = 1 \) has already been studied in (1.4) and if \( s = 1 \), then it is readily seen that \( A \) is genetic by using (1.8)(ii) or (iii) and the formula about the product in subalgebras generated by a single element given in Section 0.

We will prove that these are the only possibilities. Suppose \( r \geq 2 \) and \( s \geq 2 \).
Let $e$ be a non-zero idempotent in $A$, $A = Ke + U_e + V_e$.

Let us consider $K(X)$, the simple transcendental extension of $K$, and $K(X) \otimes_K U_e$ as a vector space over $K(X)$. We know that if we take $0 \neq v \in V_e$, the elements $v^2, ..., v^{r+1}$ are a basis for $U_e$. Thus, in $K(X) \otimes_K U_e$ we have the basis \{1 \otimes v^2, ..., 1 \otimes v^{r+1}\}.

Since $r \geq 2$, we can find $w$ in $V_e$, such that $v$ and $w$ are linearly independent. Hence, using (1.8)(ii), for any $\lambda$ in $K$,

$$U_e = K(v + \lambda w)^2 + \ldots + K(v + \lambda w)^{r+1} \quad \text{and} \quad (v + \lambda w)^{r+2} = 0.$$

Denote $f_\lambda = R_{v + \lambda w}$, the multiplication by $v + \lambda w$, restricted to $U_e$. It is clear that $f'_\lambda \neq 0$ and $f''_\lambda \neq 0$, since $f''_\lambda = (v + \lambda w)^2 = (v + \lambda w)^{r+1} \neq 0$.

Put $g = R_w|_{U_e}$, thus $f_\lambda = f_0 + \lambda g$.

The matrix of $f_0$ relative to the basis \{v^2, ..., v^{r+1}\} is

$$E = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{bmatrix}.$$

Put $B$ for the matrix of $g$ relative to the same basis.

Hence, the matrix of $f_\lambda$ relative to the basis \{v^2, ..., v^{r+1}\} is $E + \lambda B$ and it satisfies \((E + \lambda B)^r = 0\) and \((E + \lambda B)^{r-1} \neq 0\).

Define $f \in \text{End}_{K[X]}[K(X) \otimes_K U_e]$ given by the matrix $E + XB$ relative to the basis \{1 \otimes v^2, ..., 1 \otimes v^{r+1}\}.

It is clear that $f' = 0$, $f'' \neq 0$, and, since $K(X) \otimes_K U_e$ has dimension $r$, $\text{Ker} f = \text{Im} f''$ has dimension 1 over $K(X)$. In particular, there exists $j$ such that $f'' \neq 0$ is in $\text{Ker} f$.

The elements in the matrix $E + XB$ (the matrix of $f$) are polynomials in $X$ of degree less than or equal to 1. Thus, with respect to the same basis, the entries of the matrix of $f''$ are polynomials in $X$ of degree less than or equal to $r - 1$. Put

$$0 \neq w(X) = f''(1 \otimes v^j) = \sum_{i = 2}^{r+1} p_i(X)(1 \otimes v^j) \in \text{Im} f'' = \text{Ker} f,$$

where $p_i(X)$ is in the polynomial ring $K[X]$ and has degree less than or equal to $r - 1$, for $i = 2, ..., r + 1$.

On the other hand, for all $\lambda$ in $K$, $0 \neq (v + \lambda w)^{r+1} \in \text{Ker} f_\lambda$.

Put $(v + \lambda w)^{r+1} = \sum_{i = 2}^{r+1} q_i(X) v^i$, where $q_i(X) \in K[X]$.

Hence, the vector $0 \neq \tilde{w}(X) = \sum_{i = 2}^{r+1} q_i(X)(1 \otimes v^i) \in K(X) \otimes_K U_e$ is an element in $\text{Ker} f$. As $\text{Ker} f$ has dimension 1, we get $\tilde{w}(X) = (r(X)/s(X)) w(X)$,
with \( r(X)/s(X) \in K(X) \), \( r(X), s(X) \in K[X] \), relatively prime polynomials, \( r(X) \neq 0 \). That is to say,

\[
\sum_{i=2}^{r+1} q_i(X)(1 \otimes v') = \frac{r(X)}{s(X)} \sum_{i=2}^{r+1} p_i(X)(1 \otimes v').
\]

(\( \ast \))

We can “spend” \((v + \lambda w)^{r+1} \), for all \( \lambda \) in \( K \), obtaining \((v + \lambda w)^{r+1} = \sum_{i=2}^{r+1} \alpha_i(\lambda) v^i + \lambda^{r+1} w^{r+1} = \sum_{i=2}^{r+1} q_i(\lambda) v^i \), where \( \alpha_i(X) \) has degree less than or equal to \( r \), for all \( i \).

We know that \( w^{r+1} \neq 0 \), hence there must exist \( j \) such that \( q_j \) has degree \( r+1 \).

From equality (\( \ast \)), we can assure that \( \text{deg}(r(X)) - \text{deg}(s(X)) \geq 2 \), since \( \text{deg}(p_i(X)) \leq r - 1 \), for all \( i = 2, \ldots, r+1 \). In particular, \( r(X) \) is not constant and, using that \( K \) is algebraically closed, there exists \( \lambda_0 \) in \( K \), such that \( r(\lambda_0) = 0 \). Thus, \((v + \lambda_0 w)^{r+1} = 0 \), which is a contradiction. 

(2.3) **Corollary.** Let \( K \) be an algebraically closed field of characteristic different from two and \( A \) a non-trivial Bernstein algebra of dimension \( n+1 \) over \( K \). Then \( A \) is modular if and only if \( A \) is isomorphic to the commutative algebra

\[
Ke \oplus Ku_1 \oplus \cdots \oplus Ku_{n-1} \oplus Kv,
\]

where

\[
e^2 = e, \quad eu_i = \frac{1}{2} u_i, \quad 1 \leq i \leq n - 1, \quad ev = 0,
\]

\[
u_i u_j = 0, \quad 1 \leq i, j \leq n - 1,
\]

\[
u_i v = u_{i+1}, \quad 1 \leq i \leq n - 2, \quad u_{n-1} v = 0, \quad v^2 = u_1.
\]

**Proof.** We have seen in the proof of the previous result that if the type of \( A \) is \((r+1, s)\), then either \( r = 1 \) or \( s = 1 \). If \( s \geq 2 \), then \( r = 1 \). Put \( A = Ke + U_e + V_e \), for \( e \) a non-zero idempotent in \( A \). The subspace \( U_e \) has dimension 1 and the product of the algebra restricted to \( V_e \) can be considered, once we have fixed a basis of \( U_e \), as a symmetric bilinear form defined in \( V_e \). Since \( K \) is algebraically closed and \( V_e \) has dimension bigger than one, there exists some non-zero isotropic vector, i.e., there exists a non-zero element, \( v \) in \( V_e \), whose square is zero, which contradicts (1.8). Thus, \( s = 1 \). Put \( A = Ke + U_e + V_e \), for \( e \) a non-zero idempotent in \( A \), \( V_e = Kv \). Take \( u_e = v^{n-1} \). We claim that the multiplication is as we wanted, from (1.8)(ii) and the formula for the product in the subalgebras generated by a single element, given in Section 0. 

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