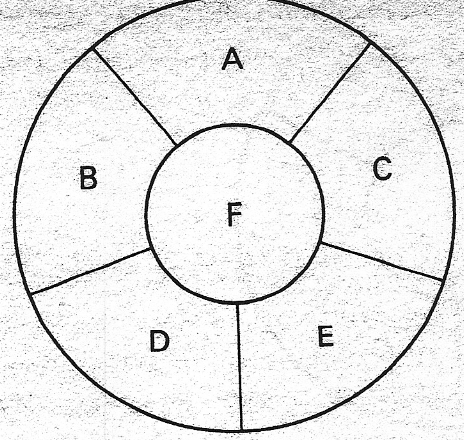


Figure 33. Highlighting a false argument. Many people have assumed that because no map allows the configuration in which each of five countries shares a common border with the other four, four colours will suffice to colour the map. This implication is not valid. In the map shown, there is no configuration in which each of four countries shares a common border with the three adjoining ones, and yet the map as a whole cannot be coloured using three colours, so the number of colours needed to colour a map is *not* the same as the highest number of mutually adjacent countries in the map.

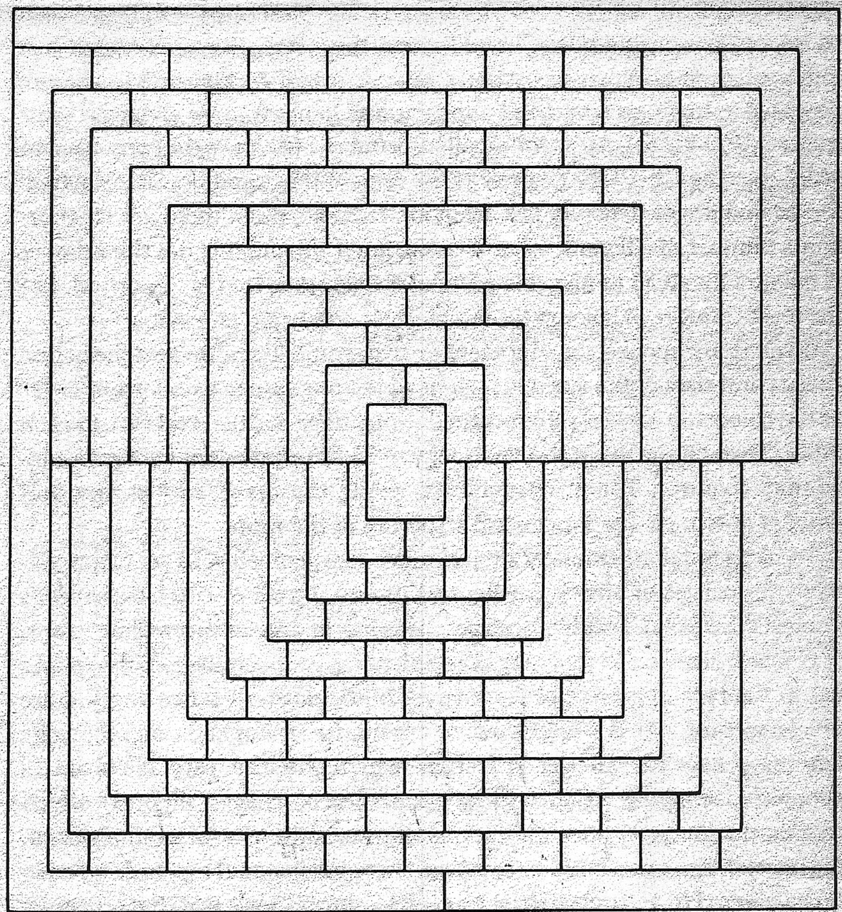
seem to have aroused very much interest until, on 13 June 1878, the English mathematician Arthur Cayley asked the assembled members of the London Mathematical Society if they knew of a proof of the conjecture. (Cayley's question was published in the Society's Proceedings, and this was the first mention of the problem in print.) With this act the hunt was about to begin.

Maps, Networks, and Topology

The first major difficulty facing anyone who sets out to prove the four-colour conjecture is that it refers to *all* maps – not just all the maps in all the atlases around the world, but all conceivable maps, maps with millions (and more) of countries of all shapes and sizes. Knowing that you can colour some particular map using four colours does not help you at all.

You need to produce an argument that will work in all cases. Which means that you have very little to go on. In fact, just what is there to go on? At this stage it is wise to make sure that we are certain what is involved in the problem.

Figure 34. A complicated map that would be difficult (though not impossible!) to colour using just four colours. (Try it!) This particular map was published as part of an April Fool's joke in the magazine *Scientific American* on 1 April 1975. It was included in an article by the celebrated mathematical columnist Martin Gardner who, with his tongue placed firmly in his cheek, claimed that it was an example of a map which refuted the long-standing four-colour conjecture.



For the purposes of Guthrie's problem, a *map* consists of an arbitrary number of regions of the plane ('countries' if you like) separated from each other by lines (or 'borders'). This general definition includes maps of the real world, as in Figure 31, as well as artificial, 'mathematical' maps as in Figures 32, 33, and 34. Actually there is a potential problem with the map of the USA – Figure 31 – in that some states occupy two distinct regions. For instance, Michigan consists of two separate regions on the map, separated by Lake Michigan. As they are physically separate regions they would have to be regarded as such as far as map colourings are concerned. Likewise Long Island, New York, would have to be considered a separate entity from the rest of New York State. Thus as far as a mathematical study of maps is concerned, geometry is the dominant notion, not politics.

The four-colour conjecture concerns the colouring in of the (geographical, not political!) regions of a map in such a way that no two regions which share a common boundary are coloured the same. (Regions which just touch at a point, such as Arizona and Colorado in Figure 31, are not regarded as having a common boundary, and hence may be given the same colour.) What is at issue is the smallest number of colours that you need in order to colour in all the regions of the map in this manner. This is where the second major difficulty lies. Even for a specific map, there are an enormous number of different ways of colouring it in, and it is not the number of colours involved in any one particular colouring that is important, but the least number of colours for which *some* colouring is possible.

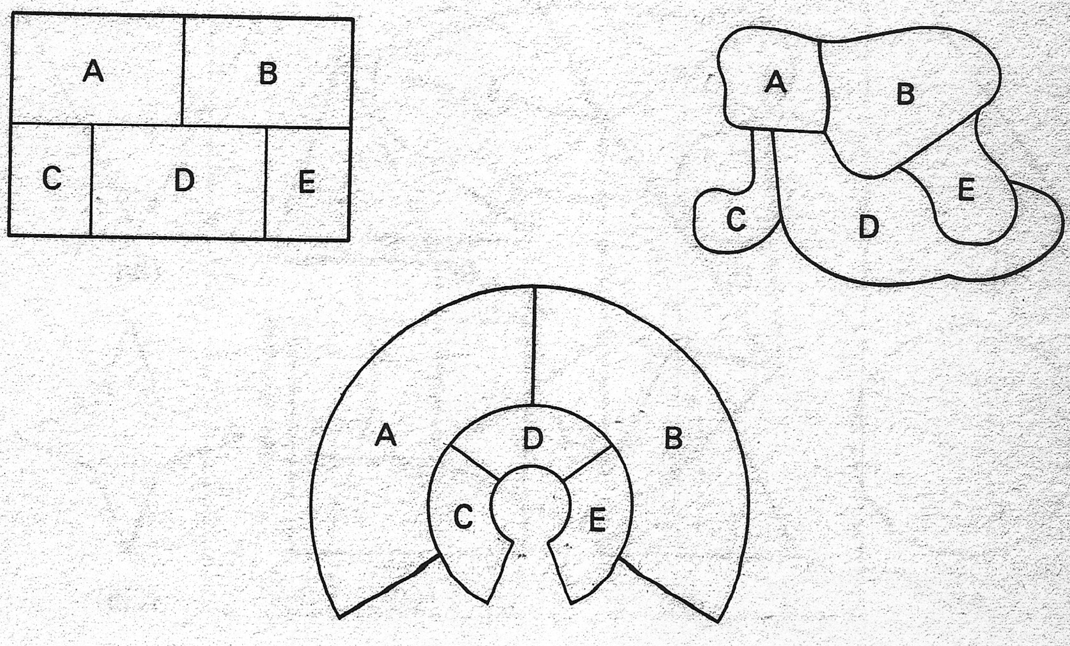
If you think about it for a moment or two, you will realize that the actual shapes and sizes of the various regions of the map are not important as far as the colouring problem is concerned – only their positions relative to each other. Thus all the maps shown in Figure 35 are equivalent for the would-be map colourer. The mathematician would express this by saying that what is at issue is the topological structure of the map.

Topology is a mathematical subject much like geometry. In geometry you study properties of objects (or figures) in two, three, or more dimensions ('objects' takes on a highly abstract meaning in four or more dimensions, of course). Likewise in topology. The difference between the two disciplines lies in the type of properties considered. In topology, distance and size are not important, nor is straightness or circularity or angle. In fact topology ignores practically all the properties which are the very life-blood of geometry, studying instead those properties of objects (figures) which remain unchanged under continuous transformations – for example bending, stretching, squashing, or twisting. Two-dimensional topology is sometimes referred to as 'rubber-sheet geometry', since it deals with the properties of figures which would not be changed if the figures were drawn on a 'perfectly elastic' rubber sheet which was then stretched and twisted about (see Figure 36).

To anyone meeting the idea of topology for the first time it would appear that there is not nearly enough to enable a reasonable mathematical study to be made, but in fact nothing could be further from the truth. Topology is a vast area of mathematics in which there are many deep and profound results (see Chapter 10). Indeed, the four-colour problem is itself a problem of topology, though its solution does not use any of the deeper techniques of the subject. Figure 35 illustrates why this is so – the shapes and sizes of the countries that make up a map are not important, only their configuration. You will find it helpful in understanding what follows if you try to bear in mind that it is this *topological* nature of a map that counts, not its superficial 'shape'.

Once this point is grasped, the notion of the *neighbouring network* of a map seems a sensible alternative way of looking at the four-colour problem. Given a map, the neighbouring network is obtained as follows (see Figure 37). Within each region of the map you place a single point, known as a *node* of the network. (You can think of these points as the capital cities

Figure 35. Topological equivalence. Each of the maps shown is equivalent as far as the four-colour problem is concerned; topologically there is no difference between any of them.

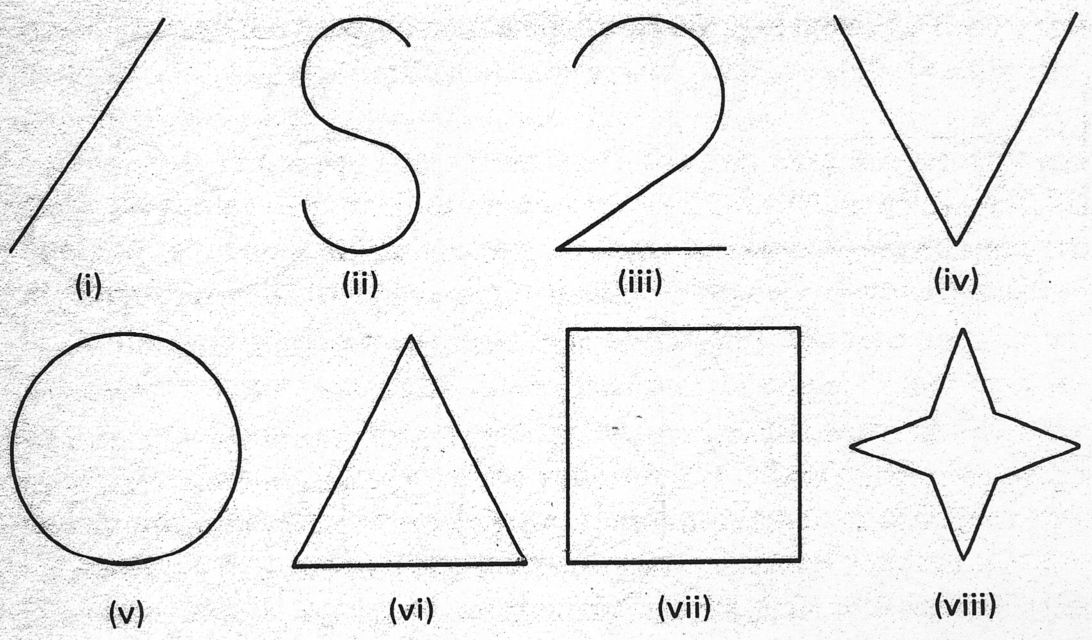


of the countries, if you wish.) You then join up the nodes in a certain way to form a network (in much the same way that you might link cities by a rail network). The rule is that two nodes are joined together if, and only if, their respective map regions share a common boundary, in which case the line joining them has to lie entirely within the two regions, crossing over the common boundary. (In terms of a rail link this would mean that the line cannot cross the territory of any third country.)

The neighbouring network shows at a glance the topological structure of the map it represents. Indeed, the problem of colouring the map (in the sense of Guthrie's problem) can be reformulated in terms of colouring the network: colour the nodes of the network in such a way that any two nodes which are connected together must have different colours. If all networks can be so coloured using four colours, so can all maps, and vice versa. So the network formulation of the four-colour problem provides an alternative way of looking at it which is entirely equivalent to the original formulation, and it makes sense to investigate such networks.

This brings the problem into the area known as *graph theory*. Notice that as a consequence of the way a neighbouring network was defined, no two paths in the network may cross (or intersect). A *graph* is similar to a

Figure 36. Topological properties in two dimensions. The objects (i) to (iv) are topologically the same, and the objects (v) to (viii) are topologically the same, but none of (i) to (iv) is the same as any of (v) to (viii).



neighbouring network except that this restriction on paths not crossing is removed. (This use of the word 'graph' is not really connected with its other use in mathematics to refer to curves representing equations drawn on 'graph paper'.) Though much of the original impetus behind the subject of graph theory was provided by the four-colour problem (Hamilton, to whom de Morgan communicated the problem, did a lot of the early work in graph theory), the study of arbitrary 'graphs' is now a large and thriving subject in its own right.

Figure 37. Neighbouring networks. To obtain the neighbouring network for a given map, place a point inside each region of the map and connect together these 'nodes' by lines which lie entirely within the two regions concerned. This is possible only when the two nodes lie in regions which share a common border, in which case the connecting line will cross that border. So the connections reflect the existence of common borders. Colouring the map so that no two adjacent countries are coloured the same is equivalent to colouring the nodes of the neighbouring network so that no two nodes joined by a path of the network are coloured the same. In the example shown it is possible to join each of the nodes with straight lines, but this is not always the case and curved connections are permitted. (Straightness and curving are not topological properties.)