7 The Four-Colour Problem

Computer Mathematics Comes of Age

In 1976, two mathematicians at the University of Illinois, Kenneth Appel and Wolfgang Haken, announced that they had solved a century-old problem to do with the colouring of maps. They had, they said, proved the *four-colour conjecture*. This in itself was a newsworthy event. The four-colour problem was, after Fermat's last theorem (see Chapter 8), probably the second most famous unsolved problem in mathematics. But for mathematicians the really dramatic aspect of the whole affair was the way the proof had been achieved. Large and crucial parts of their argument were carried out by a computer, using ideas which had themselves been formulated as a result of computer-based evidence. So great was the amount of computing required that it was not feasible for a human mathematician to check every step. This meant that the whole concept of a 'mathematical proof' had suddenly changed. Something that had been threatening to occur ever since electronic computers were first developed in the early 1950s had finally happened; the computer had taken over from the human mathematician part of the construction of a real mathematical proof.

Until then, a *proof* had been a logically sound piece of reasoning by which one mathematician could convince another of the truth of some assertion. By reading a proof, a mathematician could become convinced of the truth of the statement in question, and also come to understand the reasons for its truth. Indeed, a proof worked as a proof precisely because it did provide those reasons!

Very long proofs such as the classification theorem for simple groups (described in Chapter 5) tend to stretch this simplistic view of a proof to some extent, since the average mathematician, faced with a proof which occupies (say) two 500-page volumes, would be tempted to skip over a great many of the details. But this is really only a matter of economy of effort. Secure in the knowledge that others have checked the various parts of the argument, the busy mathematician need not examine every step in detail. Such proofs are still the product of human endeavour alone. Though computers were used in proving parts of the classification theorem for simple groups, the results they produced could all be checked by hand. The role played by the computer was in no way an 'essential' one.

However, in the proof of the four-colour conjecture the use of the computer was absolutely essential – the proof hinged directly on it. In order to accept the proof you have to believe that the computer program used does what its authors claim of it. When Appel and Haken submitted their proof for publication in the *Illinois Journal of Mathematics*, its editors arranged for the computer part of the proof to be checked *by running an independently produced computer program on another machine*! So a critical part of the proof remained hidden from human eyes.

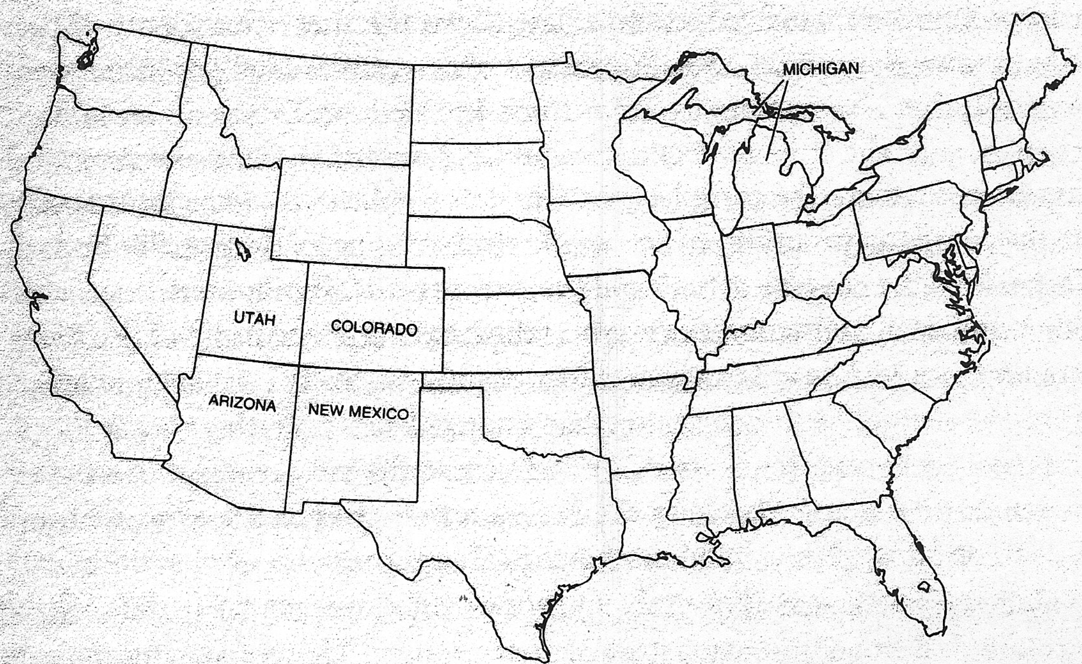
At first considerable scepticism was voiced by a great many mathematicians. 'Such a procedure, which makes essential use of the results obtained on a computer, results which by their very nature cannot be checked by human hand, cannot be regarded as a mathematical proof,' argued one critic. For such people, the four-colour problem remained unsolved. And indeed, the question of whether or not a 'standard' proof can be found remains open to this day. Given the sheer complexity of the computations involved, even supporters of computer-aided proofs have to concede that the opponents have some justification in their views, and even now at the time of writing, some ten years after the proof was first announced, there are periodic rumours that a subtle error has been found in the computer program, which would render the proof useless. But by and large, with the passage of time and the growing use of computers in society, the number of mathematicians who refuse to accept the proof of the four-colour theorem has gradually declined, and the majority now acknowledge that the arrival of the computer has changed not only the way a lot of mathematical research is carried out, but also the very concept of what is regarded as a proof. Checking the program that produces the 'proof' now has to be allowed as a valid mathematical argument.

And so what was this problem whose solution was to have such a profound effect on the very nature of mathematics? The story begins almost exactly one hundred years before the first commercial computers were built.

Guthrie's Problem

One day in October 1852, shortly after he had completed his studies at University College, London, the young mathematician Francis Guthrie (who was to go on to become Professor of Mathematics at the South African University in Cape Town) was colouring in a map showing the

Figure 31. Map of the USA. By using four colours it is possible to colour in all the states so that no two states which share a common border are coloured the same. Thus Colorado and New Mexico (for example) have to be coloured differently, though Colorado and Arizona may be coloured the same since they touch only at a point. In a mathematical study states such as Michigan, which consist of two physically separate regions, must be regarded as separate entities. You should have no difficulty in demonstrating for yourself that the map cannot be coloured (in the manner specified) using only three colours.



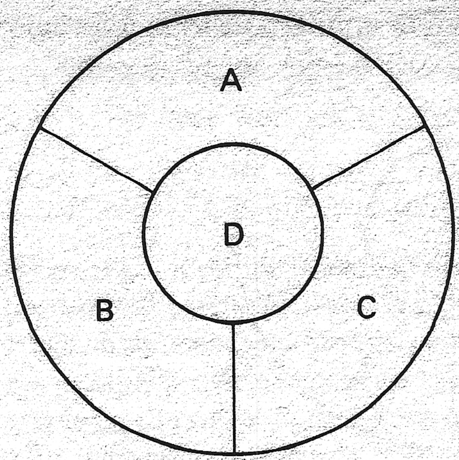


Figure 32. Three colours are not enough. In order to colour the map shown in such a way that no two adjacent countries receive the same colour, you have to colour the countries A, B, C, D using four different colours.

counties of England. As he did so it occurred to him that, in order to colour *any* map (drawn on a plane) subject to the obviously desirable requirement that no two regions (countries, counties, or whatever) sharing a length of common boundary should be given the same colour, the maximum number of colours required seemed likely to be four (see Figure 31). Being unable to prove this, he communicated the problem to his brother Frederick, still at University College as a student of physics. Frederick passed it on to his mathematics professor, the great English mathematician Augustus de Morgan.

Like Francis Guthrie, de Morgan had no difficulty proving that at least four colours are necessary (i.e. that there are maps for which three colours are not sufficient – see Figure 32). He also proved (see later) that it is not possible for five countries to be in a position such that each of them is adjacent to the other four, which at first glance might appear to imply that four colours are always sufficient, but which does not in fact imply this at all (see Figure 33), as de Morgan himself appears to have realized. (Many of the numerous false 'proofs' of the four-colour conjecture that appeared between its formulation in 1852 and its eventual proof in 1976 were based upon a belief in this invalid implication. Indeed, it seems that Francis Guthrie himself at one stage fell into this particular trap.)

Unable to solve the problem, de Morgan passed it on to his students and to other mathematicians (among them Sir William Hamilton of Trinity College, Dublin, the inventor of quaternions – see Chapter 3), giving credit to Guthrie for raising the question. But by and large the problem does not

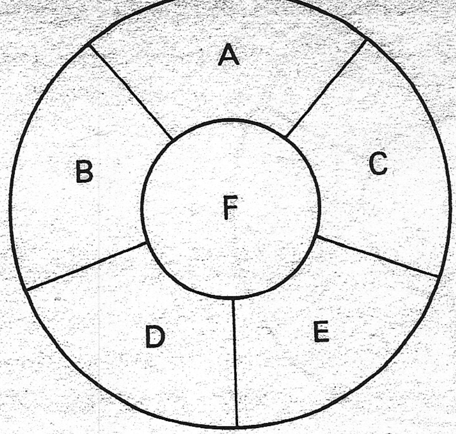


Figure 33. Highlighting a false argument. Many people have assumed that because no map allows the configuration in which each of five countries shares a common border with the other four, four colours will suffice to colour the map. This implication is not valid. In the map shown, there is no configuration in which each of four countries shares a common border with the three adjoining ones, and yet the map as a whole cannot be coloured using three colours, so the number of colours needed to colour a map is *not* the same as the highest number of mutually adjacent countries in the map.

seem to have aroused very much interest until, on 13 June 1878, the English mathematician Arthur Cayley asked the assembled members of the London Mathematical Society if they knew of a proof of the conjecture. (Cayley's question was published in the Society's Proceedings, and this was the first mention of the problem in print.) With this act the hunt was about to begin.

Maps, Networks, and Topology

The first major difficulty facing anyone who sets out to prove the four-colour conjecture is that it refers to *all* maps – not just all the maps in all the atlases around the world, but all conceivable maps, maps with millions (and more) of countries of all shapes and sizes. Knowing that you can colour some particular map using four colours does not help you at all.